# Quantum theory of a one-dimensional laser with output coupling: Linear approximation

J. C. Penaforte and B. Baseia
*Departamento de Física, Centro de Ciências Exatas e da Natureza, Universidade Federal da Paraíba, 58000 João Pessoa, Paraíba, Brazil*
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A simple model of a one-dimensional optical cavity having output coupling is used to develop a fully quantized linear laser theory. The present model and procedure, which work directly within the continuous spectrum, allow us to find a simple treatment that leads to explicit solutions for the field inside and outside the laser cavity. The threshold condition for laser light onset is obtained by assuming the field operating in a single collective mode and the usual adiabatic hypothesis. A development that removes this later restriction is also discussed and outlined.

## I. INTRODUCTION

Laser theory conventionally deals with discrete cavity modes and introduces artificial mechanisms to simulate the field loss due to beam extraction. More recently, however, several realistic models[1–4] have been introduced in theoretical treatments which are able to describe properly the output coupling, i.e., the coupling of the field inside the laser cavity with the field outside this region due to partial transmission at one of the mirrors.

In this paper, which is a natural extension of a previous semiclassical treatment,[4] we develop a fully quantized version using a simple model of a one-dimensional optical cavity where the field quantization is carried out in terms of the modes of the continuous spectrum defined throughout the space.[4] The coupling of the radiation field with the active atoms is confined in the laser cavity, which is the subspace defined in the interval $z\in[0,l]$ of the entire cavity defined in the interval $z\in(-\infty,l]$.

The present treatment becomes simple and compact due to the model and procedure which work directly within the continuous spectrum. This yields an integral equation for the field inside the laser cavity, within the linear approximation. The application of standard techniques leads to an explicit solution that exhibits the role of excitations in the buildup of the laser field from vacuum.

In Sec. II, we briefly discuss the model and normal modes of the entire space $z\in(-\infty,l]$. Section III treats the field quantization. In Sec. IV, the basic equation of motion for the field in the presence of active atoms is derived. As usual, we assume a model of two levels and noninteracting atoms with the population inversion kept constant. This later assumption leads to the linear theory. The damping term for the atomic polarization is added phenomenologically, while the damping term for the field, due to beam extraction, is automatically built in the present model. In Sec. V we solve the field inside and *also* outside the laser cavity, under the following usual approximations: the laser field in single-model operation, homogeneously broadened atoms, and slowly varying amplitude approximation. Discussions, conclusions, and the outlook on further extensions of the present treatment are presented in Sec. VI.

## II. CAVITY MODEL AND FIELD MODES

Here we summarize the model of a laser cavity considered in Ref. 4; the reader must refer there for further details. This model is realistic with respect to the inclusion of the loss in the radiation field, by taking into account the beam-light extraction in the laser, allowing the natural inclusion of the dissipation in the quantized electromagnetic field.[5]

The laser cavity in this model essentially consists of two parallel plates, one of which is totally reflecting, whereas the other one is semitransparent. The origin is taken at the semitransparent plate and we put the other plate at a distance $l$; the $z$ axis is made to be perpendicular to the plates. Then, the optical cavity is the region $0\le z\le l$ and the outside region is the left half-space $-\infty<z\le 0$. The semitransparent plate simulates an ideal limiting case of a dielectric medium placed at $z=0$, having a very small thickness with a large dielectric constant $\eta$. So the model is analytically described by[1,4]

$$\epsilon(z)=\epsilon_0[1+\eta\delta(z)]\ , \tag{1}$$

where $\epsilon_0$ is the electrical permissivity of vacuum; $\eta$ is a real parameter with dimension of length, which determines the transparency of the window at $z=0$; and $\delta(z)$ is the Dirac $\delta$ function which represents the ideal limiting case of narrowness for the semitransparent plate.

