Squeezing of intracavity and traveling-wave light fields produced in parametric amplification

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A general input-output theory for quantum dissipative systems is developed in which it is possible to relate output to input via the internal dynamics of a system. This is applied to the problem of computing the squeezing produced by a degenerate parametric amplifier located inside a cavity. The results for the internal modes are identical with those of Milburn and Walls [Opt. Commun. **39**, 401 (1981)]. The output field is also found to have only 50% of maximal squeezing. However, by taking the output for a degenerate parametric amplifier inside a single-ended cavity and feeding this into an empty single-ended cavity, one can produce a maximally squeezed state inside this second cavity.

I. INTRODUCTION

Recent calculations by Milburn and Walls[1] and by Yurke[2] have shown that squeezing in a parametric amplifier is a subject of great subtlety and possible ambiguity. Milburn and Walls made calculations using master-equation techniques which give a maximum squeezing of only $\frac{1}{8}$ compared with a theoretical maximum of $\frac{1}{4}$, while Yurke has carried out a single-mode analysis which appears to show that the actual output from the cavity is not so limited.

We show that a more careful formulation of input, output, and internal fields in such a system is needed. The behavior of the light field inside a cavity can be described by standard master-equation techniques, which treat the external field only in its role as a heat bath. This approach is incomplete in two ways: first, it does not allow for the possibility that the incoming part of the field may be other than a vacuum or thermal, although the inclusion of a classical driving field is equivalent to allowing the incoming field a coherent amplitude; second, and more importantly, it contains no prescription for calculating the properties of the light emitted from the cavity, despite the fact that it is precisely this emitted light which is normally accessible to measurement. The approach presented here aims to provide such a method. The internal field is linked with the input by identification of the "noise" with the incoming field, and the output can then be calculated using the boundary conditions at the cavity mirror. Yurke and Denker[3] have treated the case of an electronic circuit connected to a transmission line from this viewpoint. To the circuit this looks just the same as a resistor, but it is clearly capable of carrying signals in and out.

II. INPUT-OUTPUT EQUATIONS FOR A MODEL CAVITY

We present here a phenomenological derivation of the input-output theory for a light field interacting with a cavity. In a later paper we will present a rigorous development, which is, however, not so intuitively appealing or instructive.

An optical cavity is commonly described by a Hamiltonian of the form

$$H_{\text{tot}}=H_{\text{sys}}+H_b+H_{\text{int}} \ , \tag{1}$$

where H_{sys} is a function of internal-mode operators only, H_b is the free Hamiltonian of the bath, and H_{int} describes the interaction between bath and cavity field, which is taken to be linear. The behavior of the internal mode or modes may then be calculated by master-equation methods. Alternatively, one may obtain quantum Langevin equations[4] which for a single-mode cavity becomes

$$\frac{da}{dt}=-\frac{i}{\hbar}[a,H_{\text{sys}}]-\frac{\gamma}{2}a+\Gamma \ , \tag{2}$$

where a is the annihilation operator for the internal mode, γ is the cavity damping constant, and Γ is the noise operator. For a single-ended cavity, the bath is simply the radiation field outside the mirror and the inhomogeneity Γ must therefore be ascribed to the incoming part of this external field

$$\Gamma=\gamma' a_{\text{in}} \ , \tag{3}$$

where a_{in} describes the incoming field and γ' is as yet undetermined. Time reversal of (2) must be equivalent to a change of sign in the systematic part, and replacement of the incoming field by the outgoing one, to give

$$\frac{da}{d(-t)}=\frac{i}{\hbar}[a,H_{\text{sys}}]-\frac{\gamma}{2}a+\gamma' a_{\text{out}} \ . \tag{4}$$

