Photon statistics of radiation scattered by relativistic electrons in an interfering electromagnetic field

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(Received 13 July 1983)

The statistical properties of radiation emitted in a scattering process between a relativistic electron beam and two interfering crossed laser fields are discussed, when the radiation system is coupled to a reservoir (electron-beam system). The conditions to obtain anticorrelation effect are presented, and the presence of super-Poissonian statistics, when the short-time approximation is adopted, is shown.

I. INTRODUCTION

The scattering between a relativistic electron beam and two interfering laser beams has already been discussed.[1,2] The emitted photon flux has been shown to be increased, when some "angle conditions" are fulfilled, if electrons are bunched. The bunching effect is produced by a ponderomotive force[3] which creates an "electron grating" from which light is scattered in the Bragg directions.

In the following the statistical properties of backscattered radiation in the electron rest frame (ERF) are studied when the radiation system is coupled to a "reservoir" (electron beam). Anticorrelation effects are found to occur between scattered modes; super-Poissonian statistics also occurs in each emitted mode when the short-time approximation is adopted.

In the radiation field the following modes are considered: two incident modes at nearly the same frequency (in the ERF) (modes $1i$ and $2i$ which are the interfering incident modes), and two backscattered modes ($1s$ and $2s$) at nearly equal frequency ω_s (the scattering is a quasielastic one in the ERF). When the phase difference between modes $1s$ and $2s$ is zero, the emitted intensity is higher; this means that the scattered field also interferes.

The coherent state technique and the q-c number correspondence have been employed, working in the Schrödinger picture with the generalized Fokker-Planck equation for the antinormal quasidistribution function. The factorial moments are derived from the solution of the Fokker-Planck equation. Losses are not accounted for.

II. MASTER EQUATION

In the following we suppose that an accelerated electron beam can be considered as a reservoir in which the interaction with the radiation system produces transitions between the two electron states. We can consider the electron beam a reservoir if we assume that the electron beam is not fully monoenergetic, but has an energy spread ΔE (in the hypothesis of $\Delta E_{\rm elec} \gg \Delta E_{\rm rad}$, see Appendix A). The statistical properties of radiation are obtained from the motion equation of the reduced density operator for the field, under the Markov approximation.

We study the problem in the electron rest frame, in which the scattering is a quasielastic one, in two cases:

(a) small gain, i.e., the difference between number of emission and absorption processes is very small and

(b) high gain, in which the number of emission processes is greater than the number of absorption processes.

The total Hamiltonian $H_{\rm tot}$ describing the interaction is given by

$$H_{\rm tot} = H_0 + H_{\rm int} \;, \tag{1}$$

where H_0 is the free Hamiltonian, i.e.,

$$H_0 = H_f + H_e \;, \tag{2}$$

where $H_f = \sum_{k=1}^{4} \hbar \hat{a}_k^\dagger \hat{a}_k$ is the free radiation Hamiltonian, and

$$H_e = E \sum_n \sum_p (\hat{c}_{pn}^\dagger \hat{c}_{pn} - \hat{d}_{pn} \hat{d}_{pn}^\dagger) \tag{3}$$

is the free Hamiltonian of the electron system in second quantization, where $\hat{c}^\dagger$ and $\hat{c}$ are fermion creation and annihilation operators, for fixed spin value, and $\hat{d}$ and $\hat{d}^\dagger$ are antiparticle operators. $\sum_p$ is the sum on the momentum and $\sum_n$ is the sum on the total number of electrons. The interaction Hamiltonian for the scattering in the interfering field is given by

$$H_{\rm int} = \sum_{l,m} \sum_n \hbar K^{(2)} \hat{c}_{ln}^\dagger \hat{c}_{mn} (\hat{a}_{1s}^\dagger \hat{a}_{1i} + \hat{a}_{1s}^\dagger \hat{a}_{2i} + \hat{a}_{2s}^\dagger \hat{a}_{1i} + \hat{a}_{2s}^\dagger \hat{a}_{2i}) + \text{c.c.} \;, \tag{4}$$

where l and m are electron states, and $K^{(2)}$ is the transition matrix element. We remember that for the photon momentum uncertainty in an interfering field, each one of the products $\hat{c}_l^\dagger \hat{c}_m \hat{a}_s^\dagger \hat{a}_i$ gives a nonzero contribution to the scattering probability (the photon momentum uncertainty makes sure of the momentum conservation law in the scattering).[1,2] Introducing the new variables $\hat{A}_{1i,s} = (\hat{a}_{1i,s} + \hat{a}_{2i,s})/\sqrt{2}$ and $\hat{A}_{2i,s} = (\hat{a}_{1i,s} - \hat{a}_{2i,s})/\sqrt{2}$ that satisfy the standard commutation rules, the interaction Hamiltonian becomes

$$H_{\rm int}=\sum_{l,m}\sum_{n}\hbar K^{(2)}\hat{c}^{\dagger}_{ln}\hat{c}_{mn}\hat{A}^{\dagger}_{1s}\hat{A}_{1i}\ , \qquad (5)$$

where

$$\sum_{l,m}=\sum_{p}\ .$$

The statistical properties of the radiation—two-state-electron system are described by a density operator $\hat{\rho}(t)$ which satisfies the following motion equation:

$$\frac{\partial\hat{\rho}}{\partial t}=\frac{1}{i\hbar}[H_{\rm tot},\hat{\rho}] \qquad (6)$$

[in the Schrödinger picture (SP)].