The normal field modes are stationary solutions of Maxwell equations, which satisfy certain boundary conditions. They are given by[4] (see Appendix)

$$U_k(z)=\begin{cases}L_k\sin[k(z-l)], & 0\le z\le l\\ (2/\pi)^{1/2}\sin(kz-\phi_k), & -\infty<z\le 0\end{cases} \tag{2}$$

where the $\delta$-function normalization

$$\int_{-\infty}^{l}U_k(z)U_k'(z)[\epsilon(z)/\epsilon_0]dz=\delta(k-k') \tag{3}$$

has been employed. $\phi_k$ is a phase shift: $\phi_k=\sin^{-1}[(\pi/2)^{1/2}L_k\sin(kl)]$ for $L_k$ as given by Eq. (4) below. The plot of $L_k^2$ as a function of $\omega_k$ shows resonance peaks centered approximately around the Fox-Li quasimodes frequencies $\omega_{0n}=n(c\pi/l)$ with spacing $\Delta\omega=c\pi/l$.


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We assume the transparency of the transmitting window is very small, in such a way that the function $L_k^2$ will be strongly peaked around the Fox-Li quasimodes frequencies. In this case, the linewidth $\Gamma_n$ associated with a given Fox-Li resonance frequency $\omega_{0n}$ is much smaller than the spacing between the neighboring resonances, i.e., $\Gamma_n \ll \Delta\omega = c\pi/l$, and we can approximate the line-shape function $L_k$ by a Lorentzian function $M_k(n)$ as[2,4] (hereafter, for brevity, we drop the band index: $\Gamma_n \to \Gamma$, $\Lambda_{0n} \to \Lambda$, and $\omega_{0n} \to \omega_0$)

$$L_k \simeq M_k(n) = (2/\pi)^{1/2}\Gamma\Lambda/[(\omega_k-\omega_0)^2+\Gamma^2]^{1/2} , \quad (4)$$

where $\Gamma$ is determined by the window transparency,

$$\Gamma = c/\Lambda^2 l , \quad (5)$$

with

$$\Lambda = \eta k_0 \simeq n(\eta\pi/l) \gg 1 \quad (6)$$

and $\omega_0$ the resonance frequency associated with the $n$th Fox-Li quasimode[4] (in the optical case $n \simeq 10^6 \gg 1$)

$$\omega_0 = ck_0 \simeq (n\pi + \Lambda^{-1})c/l . \quad (7)$$

Equations (5) and (6) together express the requirement that the transparency in the window be very small.

## III. FIELD QUANTIZATION

The classical Hamiltonian for the free radiation field in the present case is[6]

$$H_F = \tfrac{1}{2}\int_{-\infty}^{l} [\epsilon(z)(\partial\vec{A}/\partial t)^2 + \mu(z)(\vec{\nabla}\times\vec{A})^2]dz , \quad (8)$$

where $\epsilon(z)$ is given in Eq. (1), $\mu(z)$ is the magnetic permeability of medium, and $\vec{A} = A(z,t)\hat{x}$ is the vector potential. Since we are dealing with a nonmagnetic medium we set $\mu(z)=1$. Expanding the vector potential in terms of the complete set of the normal modes $U_k(z)$ in the whole cavity, we have

$$\vec{A}(z,t) = \int q_k(t)U_k(z)dk\,\hat{x} . \quad (9)$$

Using Eqs. (3), (8), and (9) the Hamiltonian for the radiation field in the whole cavity can be written as

$$H_F = \tfrac{1}{2}\int_0^\infty (p_k^2 + \omega_k^2 q_k^2)dk , \quad (10)$$

where $p_k(t) = \dot{q}_k(t)$. The field is quantized by imposing on the $q_k$'s and $p_k$'s the standard commutation relations

$$[q_k, p_{k'}] = i\hbar\delta(k-k'), \quad [q_k, q_{k'}] = 0 = [p_k, p_{k'}] . \quad (11)$$

Introducing the annihilation and creation operators $a_k, a_k^\dagger$ through the conventional canonical transformations

$$q_k = (\hbar/2\omega_k)^{1/2}(a_k + a_k^\dagger) , \quad (12a)$$

$$p_k = -i(\hbar\omega_k/2)^{1/2}(a_k - a_k^\dagger) , \quad (12b)$$

we have

$$[a_k, a_{k'}^\dagger] = \delta(k-k'), \quad [a_k, a_{k'}] = 0 = [a_k^\dagger, a_{k'}^\dagger] , \quad (13)$$

and Eq. (10) becomes

$$H_F = \int_0^\infty \hbar\omega_k a_k^\dagger a_k dk , \quad (14)$$

where the zero-point energy has been neglected. From Eqs. (9) and (12b) and the definition $p_k(t) = \dot{q}_k(t)$, we find the electric field operator in terms of the operators $a_k, a_k^\dagger$:

$$E(z,t) = -(\partial/\partial t)A(z,t) = i\int_0^\infty (\hbar\omega_k/2)^{1/2}(a_k - a_k^\dagger)U_k(z)dk . \quad (15)$$