There will be a boundary condition at the mirror which will take to be of the form

$$a=k(a_{\text{in}}+a_{\text{out}}) \tag{5}$$

and consistency in (2)—(5) then requires $\gamma'=k\gamma$, giving

$$\begin{aligned}\frac{da}{dt}&=-\frac{i}{\hbar}[a,H_{\text{sys}}]-\frac{\gamma}{2}a+k\gamma a_{\text{in}}\\&=-\frac{i}{\hbar}[a,H_{\text{sys}}]+\frac{\gamma}{2}a-k\gamma a_{\text{out}} \ .\end{aligned} \tag{6}$$

For a linear system this can be rewritten as

$$\frac{d\underline{a}}{dt}=\left[\underline{A}-\frac{\gamma}{2}\mathbb{1}\right]\underline{a}+k\gamma\underline{a}_{\text{in}}$$
$$=\left[\underline{A}+\frac{\gamma}{2}\mathbb{1}\right]\underline{a}-k\gamma\underline{a}_{\text{out}}\ , \tag{7}$$

where

$$\underline{a}=\begin{bmatrix}a\\a^{\dagger}\end{bmatrix} \tag{8}$$

and $\underline{A}$ is a matrix. This system is linear in a quite general sense, including, for instance, the possibility of phase conjugation (on phase-conjugating and phase-preserving amplifiers, see Caves[5]). In terms of frequency components, defined by

$$\tilde{a}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\omega t}a(t)dt\ , \tag{9}$$

Eq. (7) becomes

$$-i\tilde{\underline{a}}(\omega)=(\underline{A}-\tfrac{1}{2}\gamma\mathbb{1})\tilde{\underline{a}}(\omega)+k\gamma\tilde{\underline{a}}_{\text{in}}(\omega)$$
$$=(\underline{A}+\tfrac{1}{2}\gamma\mathbb{1})\tilde{\underline{a}}(\omega)-k\gamma\tilde{\underline{a}}_{\text{out}}(\omega)\ , \tag{10}$$

where

$$\tilde{\underline{a}}(\omega)=\begin{bmatrix}\tilde{a}(\omega)\\ \tilde{a}^{\dagger}(-\omega)\end{bmatrix}\ , \tag{11}$$

and for simplicity, the commutation relations of the operators will be taken as

$$[\tilde{a}_{\text{in}}(\omega),\tilde{a}_{\text{in}}(\omega')]=0\ ,$$
$$[\tilde{a}_{\text{in}}(\omega),\tilde{a}^{\dagger}_{\text{in}}(\omega')]=\delta(\omega-\omega')\ . \tag{12}$$

This is really an approximate form, valid only for the case that one is dealing with a very narrow band of frequencies around a high frequency, which is always the case in quantum optics. In a future paper, we will show this is not an essential simplification. Rearranging to eliminate the internal mode,

$$\tilde{\underline{a}}_{\text{out}}(\omega)=[\underline{A}+(\tfrac{1}{2}\gamma+i\omega)\mathbb{1}][-\tilde{\underline{A}}+(\tfrac{1}{2}\gamma-i\omega)\mathbb{1}]^{-1}\tilde{\underline{a}}_{\text{in}}(\omega)\ . \tag{13}$$

III. A ONE-SIDED CAVITY

If the systematic part of the Hamiltonian is taken as that of a free Harmonic oscillator, we get a model of a single mode in a one-sided cavity, i.e., a cavity with significant loss through only one mirror. Thus we take

$$H_{\text{sys}}=\hbar\omega_0 a^{\dagger}a\ . \tag{14}$$

The equation for the internal-mode operator is

$$\frac{da}{dt}=-i\omega_0 a-\frac{\gamma}{2}a+k\gamma a_{\text{in}} \tag{15}$$

which has the solution for the frequency components

$$\tilde{a}(\omega)=\frac{\gamma}{\frac{1}{2}\gamma-i(\omega-\omega_0)}k\tilde{a}_{\text{in}}(\omega) \tag{16}$$

which is an ideal Lorentzian with width $\gamma/2$. The factor k can be seen to be just a normalization constant for the internal mode. It can be determined by the requirement that the internal mode have the usual equal-time discrete boson commutator

