RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 30, NUMBER 3 SEPTEMBER 1984

Degree of randomness of the sequence of eigenvalues

Giulio Casati
Dipartimento di Fisica dell'Università, Via Celoria 16, I-20133 Milano, Italy

Italo Guarneri
Dipartimento di Fisica Teorica e Nucleare dell'Università, Via Bassi, I-27100 Pavia, Italy

Fausto Valz-Gris
Dipartimento di Fisica dell'Università, Via Celoria 16, I-20133 Milano, Italy

(Received 21 May 1984)

Algorithmic complexity theory is used to discuss the degree of randomness of the sequence of energy eigenvalues of conservative quantum systems.

The identification of the features of conservative, finite quantum systems whose classical limit is chaotic (QCS) or integrable is of great interest for the foundations of quantum statistical mechanics and for several applications.

Here, we will consider the possibility to distinguish between these two categories of quantum systems just by looking at the sequence of eigenvalues. This last point is quite controversial. Some arguments based on numerical computations[1-5] lead to the conclusion that for QCS there is level repulsion and this is taken as an indication that the levels are not at "random;" on the other hand it is claimed that, for integrable systems there is no level repulsion, and that the nearest spacing distribution follows the Poisson law;[6,7] this may suggest that levels for integrable systems are, in some sense, more random.

Connecting disorder in quantum mechanics with the degree of randomness of the sequence of eigenvalues may be a fruitful research program, but all its chances rely on the possibility of removing that intriguing notion of randomness from the sphere of one's private feelings into a more objective domain.

While "random" is a quite useful word, giving it an operational, objective meaning is an open and perhaps unsolvable problem, and so is not our aim here. Nevertheless, we adhere to a point of view originated by the works of Kolmogorov[8] and Martin-Loef,[9] and lately advocated by J. Ford,[10] and submit that an algorithmic complexity theory can bring substantial clarification to this particular problem. This theory does not provide, in general, an effective test for randomness of given sequences; yet, it can sometimes enable us to say that certain sequences are not random.

In this paper we deal with one such case. We consider an example of an integrable system for which analytical arguments and numerical results[6] indicate that the corresponding level spacing distribution obeys the Poisson law $P(s)=e^{-s}$ and we show that the sequence of spacings has zero algorithmic complexity. The proof we give seems to be quite easily generalized to a much broader class of integrable systems.

We have no idea as to what could be said about the complexity of the eigenvalues of nonintegrable systems. As a matter of fact, very little is known at present about the spectral features of this class of systems. Nevertheless, there are now reasons to believe that random matrix theory may yield a model for "chaotic" behavior. In particular, statistical tests performed on "chaotic" billards gave results analogous to the Gaussian orthogonal ensemble. So, it seems useful to inquire about the complexity of the eigenvalues of a random matrix, in order to get a term for comparison for the, albeit particular, result obtained for an integrable billard. As a matter of fact, we show that the spectral sequence of a random matrix in the orthogonal circular ensemble has positive complexity.

Let us consider a billard ball in a rectangle with incommensurate sides.

The eigenvalues, in suitable units, are given by

$$E_{n,m}=\alpha m^2+n^2 \quad (\alpha>1) \ . \tag{1}$$

Let us rearrange the double sequence $E_{n,m}$ in increasing order, thus obtaining the sequence E_N. Since in this case there are no degeneracies, from Weyl's formula we know that (asymptotically)

$$E_N \sim \frac{4\pi}{A} N \ ,$$

where A is the area of the billard. Therefore, the sequence of spacings $s_N=E_N-E_{N-1}$ has the property that

$$\lim \frac{1}{N} \sum_{i=1}^{N} s_i = \frac{4\pi}{A} = \bar{s} \ .$$

A beautiful argument by Berry and Tabor,[6] supported by numerical computations, shows that the statistics of the sequence s_n should yield a Poisson distribution. This suggests the possibility that levels come at random as in a Poisson process. Of course, the knowledge of the spacing distribution alone is not sufficient to answer this question: much more detailed information concerning the correlation between different spacings are needed.

From the point of view of algorithmic complexity theory, a distinctive property of a random sequence of numbers A_N such as, e.g., the spacings in a "typical" Poisson sequence, is the following: any sequence of integers I_N obtained by truncating A_N with sufficiently high precision and to some maximum value has positive complexity—in other words, the Kolmogorov complexity[11] of finite strings of I_N's of length L is asymptotically $\sim$ const L.

In the following we show that the sequence s_n has zero

complexity, in the sense that the above constant is zero for any truncation. To this end we will devise an algorithm which gives the Nth eigenvalue E_N to within a given precision Δ and show that the required program length increases asymptotically as $\ln N$.

The algorithm can be divided into two steps. The first provides $E_{m,n}$ for $0 \leq m,n \leq K(N)$. The second step rearranges the $K^2(N)$ numbers $E_{m,n}$ in increasing order thus obtaining the string $E_n'[n \leq K^2(N)]$.