## IV. LASER EQUATION OF MOTION

### A. Derivation of the general expression

We assume the active two-level atoms inside the laser cavity distributed with a uniform density per unit length in the $z$ direction; we also assume that they are coupled to the field at $t = t_0$. We introduce two sets of operators $[\sigma_{m_{2(1)}}, \sigma_{m_{2(1)}}]$, which satisfy the anticommutation relations

$$[\sigma_{m_i}, \sigma_{m_j}^\dagger]_+ = \delta_{ij}, \quad [\sigma_{m_i}, \sigma_{m_j}]_+ = [\sigma_{m_i}^\dagger, \sigma_{m_j}^\dagger]_+ = 0 \quad (16)$$

and are the creation and annihilation operators for the upper (lower) energy state of the $m$th atom. The atomic energy levels are separated by a gap $\hbar\Omega_m$.

The complete Hamiltonian for the whole system is

$$H = H_F + H_A + H_I . \quad (17)$$

$H_F$ stands for the Hamiltonian of the free field and is given by Eq. (14);

$$H_A = \sum_m \hbar\Omega_m \sigma_{m_2}^\dagger \sigma_{m_2} \quad (18)$$

is the Hamiltonian of the atoms; and

$$H_I = \sum_m \int_0^\infty \hbar(g_{km} a_k^\dagger \sigma_{m_1}^\dagger \sigma_{m_2} + \text{H.c.})dk \quad (19)$$

is the interaction Hamiltonian, where H.c. means Hermitian conjugate and

$$g_{km} = i\Omega_m \left[\frac{1}{2\hbar\omega_k}\right]^{1/2} U_k(z_m)p_m \quad (20)$$

is a coupling constant, representing the interaction. In Eq. (20) $p_m$ is the $z$ component of the dipole matrix element of the $m$th atom.

Using Eqs. (13), (14), and (16)–(19), we find the Heisenberg equations for the field and atoms operators:

$$(d/dt)a_k^\dagger = i\omega_k a_k^\dagger + i\sum_m g_{km}^* \sigma_{m_2}^\dagger \sigma_{m_1} , \quad (21)$$

$$(d/dt)(\sigma_{m_2}^\dagger \sigma_{m_1}) = i\Omega_m(\sigma_{m_2}^\dagger\sigma_{m_1}) - \gamma_m(\sigma_{m_2}^\dagger\sigma_{m_1}) - i\int_0^\infty g_{km} a_k^\dagger \mathscr{P}_m dk , \quad (22)$$

where

$$\mathscr{P}_m = \sigma_{m_2}^\dagger\sigma_{m_2} - \sigma_{m_1}^\dagger\sigma_{m_1} \quad (23)$$

is the atomic population inversion and $\gamma_m$ is the phenomenological damping term for the atomic polarization. Similar equations for $a_k$ and $\sigma_{m_1}^\dagger\sigma_{m_2}$ can also be found.

Equations (21) and (22) turned out to be a set of coupled equations. However, under the assumption that the population inversion is kept constant for every atom, the mentioned equations decouple and one is led to the linear approximation. Using Eq. (15) and taking into account only the negative frequency part of the radiation field, we have

$$E^-(z,t)=-i\int_0^\infty(\hbar\omega_k/2)^{1/2}a_k^\dagger(t)U_k(z)dk\ . \tag{24}$$