$$[a(t),a^{\dagger}(t)]=1\ . \tag{17}$$

Using the commutation relations (12) we find

$$[a(t),a^{\dagger}(t')]=k^2\gamma e^{-\gamma|t-t'|}e^{-i\omega(t-t')} \tag{18}$$

giving

$$k^2=\gamma^{-1} \tag{19}$$

so that

$$\tilde{a}(\omega)\frac{\sqrt{\gamma}}{\frac{1}{2}\gamma-i(\omega-\omega_0)}\tilde{a}_{\text{in}}(\omega)\ . \tag{20}$$

This relationship between k and γ is nothing other than a quantum fluctuation-dissipation theorem (on such theorems in general see, e.g., Gardiner[6]). It will be shown in our forthcoming paper that (19) is quite general. It is, in any case, clear that since k is defined in terms of the boundary condition at the mirror, it should be independent of the nature of the internal system and also of any other boundaries (mirrors). For the outgoing field one may then use the frequency-space equivalent of (5) to give

$$\tilde{a}_{\text{out}}(\omega)=\sqrt{\gamma}\tilde{a}(\omega)-\tilde{a}_{\text{in}}(\omega) \tag{21}$$
$$=\frac{\frac{1}{2}\gamma+i(\omega-\omega_0)}{\frac{1}{2}\gamma-i(\omega-\omega_0)}\tilde{a}_{\text{in}}(\omega)\ , \tag{22}$$

that is, the output field differs from the input by a frequency-dependent phase shift and the "out" commutation relations are the same as those for the input. Though simpler in form, this is essentially the same result as obtained by Yurke and Denker[3] for signals along a transmission line connected to an LC circuit.

IV. A TWO-SIDED CAVITY

In most cavities there is a possibility of input and output in two directions. Using H_{sys} as in (14) and generalizing (6) to allow for a second external field gives

$$\frac{da}{dt}=-\frac{i}{\hbar}[a,H_{\text{sys}}]-\frac{\gamma_1}{2}a-\frac{\gamma_2}{2}a+\sqrt{\gamma_1}a_{\text{in}}+\sqrt{\gamma_2}b_{\text{in}}\ . \tag{23}$$

In frequency space we obtain

$$\tilde{a}(\omega)=\frac{\sqrt{\gamma_1}\tilde{a}_{\text{in}}(\omega)+\sqrt{\gamma_2}\tilde{b}_{\text{in}}(\omega)}{\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-i(\omega-\omega_0)}\ . \tag{24}$$

The output field components $\tilde{a}_{\text{out}}(\omega)$ are then

$$\tilde{a}_{\text{out}}(\omega)$$
$$=\sqrt{\gamma_1}\tilde{a}(\omega)-\tilde{a}_{\text{in}}(\omega)$$
$$=\frac{[\frac{1}{2}\gamma_1-\frac{1}{2}\gamma_2+i(\omega-\omega_0)]\tilde{a}_{\text{in}}(\omega)+\sqrt{\gamma_1\gamma_2}\tilde{b}_{\text{in}}(\omega)}{\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-i(\omega-\omega_0)}\ . \tag{25}$$

If γ_2 becomes very small we regain the results for the single-ended cavity, that is to say, a small loss has a comparably small effect on the system. If, on the other hand, the two mirrors are the same, $\gamma_1=\gamma_2=\gamma$, the result is

$$\tilde{a}_{\text{out}}(\omega)=\frac{i(\omega-\omega_0)\tilde{a}_{\text{in}}(\omega)+\gamma\tilde{b}_{\text{in}}(\omega)}{\gamma-i(\omega-\omega_0)} . \quad (26)$$

Near resonance ($\omega\simeq\omega_0$) this is approximately a through-pass Lorentzian filter

$$\tilde{a}_{\text{out}}(\omega)\approx\frac{\gamma}{\gamma-i(\omega-\omega_0)}\tilde{b}_{\text{in}}(\omega) . \quad (27)$$