The number $K(N)$ must be chosen in such a way to ensure that

$$E_J' = E_J \quad \text{for } J \leq N \ .$$

The length of the program needed for the above is, basically, the length needed to specify (1) the number $K(N)$ and (2) the irrational number α with the accuracy required to compute the eigenvalue E_N with precision Δ.

One way of finding $K(N)$ is illustrated in Fig. 1. $K(N)$ can be any integer such that the curve $H = E_N$ lies completely inside the square of side $K(N)$. Therefore, we may assume $K(N) \geq [\sqrt{E_N}] + 1$, where [ ] denotes the integral part. By looking at the dashed triangle we see that $[(\alpha^{-1}E_N)^{1/2}]^2 < 2N$ whence $\sqrt{E_N} < \sqrt{\alpha}(1+\sqrt{2N})$. Therefore, one possible choice of $K(N)$ will be just $2 + [\sqrt{\alpha}(1+\sqrt{2N})]$.

As to the second point, the error in computing $E_{m,n}$ is $\delta E_{m,n} = \delta\alpha m^2$. Since $m^2 \leq K^2(N)$ one has that $\delta E_{m,n} \leq \Delta$ if

$$\delta\alpha \leq \frac{\Delta}{K^2(N)} \sim \frac{\Delta}{2N} \ .$$

This shows that in order to compute N eigenvalues with precision Δ the number of digits in α must increase as $\ln N$. Therefore, the minimum program length required to compute N eigenvalues within Δ increases no faster than const $\ln_2 N$, and the sequence of eigenvalues is not random.

Our construction seems easily generalizable to a wider class of integrable systems. In particular, the following conditions seem sufficient in order that the same conclusion holds: (i) $H(I_1, \ldots, I_m)$ is a smooth function of algebraic growth at infinity; (ii) the ratio of the volume V_E of the smallest hypercube containing the hypersurface $H = E$ to the volume S_E of the region where $H \leq E$, is bounded.

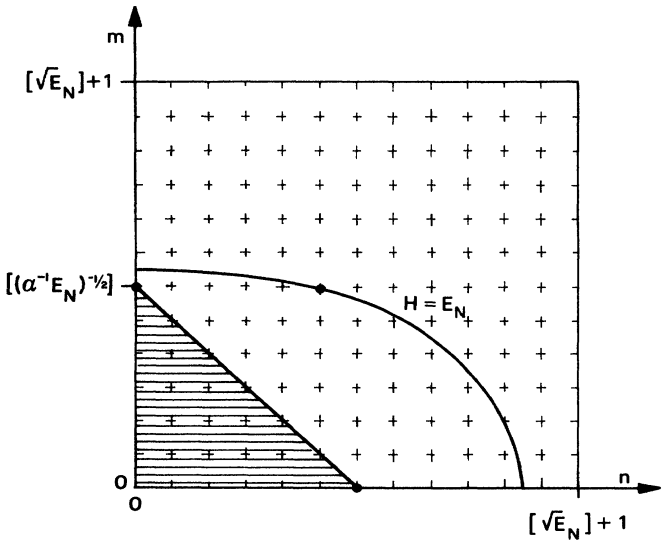

FIG. 1. Illustrating the choice of $K(N)$.

Independently of the above conclusions, empirical statistical tests commonly used in nuclear physics, such as the Δ_3 Dyson-Mehta statistics, might be applied to the present case. These tests involve an "unfolding" procedure, and are sensitive to the particular choice of the "smoothed" level density used in that procedure. Our result holds, independently of this choice, as long as the "smoothed" density is a computable function and introduces no complexity of its own in the algorithm. On the other hand, it seems hardly reasonable that this requisite of computability of the smoothed density can be dispensed with, whatever definition of the latter one may wish to adopt.

It is a known fact that sequences of eigenvalues of random matrices look quite different than, e.g., Poisson sequences. They are much more "rigid" because of correlations between different levels. This is particularly apparent in the orthogonal circular ensemble $\mathscr{E}_N$ which consists of symmetric unitary matrices of rank N, whose eigenvalues are distributed on the unit circle according to the density

$$W_N(\theta_1, \ldots, \theta_N) = C_N \prod_{1 \leq j < K \leq N} |e^{i\theta_j} - e^{i\theta_K}| \qquad (2)$$

with C_N a normalization constant. W_N has a maximum $W_{N_0} = C_N e^{N \ln N/2}$ when the θ_l are equally spaced and, therefore, are very regularly distributed.[12] On account of the regularity of this "most likely" configuration, one might conjecture that the string obtained by ordering the θ_l is a somewhat "ordered" string. Then, suppose that $\{\theta_1, \ldots, \theta_N\}$ are measured in units of the average spacing $2\pi/N$ with an approximation δ. (In the sequel we take $\delta = 1$.) By ordering them, we get a string of N integer numbers, ranging from 1 to N. We are going to investigate the complexity of this string for $N \to \infty$.