The positive frequency part $E^+(z,t)$ is obtained through a similar procedure used in the calculation for $E^-(z,t)$ and the results do not depend on the choice between $E^-(z,t)$ and $E^+(z,t)$. Keeping in mind the definition in Eq. (24), after some straightforward calculations on Eqs. (21) and (22) we get

$$E^-(z,t)=F(z,t)+\int_0^\infty\int_{t_0}^t(\hbar\omega_k/2)^{1/2}U_k(z)e^{i\omega_k(t-t')}\sum_m g_{km}^*\sigma_{m_2}^\dagger\sigma_{m_1}(t')dt'dk\ , \tag{25}$$

where

$$F(z,t)=-i\int_0^\infty(\hbar\omega_k/2)^{1/2}a_k^\dagger(t_0)U_k(z)e^{i\omega_k t}dk \tag{26}$$

is a noise operator arising in the initial values of the field operator $a_k^\dagger(t)$. Furthermore, using the approximation[7]

$$\int_0^\infty g_{km}a_k^\dagger(t)dk\simeq\frac{\Omega_m p_m}{\hbar_{\omega_0}}E^-(z_m,t)\ , \tag{27}$$

we find

$$E^-(z,t)=F(z,t)+G(z,t)+\sum_m\frac{|p_m|^2\Omega_m^2\mathcal{P}_m}{2\hbar\omega_0}\int_{t_0}^t\int_0^\infty U_k(z)U_k(z_m)e^{i\omega_k(t-t')}dk\int_{t_0}^{t'}\exp[(i\Omega_m-\gamma_m)(t'-t'')]E^-(z_m,t'')dt''dt'\ , \tag{28}$$

where

$$G(z,t)=\int_0^\infty\left[\frac{\hbar\omega_k}{2}\right]^{1/2}U_k(z)e^{i\omega_k t}\sum_m g_{km}^*\sigma_{m_2}^\dagger\sigma_{m_1}(t_0)\left[\frac{\exp\{[i(\Omega_m-\omega_k)-\gamma_m]t\}-\exp\{[i(\Omega_m-\omega_k)-\gamma_m]\}}{i(\Omega_m-\omega_k)-\gamma_m}\right]dk \tag{29}$$

is another noise operator which can be shown to be negligible after a long time in comparison to the lifetime of the atom or the field inside the laser cavity. Unlike $F(z,t)$, the noise operator $G(z,t)$ depends on the initial values of the atomic operators $\sigma_{m_2}^\dagger\sigma_{m_1}(t)$ and also on the coupling constant, as is clear in Eq. (29), which correlates the two interacting systems of atoms and field.

Equation (28) is the basic equation for our laser field and is valid both inside and outside the laser cavity. It can be written in a different form by decomposing $E^-(z,t)$ into a slowly varying amplitude (in time domain) and an oscillating factor as

$$E^-(z,t)=\mathscr{E}^-(z,t)e^{i\omega t}\ , \tag{30}$$

where $\omega$ [$\omega\simeq\omega_0$, by consistency with Eq. (27)] is the approximate center frequency of mode oscillation and $E^-(z,t)$ is a complex amplitude denoting the slowly varying field component. Hereafter, for simplicity, we approximate $\omega\to\omega_0$. Substituting Eq. (30) into Eq. (28) and using the adiabatic hypothesis we find

$$\mathscr{E}^-(z,t)=\mathscr{F}(z,t)+\int_{t_0}^t f_1(z,z_m,t-t')\mathscr{E}^-(z_m,t')dt'-\mathscr{E}(z_m,t_0)\times\int_{t_0}^t f_1(z,z_m,t-t')\exp\{[i(\omega_0-\Omega_m)+\gamma_m](t_0-t')\}dt'\ , \tag{31}$$

where $\mathscr{F}(z,t)=F(z,t)\exp(-i\omega_0 t)$ and the terms $f_1(z,z_m,\tau)$ and $\mathscr{E}^-(z_m,t_0)$ are defined by

$$f_1(z,z_m,\tau)=\sum_m\frac{|p_m|^2\Omega_m^2\mathcal{P}_m}{2\hbar\omega_0}\times\int_0^\infty U_k(z)U_k(z_m)\frac{e^{i(\omega_0-\omega_0)\tau}}{i(\omega_0-\Omega_m)+\gamma_m}dk\ , \tag{32}$$

$$\mathscr{E}^-(z_m,t_0)=-i\int_0^\infty(\hbar\omega_k/2)^{1/2}a_k^\dagger(t_0)U_k(z_m)dk\ . \tag{33}$$