Further away from resonance there is an increasing element of back-reflection in the output. Eventually ($|\omega-\omega_0|\gg\gamma$) the field is completely reflected,

$$\tilde{a}_{\text{out}}(\omega)\approx-\tilde{a}_{\text{in}}(\omega) . \quad (28)$$

A truly ideal Lorentzian through-pass filter is not, of course, possible as it would also "filter out" the commutation relations and hence the quantum noise. The element of reflection that appears in (26) is exactly sufficient for their preservation.

V. THE DEGENERATE PARAMETRIC AMPLIFIER

The systematic Hamiltonian for degenerate parametric amplification with a classical pump can be written as[1,2,5]

$$H_{\text{sys}}=\hbar\omega_0 a^\dagger a+\tfrac{1}{2}i\hbar[\epsilon e^{-i\omega_p t}(a^\dagger)^2-\epsilon^* e^{i\omega_p t}a^2] , \quad (29)$$

where ω_p is the frequency of the pump beam and ϵ a measure of the effective pump intensity. For now the pump and cavity will be considered to be tuned so that $\omega_p=2\omega_0$: analysis of the effect of finite detuning is postponed for a later work.

If this system is inside a double-ended cavity the equation of motion is, from (23),

$$\frac{da}{dt}=-i\omega_0 a+\epsilon e^{-i\omega_p t}a^\dagger-\left[\frac{\gamma_1}{2}+\frac{\gamma_2}{2}\right]a +\sqrt{\gamma_1}a_{\text{in}}+\sqrt{\gamma_2}b_{\text{in}} . \quad (30)$$

We transform to a rotating frame with

$$a\to e^{i\omega_p/2}a \quad (31)$$

and similarly for the input operators.

For simplicity, we now use only these operators in the rotating frame, without any distinctive notation. In matrix notation, the equations become

$$\frac{d\underline{a}}{dt}=[\underline{A}-\tfrac{1}{2}(\gamma_1+\gamma_2)\mathbb{1}]\underline{a}+\sqrt{\gamma_1}\underline{a}_{\text{in}}+\sqrt{\gamma_2}\underline{b}_{\text{in}} , \quad (32)$$

where

$$\underline{A}=\begin{bmatrix}0 & \epsilon\\ \epsilon^* & 0\end{bmatrix}=\begin{bmatrix}0 & |\epsilon|e^{i\theta}\\ |\epsilon|e^{-i\theta} & 0\end{bmatrix} . \quad (33)$$

In frequency space Eq. (32) transforms to

$$-i\omega\underline{\tilde{a}}(\omega)=[\underline{A}-\tfrac{1}{2}(\gamma_1+\gamma_2)\mathbb{1}]\underline{\tilde{a}}(\omega) +\sqrt{\gamma_1}\underline{\tilde{a}}_{\text{in}}(\omega)+\sqrt{\gamma_2}\underline{\tilde{b}}_{\text{in}}(\omega) , \quad (34)$$

where now to allow for the rotating frame,

$$\underline{\tilde{a}}(\omega)=\begin{bmatrix}\tilde{a}(\omega_s+\omega)\\ \tilde{a}^\dagger(\omega_s-\omega)\end{bmatrix} , \quad (35)$$

with $\omega_s=\omega_p/2$, and similarly for the input operators. After performing the matrix inversion

$$\tilde{a}(\omega_s+\omega)=\frac{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-i\omega)[\sqrt{\gamma_1}\tilde{a}_{\text{in}}(\omega_s+\omega)+\sqrt{\gamma_2}\tilde{b}_{\text{in}}(\omega_s+\omega)]}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-i\omega)^2-|\epsilon|^2}+\frac{\epsilon[\sqrt{\gamma_1}\tilde{a}^\dagger_{\text{in}}(\omega_s-\omega)+\sqrt{\gamma_2}\tilde{b}^\dagger_{\text{in}}(\omega_s-\omega)]}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-i\omega)^2-|\epsilon|^2} . \quad (36)$$

