Brief Reports

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Schrödinger equation for the helium atom

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It is shown that the Schrödinger equation for the helium atom does not have a Frobenius-type solution in the variables r_1, r_2, and r_{12}.

I. INTRODUCTION

In an attempt to detect the Lamb effect for the He atom, Kinoshita[1] has extended Hylleraas's solution to the wave equation. Allowing for mass polarization and relativistic effects and a Lamb shift, he obtained a theoretical ionization potential in good agreement with the best observed value. His solution, like Fock's solution,[2] satisfies the wave equation, whereas Hylleraas's series does not. However, unlike Fock's solution, his solution is not defined at $r_1=r_2=0$. Kinoshita showed that the wave equation has infinitely many solutions[3] and emphasizes the importance of obtaining a suitable type of expansion. Earlier Bartlett, Gibbons, and Dunn[4] showed that the wave equation has no power-series solution. This implies that Kinoshita's series (2.11) has no solution satisfying $l \geq m$. Here, we prove a more general result: that the wave equation has no solution of Frobenius type.

II. THE FROBENIUS EXTENSION TO THE HYLLERAAS EXPANSION

Bartlett *et al.*[4] showed that the wave equation

$$\psi_{xx}+2\psi_x/x+\psi_{yy}+2\psi_y/y+2\psi_{zz}+4\psi_z/z+(x^2-y^2+z^2)\psi_{xz}/(xz)+(y^2-x^2+z^2)\psi_{yz}/(yz)+(\tfrac{1}{4}\lambda+x^{-1}+y^{-1}-\tfrac{1}{2}z^{-1})\psi=0$$

has no power-series solution in the variables $x=r_1$, $y=r_2$, and $z=r_{12}$.

However, the Frobenius series

$$\psi=x^L y^M z^N \sum_{l,m,n=0}^{\infty} C_{l,m,n}x^l y^m z^n \ ,$$

with $C_{0,0,0}\neq 0$, does give a formal solution, provided that, for $l,m,n \geq -2$,

$$(\bar{l}+2)(\bar{l}+3+\bar{n})C_{l+2,m,n}+(\bar{m}+2)(\bar{m}+3+\bar{n})C_{l,m+2,n}+(\bar{n}+2)(2\bar{n}+6+\bar{l}+\bar{m})C_{l,m,n+2}-(\bar{l}+2)(\bar{n}+2)C_{l+2,m-2,n+2}$$
$$-(\bar{m}+2)(\bar{n}+2)C_{l-2,m+2,n+2}+\lambda C_{l,m,n}/4+C_{l+1,m,n}+C_{l,m+1,n}-\tfrac{1}{2}C_{l,m,n+1}=0 \ ,$$

where $\bar{l}=l+L$, $\bar{m}=m+M$, $\bar{n}=n+N$.

We now show that this system has no solution. Putting $(l,m,n)=(-2,0,0)$, $(0,-2,0)$, and $(0,0,-2)$ yields

$$L(L+1+N)=M(M+1+N)$$
$$=N(2N+2+L+M)=0 \ ,$$

so that

$$(L,M,N)=(0,0,0),\ (0,-1,0),\ (-1,0,0) \ ,$$

or

$$(-1-N,\ -1-N,N) \ .$$

On putting $(l,m,n)=(-2,2,-2)$ and $(2,-2,-2)$ this reduces to $(L,M,N)=(0,0,-1)$ or $(-1,-1,0)$ or $(0,-1,0)$.

Putting $(l,m,n)=(0,0,-1)$ now yields a contradiction.

Hence, the wave equation does not have a Frobenius-type solution.

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[1]T. Kinoshita, Phys. Rev. **105**, 1490 (1975); **115**, 366 (1959).

[2]V. A. Fock, Izvest. Akad. Nauk S.S.S.R. Ser. Fiz. **18**, 161 (1954).

[3]There is an error on p. 1500: $C_{l,m,1}$ are also arbitrary and $C_{l,m,n}$ $(n\geq 2)$ can be expressed uniquely in terms of $(C_{l,m,0}, C_{l,m,1})$.

[4]J. H. Bartlett, J. J. Gibbons, and C. G. Dunn, Phys. Rev. **47**, 679 (1935).