The relevant pole of the integrand in Eq. (32) is given by $\xi=(\omega_0+i\Gamma)/c$. In order to solve $f_1(z,z_m,\tau)$, we use the normal modes $U_k(z)$ as defined previously in Eqs. (2) and (4), and the integral over the continuous spectrum is extended to infinity, in view of the falloff of the Lorentzian factors. It can be shown that the last term in Eq. (31) may be neglected when compared with $\mathscr{F}(z,t)$. Accordingly, the negative frequency part of the field becomes

$$\mathscr{E}^{-}(z,t)=\mathscr{F}(z,t)+\int_{t_0}^{t}f_1(z,z_m,t-t')\mathscr{E}^{-}(z_m,t')dt' . \tag{34}$$

When looking for the field inside the cavity, Eq. (33) turns out to be an integral equation. On the other hand, once the field inside the laser cavity is obtained, the field outside can be found from Eq. (34) by direct integration. In the remainder of this section we devote ourselves to finding the field inside the laser cavity.

### B. Field inside the laser cavity

We want an explicit expression for the above-mentioned field. If we apply the Laplace transform $\mathscr{L}[\mathscr{E}^{-}(z,t)]\to\widetilde{\mathscr{E}}^{-}(z,s)$ and set $z=z_m'$ in Eq. (34) we get, after a little algebra,

$$\widetilde{\mathscr{E}}^{-}(z_m,s)=\frac{\widetilde{\mathscr{F}}(z_m',s)}{1-\widetilde{f}_1(z_m',z_m,s)} , \tag{35}$$

where the convolution theorem has been used. $\widetilde{\mathscr{E}}^{-}$, $\widetilde{\mathscr{F}}$, and $\widetilde{f}_1$ are the Laplace transforms of $\mathscr{E}^{-}$, $\mathscr{F}$, and $f_1$, respectively. The application of the inverse Laplace transformation to the foregoing equation leads to

$$\mathscr{E}^{-}(z_m,t)=\mathscr{L}^{-1}[\widetilde{\mathscr{E}}^{-1}(z_m,s)]$$

$$=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{e^{st}\mathscr{F}(z_m',s)ds}{1-f_1(z_m',z_m,s)} , \tag{36}$$

where $\lambda$ is the abscissa of convergence. Equation (36) is a closed expression for the field inside the laser cavity. Note that $\widetilde{\mathscr{F}}(z_m',s)$ in Eq. (36) is an operator, while $f_1(z_m',z_m,s)$ is a classical function. Also, Eq. (36) shows that the laser field rises from the fluctuations through the inhomogeneous term $\mathscr{F}(z_m,t)$ of the integral equation (34).

## V. SOLUTION OF THE LASER EQUATION OF MOTION

### A. Field inside the laser cavity

As mentioned above, the expression (36) is a formal and compact expression for the laser field inside the cavity. Its exact solution is accessible by straightforward application of the residue theorem. In Eq. (36) we consider the case of homogeneous broadening and uniform population inversion for the atoms. We also assume that the atomic frequency, the dipole matrix element, and the damping constant, as well as the population inversion, are the same for all atoms. The poles of interest in Eq. (36) are the roots of the denominator in the integrand. Thus we set

$$1-\widetilde{f}_1(z_m',z_m,s)=0 . \tag{37}$$

Next, substituting $U_k(z_m')$ and $U_k(z)$ into the foregoing equation, we integrate over $k$ by using the residue technique. We find the single root

$$s_0=\alpha M^2-\Gamma , \tag{38}$$

where

$$\alpha=\frac{p^2\omega_0\mathscr{P}}{4\hbar c\gamma} \tag{39}$$

and

$$M^2=\int_0^\infty M_k^2dk=2/l . \tag{40}$$

For the sake of simplification we set $\omega_0\simeq\Omega_m$ (resonance) and substitute $\sin[k(z_m-l)]$ in Eq. (39) by its root-mean-square value, i.e., $\sin[k(z_m-l)]\to1/\sqrt{2}$. Making use of the residue theorem in Eq. (36) we find, for the field inside the cavity,

$$\mathscr{E}^{-}(z_m,t)=-i\alpha M^2e^{s_0t}\times\int_0^\infty(\hbar\omega_k/2)^{1/2}a_k^\dagger(t_0)\frac{U_k(z_m)}{s_0-i\Delta\omega_k}dk . \tag{41}$$