If both the input fields are vacuum or coherent, they will have zero normally ordered variance, that is,

$$\underline{C}_N(a_{\text{in}},a_{\text{in}}^\dagger)=\underline{C}_N(b_{\text{in}},b_{\text{in}}^\dagger)=\underline{0} , \quad (37)$$

where

$$\underline{C}_N(a,a^\dagger)=\begin{bmatrix}\langle a,a\rangle & \langle a^\dagger,a\rangle\\ \langle a^\dagger,a\rangle & \langle a^\dagger,a^\dagger\rangle\end{bmatrix} \quad (38)$$

using the notation

$$\langle a,b\rangle=\langle ab\rangle-\langle a\rangle\langle b\rangle . \quad (39)$$

In this case the only contribution to the normally ordered variance of the internal field will be from the commutator terms, giving

$$\langle\tilde{a}(\omega_s+\omega),\tilde{a}(\omega_s+\omega')\rangle=\frac{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-i\omega)e^{i\theta}}{2}\left[\frac{1}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|)^2+\omega^2}-\frac{1}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|)^2+\omega^2}\right]\delta(\omega+\omega') ,$$

$$\langle\tilde{a}^\dagger(\omega_s+\omega),\tilde{a}(\omega_s+\omega')\rangle=\frac{|\epsilon|}{2}\left[\frac{1}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|)^2+\omega^2}-\frac{1}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|)^2+\omega^2}\right]\delta(\omega-\omega') . \quad (40)$$

For the full internal mode, the variances are then

$$\langle a^\dagger,a\rangle=\frac{1}{2\pi}\int d\omega\, d\omega'\,\langle \tilde{a}^\dagger(\omega_s+\omega),\tilde{a}(\omega_s+\omega')\rangle$$
$$=\frac{1}{2}\frac{|\epsilon|^2}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2)^2-|\epsilon|^2}\,, \tag{41}$$

$$\langle a,a\rangle=\frac{1}{2\pi}\int d\omega\, d\omega'\,\langle \tilde{a}(\omega_s+\omega),\tilde{a}(\omega_s+\omega')\rangle$$
$$=\frac{1}{2}\frac{\epsilon(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2)}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2)^2-|\epsilon|^2}\,. \tag{42}$$

To see the squeezing, the field must be expressed in terms of the quadrature phases,[1,2,5] Hermitian operators defined by

$$a=e^{i\theta/2}(X_1+iX_2)\,. \tag{43}$$

The normally ordered variances of these operators are

$$\langle :X_1,X_1:\rangle=\frac{1}{4}\frac{|\epsilon|}{\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|}\,,$$
$$\langle :X_2,X_2:\rangle=-\frac{1}{4}\frac{|\epsilon|}{\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|}\,, \tag{44}$$
$$\langle :X_1,X_2:\rangle=0\,.$$

Perfect squeezing in one quadrature, corresponding to an eigenstate of the quadrature phase operator, is achieved with a normally ordered variance of $-\frac{1}{4}$. The best that can be achieved in the case of the parametric amplifier in a cavity is when on oscillation threshold, giving

$$\langle :X_2,X_2:\rangle=-\tfrac{1}{8}\,. \tag{45}$$

Squeezing by a factor of one-half can thus be obtained in the X_2 quadrature of the generating cavity with the X_1 quadrature infinitely unsqueezed. Note that the properties of the internal mode considered in this section depend only on the total damping, not on the damping through each mirror separately. This is naturally only true when the two input fields have, as assumed, identical statistics.

The degenerate parametric amplifier in a cavity has been analyzed in depth by Milburn and Walls[1] including a quantized pump beam. It is not difficult to check that their result is identical with ours for the intracavity properties, but of course it does not give any answer for the output fields.