For given N, the total number $\mathscr{N}_N$ of nondecreasing strings of length N, that one can form by using numbers from 1 to N is

$$\mathscr{N}_N = \frac{(2N-1)!}{N!(N-1)!} \ . \qquad (3)$$

The maximum complexity of one such string is, asymptotically,

$$O(\ln_2 \mathscr{N}_N) = O(N) \ .$$

The number of strings with complexity not exceeding $\ln_2 \mathscr{N}_N - l$ is, $\leq 2^{-l+1}\mathscr{N}_N$, and the statistical weight of the matrices in $\mathscr{E}_N$ associated with such strings is $\mu_N \leq 2^{-l+1}\mathscr{N}_N\sigma_N$ where σ_N is the maximum probability of one string. Suppose that we can find γ, $0 < \gamma < 1$, such that $\mathscr{N}_N^{\gamma}\sigma_N \to 0$ for $N \to \infty$ ($\mathscr{N}_N\sigma_N \geq 1$ be definition). Then assuming $l = l_N = (1-\gamma)\ln_2 \mathscr{N}_N$ we find $\mu_N(l_N) \leq 2\mathscr{N}_N^{\gamma}\sigma_N \to 0$ exponentially as $N \to \infty$, meaning that, for large N, the overwhelming majority of matrices in $\mathscr{E}_N$ will give strings of complexity greater than $\gamma \ln_2 \mathscr{N}_N$, i.e., of the same order, as $N \to \infty$, as strings of maximum complexity.

This γ we can actually find. For any given string $\{x_1, \ldots, x_N\}$ we have

$$\text{Prob}\{x_1, \ldots, x_N\} = N! \int_{\Delta_1} d\theta_1 \cdots \int_{\Delta_N} d\theta_N W(\theta_1, \ldots, \theta_N) \ ,$$

$$\theta_1 \leq \theta_2 \leq \cdots \leq \theta_N \ ,$$

where Δ_l are intervals in $(0, 2\pi)$ of width $2\pi/N$. Therefore, we may assume

$$\sigma_N \leq N! \left(\frac{2\pi}{N}\right)^N C_N e^{N \ln N/2} . \quad (4)$$

Using the known expression for C_{2m} (Ref. 12)

$$C_{2m} = \frac{1}{2^{4m} m! \pi^m}$$

in inequality (4), and relation (3) we find that $\mathcal{N}_N^{\gamma} \sigma_N \to 0$ for $N \to \infty$ provided that $0 < \gamma < \frac{1}{4} \ln_2 (2e/\pi)$.

Thus, we have shown that most matrices in $\mathcal{E}_N$ have eigenvalues with asymptotically maximal complexity.

If now one is willing to accept the idea that statistical properties of the eigenvalues of a random matrix reproduce somehow analogous properties of eigenvalues of chaotic systems, then the sense of our whole discussion is that in order to operate a distinction between integrable and chaotic systems, according to the different degree of randomness of their eigenvalues, one must consider the eigenvalues of classically chaotic systems as being generally more random than those of integrable systems. The important question remains open, whether any such distinction is possible at all, at least generically.

This work was supported by ERO and the Consiglio Nazionale delle Ricerche, Rome, under grant No. 82.02634.02.

[1]M. V. Berry, in *Chaotic Behavior of Deterministic Systems,* edited by R. H. G. Helleman and G. Joos (North-Holland, Amsterdam, 1983), and references therein.
[2]G. Zaslavsky, Phys. Rep. **80**, 157 (1981), and references therein.
[3]G. Casati, F. Valz-Gris, and I. Guarneri, Lett. Nuovo Cimento **28**, 279 (1980).
[4]S. W. McDonald and A. N. Kaufman, Phys. Rev. Lett. **42**, 1189 (1979).
[5]O. Bohigas, M. S. Giannoni, and C. Schmit, in *Chaotic Behavior in Quantum Systems,* edited by G. Casati (Plenum, New York, in press).
[6]M. V. Berry and M. Tabor, Proc. R. Soc. London, Ser. A **356**, 375 (1977).
[7]P. Pechukas, Phys. Rev. Lett. **51**, 943 (1983).
[8]A. N. Kolmogorov, IEEE Trans. Inf. Theory **14**, 682 (1966).
[9]P. Martin-Leof, J. Inf. Control **9**, 602 (1966).
[10]J. Ford, Phys. Today **36**, 40 (1983).
[11]A. K. Zvonkin and L. A. Levin, Usp Mat. Nauk **25**, 85 (1970).
[12]M. L. Mehta, *Random Matrices* (Academic, New York, 1967).