The threshold condition for laser oscillation is found from the foregoing result by setting $s_0=0$, yielding [cf. Eq. (38)] $\alpha M^2=\Gamma$, in agreement with the semiclassical result.[4] At this point, the rapid component of the field $\exp(i\omega_0t)$ may be restored [cf. (30)]. The complex amplitude is given by Eq. (41) in an operator form. Its value, however, will depend on the (initial) state of the field. Assuming, for example, the field is initially in a coherent state[8] $\{|v_k\rangle\}$, i.e., $a\{|v_k\rangle\}=v_k\{|v_k\rangle\}$, we find

$$\langle E^{-}(z_m,t)\rangle=-i\alpha M^2e^{(i\omega_0+s_0)t}\times\int_0^\infty(\hbar\omega_k/2)^{1/2}v_k\frac{U_k(z_m)}{s_0-i\Delta\omega_k}dk \tag{42}$$

and, also, assuming that $v_k=C_0M_k$ where $C_0$ is a constant, which means an eigenvalue distribution $\{v_k\}$ for the annihilation operator $a_k$ following the Lorentzian line shape $M_k$, we obtain

$$\langle E^{-}(z_m,t)\rangle=C_0M^2\sin[\xi(z_m-l)](\hbar c\xi/2)^{1/2}\times e^{-i\pi/2}c^{-1}e^{(i\omega_0+s_0)t} , \tag{43}$$

where $\langle\mathscr{E}^{-}(z_m,t)\rangle=\langle E^{-}(z_m,t)\rangle\exp(-i\omega_0t)$, $\xi=(\omega_0+i\Gamma)/c$, and Eq. (40) has been used.

### B. Field outside the laser cavity

We recall that both Eqs. (31) and (34) are self-consistent equations. Thus the field outside the laser cavity can be found from Eq. (34) by direct integration once the field inside the cavity is already known. The substitution of Eq. (41) into Eq. (34) gives

$$\mathscr{E}^{-}(z,t)=\mathscr{F}(z,t)+\alpha\int_{t_0}^{t}\int_0^\infty U_k(z)U_k(z_m)e^{i\Delta\omega_k(t-t')}\left[-i\alpha M^2e^{s_0t}\int_0^\infty(\hbar\omega_k/2)^{1/2}a_{k'}^\dagger(t_0)\frac{U_{k'}(z_m)}{s_0-i\Delta\omega_k}dk'\right]dk\,dt' . \tag{44}$$

The solution (44) still remains in an operator form. Hence the amplitude and phase for this field can be obtained only after the state of the field is given. Under the above assumption of a field in a coherent state, we find the expectation value of the field outside the laser cavity:

$$\langle \mathscr{E}^-(z,t)\rangle=(\hbar\omega_k/2)^{1/2}e^{-i\xi z}e^{i\pi/2}C_0M^2e^{-\Gamma t}/c\Lambda+\alpha\langle \mathscr{E}^-(z_m,t)\rangle\sin[\xi(z_m-l)]e^{-i\xi z}M^2/c\Lambda\Gamma(1-e^{-\Gamma(t-t_0)}) . \quad (45)$$

The first term in the foregoing equations gives the free radiation field ($\alpha=0$) and has an exponential decay. When the laser action takes place above threshold, i.e., $s_0>0$, the second term becomes dominant and Eq. (45) can be written in the form

$$\langle \mathscr{E}^-(z,t)\rangle=\alpha M^2\langle \mathscr{E}^-(z_m,t)\rangle \times\sin[\xi(z_m-l)]e^{-i\xi z}/c\Lambda\Gamma , \quad (46)$$

where the decaying term $\exp[-\Gamma(t-t_0)]$ has also been neglected. As done previously for the field inside the laser cavity, the rapidly oscillating term $\exp(i\omega_0 t)$ for the external field can be restored.

The results found in Eqs. (43) and (46), or more generally, (41) and (44), allow us to find the amplitudes and phases for the field inside and outside the laser cavity. In a future paper we will employ these results in order to calculate the quantum coherence functions which correlate these fields.