VI. THE OUTPUT FIELD

The internal-mode operators having already been found in terms of those for the input field, the output operators can now be calculated with use of Eq. (21):

$$\tilde{a}_{\rm out}(\omega_s+\omega)=\frac{[(\frac{1}{2}\gamma_1)^2-(\frac{1}{2}\gamma_2-i\omega)^2+|\epsilon|^2]\tilde{a}_{\rm in}(\omega_s+\omega)+\epsilon\gamma_1\tilde{a}^\dagger_{\rm in}(\omega_s-\omega)}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-i\omega)^2-|\epsilon|^2}+\frac{\sqrt{\gamma_1\gamma_2}(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-i\omega)\tilde{b}_{\rm in}(\omega_s+\omega)+\epsilon\sqrt{\gamma_1\gamma_2}b^\dagger_{\rm in}(\omega_s-\omega)}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-i\omega)^2-|\epsilon|^2}\,. \tag{46}$$

Note that (13) could not be applied directly as it makes no allowance for the second mirror, although if one wished it could readily be generalized to correct this. As with the internal field, only antinormally ordered terms contribute to the variances so that

$$\langle \tilde{a}^\dagger_{\rm out}(\omega_s+\omega),\tilde{a}_{\rm out}(\omega_s+\omega')\rangle$$
$$=\frac{|\epsilon|\gamma_1}{2}\left[\frac{1}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|)^2+\omega^2}-\frac{1}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|)^2+\omega^2}\right]\delta(\omega-\omega')\,, \tag{47}$$

$$\langle \tilde{a}_{\rm out}(\omega_s+\omega),\tilde{a}_{\rm out}(\omega_s+\omega')\rangle$$
$$=\frac{|\epsilon|\gamma_1}{2}\left[\frac{1}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|)^2+\omega^2}+\frac{1}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|)^2+\omega^2}\right]\delta(\omega+\omega')\,.$$

Calculation of the total output field variances shows a very direct relationship with those for the internal mode:

$$\langle a^\dagger_{\rm out},a_{\rm out}\rangle$$
$$=\frac{1}{2\pi}\int d\omega\, d\omega'\,\langle \tilde{a}^\dagger_{\rm out}(\omega_s+\omega),\tilde{a}_{\rm out}(\omega_s+\omega')\rangle$$
$$=\gamma_1\langle a^\dagger,a\rangle\,, \tag{48}$$
$$\langle a_{\rm out},a_{\rm out}\rangle$$
$$=\frac{1}{2\pi}\int d\omega\, d\omega'\,\langle \tilde{a}_{\rm out}(\omega_s+\omega),\tilde{a}_{\rm out}(\omega_s+\omega')\rangle$$
$$=\gamma_1\langle a,a\rangle\,.$$

Once again it is the variances in the quadrature phases which are of most interest. Defining the output quadrature phases in the same fashion as the internal ones, from (47)

$$\langle :\tilde{X}_{1,\rm out}(\omega_s+\omega),\tilde{X}_{1,\rm out}(\omega_s+\omega'):\rangle$$
$$=\frac{|\epsilon|(\gamma_1/2)}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|)^2+\omega^2}\delta(\omega+\omega')\,, \tag{49}$$
$$\langle :\tilde{X}_{2,\rm out}(\omega_s+\omega),\tilde{X}_{2,\rm out}(\omega_s+\omega'):\rangle$$
$$=-\frac{|\epsilon|(\gamma_1/2)}{(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|)^2+\omega^2}\delta(\omega+\omega')\,,$$

while from (48)

$$\langle :X_{1,\text{out}},X_{1,\text{out}}:\rangle=\frac{\gamma_1}{4}\frac{|\epsilon|}{\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|}\,,$$
$$\langle :X_{2,\text{out}},X_{2,\text{out}}:\rangle=-\frac{\gamma_1}{4}\frac{|\epsilon|}{\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|}\,. \tag{50}$$

The maximum squeezing is still attained at threshold, $|\epsilon|=\frac{1}{2}(\gamma_1+\gamma_2)$, giving

$$\langle :\tilde{X}_{2,\text{out}}(\omega_s+\omega),\tilde{X}_{2,\text{out}}(\omega_s+\omega'):\rangle=-\frac{\gamma_1}{4}\frac{\gamma_1+\gamma_2}{(\gamma_1+\gamma_2)^2+\omega^2}\delta(\omega+\omega')\,, \tag{51}$$

$$\langle :X_{2,\text{out}},X_{2,\text{out}}:\rangle=-\frac{\gamma_1}{8}\,.$$

The most interesting thing about these results is that while the squeezing in the total field is independent of γ_2, this is not the case for the individual frequency components. The δ function in (51) can be removed by integrating over ω' to give the normally ordered spectrum of the operator $X_{2,\text{out}}$:

$$:S_{2,\text{out}}(\omega_s+\omega):=-\frac{\gamma_1}{4}\frac{\gamma_1+\gamma_2}{(\gamma_1+\gamma_2)^2+\omega^2}\,, \tag{52}$$

which is a convenient way of describing the squeezing in the output field. It may be thought of loosely as the squeezing at a particular frequency, although in this case it results from the coupling of pairs of frequencies on either side of resonance: This spectrum is, ignoring the sign, a Lorentzian with peak height $\frac{1}{4}[\gamma_1/(\gamma_1+\gamma_2)]$ and width $\gamma_1+\gamma_2$. Thus for a symmetric double-ended cavity with $\gamma=\gamma_1=\gamma_2$

$$:S_{2,\text{out}}(0):=-\tfrac{1}{8}\,. \tag{53}$$

The resonant mode of the output field is squeezed by a factor of one-half, the same as the internal field. If, however, the cavity is single-ended, with $\gamma_2=0$,

$$:S_{2,\text{out}}(0):=-\tfrac{1}{4} \tag{54}$$

and this corresponds to Yurke's[2] single-mode analysis. Thus our multimode analysis agrees with Yurke's result, which is correct for the case that we measure only the output field corresponding to $\omega=0$. Although the preferred method of measuring squeezing is by homodyne detection, we would like here to consider the effect of passive filtering of the output, to see what the effect on the squeezing is of trying to isolate the squeezed mode by means of such a passive filter.

VII. FILTERING OF THE OUTPUT

As a model of a passive filter, we consider passing the output through a second cavity resonant with the first, so that the system is now such that a rotator or equivalent is used to isolate the input a_{in} from any feedback effects. The output field through the filter is given by (25) as

$$\tilde{c}_{\text{out}}(\omega_s+\omega)=\frac{(\frac{1}{2}\kappa_1-\frac{1}{2}\kappa_2+i\omega)\tilde{c}_{\text{in}}(\omega_s+\omega)+\sqrt{\kappa_1\kappa_2}\tilde{d}_{\text{in}}(\omega_s+\omega)}{\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2-i\omega}\,, \tag{55}$$

where κ_1 and κ_2 describe the mirrors of the filtering cavity. Using $d_{\text{in}}=a_{\text{out}}$ and assuming c_{in} to be a vacuum field, the variances of the output field c_{out} can be computed, and we find for the quadrature phases of the total field

$$\langle :\tilde{Y}_{1,\text{out}},\tilde{Y}_{1,\text{out}}:\rangle=\frac{1}{4}\frac{\kappa_1\kappa_2|\epsilon|\gamma_1}{(\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2)(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|)(\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2+\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|)}\,,$$
$$\langle :\tilde{Y}_{2,\text{out}},\tilde{Y}_{2,\text{out}}:\rangle=-\frac{1}{4}\frac{\kappa_1\kappa_2|\epsilon|\gamma_1}{(\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2)(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|)(\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2+\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|)}\,. \tag{56}$$