## VI. COMMENTS AND CONCLUSION

The present treatment deals with the modes of the laser cavity as a resonance in a continuous spectrum. This procedure comes from the use of a realistic model which properly includes the dissipation in the field inside the cavity taking into account the beam extraction at one of the mirrors. Unlike the conventional discretized version where the field loss is unrealistically simulated by artificial loss reservoirs, the present model has appropriate boundary conditions yielding the continuous set of normal modes $U_k(z)$, as given by Eq. (2).

By expanding the field as a linear superposition of the modes $U_k(z)$ [cf. Eq. (9)], we developed a quantum theory of a laser having output coupling, with the following approximations: (i) the transparency of the transmitting window is required to be very small [cf. Eq. (6)], (ii) we treat the laser in single-mode operation, (iii) we assume two-level homogeneously broadened atoms, and (iv) uniform population inversion is kept constant, yielding the linear approximation. In this way, we derived the basic equation (28) for the laser field. This equation turns out to be an integral equation when one looks for the field inside the laser cavity. In this case it was solved by assuming the traditional adiabatic hypothesis plus the application of the Laplace transform and the convolution theorem, leading to the solution given by Eq. (41). Once the field inside the laser cavity is known, Eq. (28) gives the field outside the cavity by direct integration, as obtained in Eq. (45). This latter result cannot be derived (or even postulated) in the usual treatments that employ artificial loss reservoirs to simulate the outside cavity.

A little inspection of Eq. (34), plus Eq. (26), shows the laser field buildup from vacuum. This is due to the quantized theory which gave rise to the inhomogeneous term $F(z,t)$ appearing in Eq. (25). On the other hand, the threshold condition, as given by Eq. (38), coincides with previous results as found in Refs. 2 and 4, and the reader is referred there for comparison. This result has been obtained under the conventional assumption of the adiabatic hypothesis. However, although it is a usual and justified approximation, one could shortly reexamine the procedure following Eq. (28) in the (eventual) absence of this restriction. Therefore, going back to this equation and using Eq. (30) we find

$$\mathscr{E}^-(z,t)=\mathscr{F}(z,t)+\int_{t_0}^{t}f_1'(z,z_m,t-t')g(z_m,t')dt' , \quad (47)$$

where

$$f_1'(z,z_m,\tau)=\sum_m\frac{|p_m|^2\Omega_m^2\mathscr{P}_m}{2\hbar\omega_0}\times\int_0^\infty U_k(z)U_k(z_m)e^{i(\omega_k-\omega_0)\tau}dk , \quad (48)$$

$$g(z_m,t')=\int_{t_0}^{t'}f_2(t'-t'')\mathscr{E}^-(z_m,t'')dt'' , \quad (49)$$

and

$$f_2(\tau)=e^{[i(\Omega_m-\omega_0)-\gamma_m]\tau} . \quad (50)$$

For the field inside the laser cavity we again apply the Laplace transform plus convolution theorem and, after calculations similar to those that led to Eq. (35)—except for twice applying the convolution theorem—we find

$$\tilde{\mathscr{E}}^-(z_m,s)=\frac{\tilde{\mathscr{F}}(z_m',s)}{1-\tilde{f}_1(z_m',z_m,s)\tilde{f}_2(s)} . \quad (51)$$

Following the same procedure used to obtain the result (38), the denominator in the foregoing equation has the roots

$$s_{01}'=-(\gamma_m+\Gamma)/2+\{[(\gamma_m+\Gamma)/2]^2+\gamma_m s_0\}^{1/2} , \quad (52)$$

$$s_{02}'=-(\gamma_m+\Gamma)/2-\{[(\gamma_m+\Gamma)/2]^2+\gamma_m s_0\}^{1/2} . \quad (53)$$

Here only the root $s_{01}'$ is meaningful for laser threshold. For this root $s_{01}'$ we obtain the threshold condition by setting $s_{01}'\to 0$, which leads to $\chi=(\chi^2+\gamma_m s_0)^{1/2}$, where $\chi=(\gamma_m+\Gamma)/2$, i.e., $s_{01}'\to 0$ implies $s_0\to 0$. Thus, the threshold condition $s_{01}'=0$ coincides with that found in Eq. (38). However, according to Eq. (52), above and below threshold we have $s_{01}'\neq s_0$ and, as expected, in the absence of adiabatic hypothesis the transient behavior of the laser field differs from that obtained under this assumption. It should be stressed that, in the particular case where the adiabatic hypothesis is valid, we have $\gamma_m\gg s_0,\Gamma$, and Eq. (52) recovers the result given by Eq.