At threshold, with the generating cavity single-ended ($\gamma_2=0$),

$$\langle :Y_{2,\text{out}},Y_{2,\text{out}}:\rangle=-\frac{1}{8}\frac{\kappa_1\kappa_2\gamma_1}{(\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2)[\gamma_1+\frac{1}{2}(\kappa_1+\kappa_2)]}\,. \tag{57}$$

For any κ_1,κ_2 this always gives less squeezing than in the initial output field a_{out}. The best that can be achieved is with $\kappa_1=\kappa_2>>\gamma_1$, giving

$$\langle :Y_{2,\text{out}},Y_{2,\text{out}}:\rangle\approx-\frac{\gamma_1}{8}\,. \tag{58}$$

This is the same as if the filter were not there at all, which is reasonable as, for large κ_1 and κ_2, the second cavity interacts strongly with the external field over a correspondingly wide bandwidth. In the alternative limit of $\kappa=\kappa_1=\kappa_2<<\gamma_1$ the squeezing reduces to

$$\langle :Y_{2,\text{out}},Y_{2,\text{out}}:\rangle\approx-\frac{\kappa}{8}\,. \tag{59}$$

Thus a through-pass filter cannot improve the squeezing in the output field. If, however, one considers the internal mode of the second cavity, the picture is rather different. From (24)

$$\tilde{c}(\omega_s+\omega)=\frac{\sqrt{\kappa_1}\tilde{c}_{\rm in}(\omega_s+\omega)+\sqrt{\kappa_2}\tilde{d}_{\rm in}(\omega_s+\omega)}{\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2-i\omega}\,, \tag{60}$$

or going directly to the quadrature phases,

$$\tilde{Y}_i(\omega_s+\omega)=\frac{\sqrt{\kappa_1}\tilde{Y}_{i,\rm in}(\omega_s+\omega)+\sqrt{\kappa_2}\tilde{X}_{i,\rm out}(\omega_s+\omega)}{\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2-i\omega}\,,\quad i=1,2\,. \tag{61}$$

With $c_{\rm in}$ a vacuum field, we find for the full internal modes

$$\langle :Y_1,Y_1:\rangle=\frac{1}{4}\frac{\kappa_2|\epsilon|\gamma_1}{(\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2)(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|)(\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2+\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2-|\epsilon|)}\,,$$
$$\langle :Y_2,Y_2:\rangle=-\frac{1}{4}\frac{\kappa_2|\epsilon|\gamma_1}{(\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2)(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|)(\frac{1}{2}\kappa_1+\frac{1}{2}\kappa_2+\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+|\epsilon|)}\,. \tag{62}$$

On threshold with the generating cavity single-ended

$$\langle :Y_2,Y_2:\rangle=-\frac{1}{4}\frac{\kappa_2}{\kappa_1+\kappa_2}\frac{\gamma_1}{\gamma_1+\frac{1}{2}(\kappa_1+\kappa_2)}\,. \tag{63}$$

If the second cavity is also single-ended ($\kappa_1=0$), and is much narrower than the first ($\kappa_2\ll\gamma_1$), this becomes

$$\langle :Y_2,Y_2:\rangle=-\tfrac{1}{4}\,, \tag{64}$$

which corresponds to perfect squeezing in the second cavity. We thus see that it is indeed possible to produce arbitrarily large squeezing inside a cavity, provided this cavity is single-ended.

VIII. CONCLUSION

In this paper we have outlined general methods of relating input, output, and internal dynamics. These methods will be developed and put on a firm theoretical foundation in a forthcoming paper. The main results are a clarification of how to calculate squeezing in a multimode situation, and the demonstration that maximal squeezing *inside* a cavity can be achieved.

The results presented here depend on the model of a cavity mode as a harmonic oscillator. However, calculations by Gardiner and Savage[7] have shown that exactly the same results arise by a detailed treatment of the motion of light waves through a cavity composed of genuine mirrors, and our forthcoming paper will unify these two treatments.

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