(38), i.e., $s'_{01}=s_0$. In a future paper we investigate the extension of this work to the nonlinear approximation, as well as the quantum coherence functions for the field inside and outside the laser cavity.

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## APPENDIX: NORMAL MODE SPECTRUM

For the sake of completeness, we add a short summary of some results found in Ref. 4. As mentioned previously, we consider only longitudinal field modes, so that the model becomes effectively one dimensional, in the sense that the field depends only on the $z$ coordinate. For monochromatic waves of circular frequency $\omega_k$,

$$E(z,t)=U_k(z)\exp(-i\omega_k t) , \tag{A1}$$

and using Maxwell equations we find

$$\ddot{U}_k(z)+k^2U_k(z)=0, \quad k=\omega_k/c \tag{A2}$$

where the mode functions are subject to the boundary conditions $U_k(l)=0$ and

$$\dot{U}_k(0^+)-\dot{U}_k(0^-)=-\eta k^2U_k(0) . \tag{A3}$$

In deriving this equation, the continuity condition at $z=0$ has been used, i.e.,

$$U_k(0^+)=U_k(0^-)=U_k(0) , \tag{A4}$$

and the discontinuity in Eq. (A3) arises from the $\delta$ function term in Eq. (1).

The solutions of Eq. (A2) are given by (2) and are subject to the usual continuous spectrum normalization [see Eq. (3)]. The substitution of these solutions into the boundary conditions (A3) and (A4) yields

$$\sin\delta_k=(\pi/2)^{1/2}L_k\sin(kl) , \tag{A5}$$

$$(\pi/2)L_k^2=[1+\Lambda^2\sin^2(kl)-\Lambda\sin(2kl)]^{-1} , \tag{A6}$$

where

$$\Lambda=\Lambda(k)=\eta k . \tag{A7}$$

Setting $t=\tan(kl)$ we find

$$(\pi/2)L_k^2=(1+t^2)/[t^2+(\Lambda t-1)^2] . \tag{A8}$$

The function $L_k^2$ has peaks at the points

$$t_n=\tan(kl)=\Lambda^{-1}(k_n)=(\eta k_n)^{-1}\equiv\Lambda_n^{-1}\ll 1 , \tag{A9}$$

where the window transparency is very small [cf. Eq. (6)]. The peak values of $L_k^2$ are given by

$$(\pi/2)[L_k(t_n)]^2=1+\Lambda_n^2\simeq\Lambda_n^2 \tag{A10}$$

and the peak half-widths are

$$\Delta t_n=1/\Lambda_n^2\simeq l\Delta k_n . \tag{A11}$$

The resonance frequencies $k_n$ are the roots of the transcendental equation (A9). In the optical range we have

$$k_nl=n\pi+\theta_n, \quad n\gg 1, \quad |\theta|\ll 1 . \tag{A12}$$

Equation (A9) may be solved by interaction, with the following result:

$$k_n\simeq k_{0,n}=(n\pi+\Lambda_{0,n}^{-1})/l=\omega_{0,n}/c , \tag{A13}$$

where $\Lambda_{0,n}$ is given by

$$\Lambda_{0,n}=n\eta\pi/l . \tag{A14}$$

Note that the quasimode frequencies $\omega_{0,n}$ are shifted, with respect to the modes of a totally reflecting cavity, by $\Delta\omega_{0,n}=c/\Lambda_{0,n}l$.

Substituting the above approximations into Eq. (A8) and setting [see Eq. (A12)]

$$\Gamma_n=c/\Lambda_{0n}^2 l , \tag{A15}$$

we find, in the neighborhood of $\omega_k=\omega_{0,n}$,

$$L_k^2\simeq M_k^2=(2/\pi)\Gamma_n^2\Lambda_{0,n}^2/[(w_k-w_0)^2+\Gamma_n^2] , \tag{A16}$$

which is a Lorentzian line shape with linewidth $\Gamma_n$ given by Eq. (A15).

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