We are interested only in the radiation properties, therefore we eliminate the reservoir variables (electron system) obtaining the density operator for the field alone, $\hat{\rho}_f(t)$. The motion equation for the density operator may be described using the Markov approximation and the standard techniques (see Appendix A).

Thus we obtain the master equation for the reduced density operator in the SP:

$$\frac{\partial\hat{\rho}_f}{\partial t}=\frac{1}{i\hbar}[H_f,\hat{\rho}_f] +4K([\hat{A}_i\hat{A}^{\dagger}_s\hat{\rho}_f,\hat{A}^{\dagger}_i\hat{A}_s]+[\hat{A}_i\hat{A}^{\dagger}_s,\hat{\rho}_f\hat{A}^{\dagger}_i\hat{A}_s]) -4Z([\hat{A}_i\hat{A}^{\dagger}_s,\hat{A}^{\dagger}_i\hat{A}_s\hat{\rho}_f]+[\hat{\rho}_f\hat{A}_i\hat{A}^{\dagger}_s,\hat{A}^{\dagger}_i\hat{A}_s])\ , \qquad (7)$$

where

$$Z=+W^{\pm}_{1,2}=(\gamma/2)\bar{n}_l(1-\bar{n}_m)$$

is the number of emission processes, and

$$K=+W^{\pm}_{2,1}=(\gamma/2)\bar{n}_m(1-\bar{n}_l)$$

is the number of absorption processes; moreover (see Appendix A),

$$\gamma\sim|K^{(2)}|^2$$

and $|K^{(2)}|^2$ is the transition probability (2), $\bar{n}_m$ is the electron density in the m state, $\bar{n}_l$ is the electron density in the l state, $\hat{A}_{1i,s}=\hat{A}_{i,s}$. Equation (7) is equivalent to what is obtained in the optical region in the scattering between radiation and an atomic system.

III. FOKKER-PLANCK EQUATION

Making use of the relations

$$[\hat{u}^{\dagger}\hat{v},\hat{\rho}]=[\hat{u}^{\dagger},\hat{\rho}]\hat{v}+\hat{u}^{\dagger}[\hat{v},\hat{\rho}]\ ,$$
$$[\hat{u}\hat{v},\hat{u}^{\dagger}\hat{v}^{\dagger}\hat{\rho}]=\hat{u}^{\dagger}\hat{v}^{\dagger}[\hat{u}\hat{v},\hat{\rho}]+[\hat{u}\hat{v},\hat{\rho}]\hat{u}^{\dagger}\hat{v}^{\dagger}\ , \qquad (8)$$
$$[\hat{a}_k,\hat{a}^{\dagger}_{k'}]=\delta_{kk'}$$

and

$$[\hat{a}_j,\hat{N}]=\frac{\partial\hat{N}}{\partial\hat{a}^{\dagger}_j}\ ,\ \ [\hat{N},\hat{a}^{\dagger}_j]=\frac{\partial\hat{N}}{\partial\hat{a}_j}\ ,$$
$$\hat{a}_j\frac{\partial\hat{N}}{\partial\hat{a}^{\dagger}_j}=\frac{\partial\hat{N}}{\partial\hat{a}^{\dagger}_j}\hat{a}_j+\frac{\partial^2\hat{N}}{(\partial\hat{a}^{\dagger}_j)^2}\ , \qquad (9)$$
$$\frac{\partial\hat{N}}{\partial\hat{a}_j}\hat{a}^{\dagger}_j=\hat{a}^{\dagger}_j\frac{\partial\hat{N}}{\partial\hat{a}_j}+\frac{\partial^2\hat{N}}{(\partial\hat{a}_j)^2}\ ,$$

we obtain the master equation for the normally ordered operator $\hat{\rho}^{n}_{f}$. Now we apply the operator N^{-1} which transforms the operator function $\hat{\rho}^{n}_{f}$ into an ordinary function $\bar{\rho}^{(n)}(\alpha_j,\alpha^*_j,t)$ of the complex variable α_j by replacing $\hat{a}_j$ by α_j and $\hat{a}^{\dagger}_j$ by α^*_j. We must remember that α_j is the eigenvalue of $\hat{a}_j$ in the coherent state $|\{\alpha_j\}\rangle$ representation. Then, making use of the relation

$$\Phi_A(\{\alpha_j\},t)=\frac{1}{\pi^M}\bar{\rho}^{(n)}(\{\alpha_j\},t)\ , \qquad (10)$$

where $M=4$ is the number of modes, we obtain a generalized Fokker-Planck equation for the antinormal quasidistribution function $\Phi_A(\{\alpha_j\},t)$ representing the equation of the density operator in c numbers:

$$\begin{aligned}\frac{\partial\Phi_A}{\partial t}=&\left[\sum_i\left[\omega_i\frac{\partial\Phi_A}{\partial\alpha_i}\alpha_i-\text{c.c.}\right]+\sum_s\left[\omega_s\frac{\partial\Phi_A}{\partial\alpha_s}\alpha_s-\text{c.c.}\right]\right]\\&+4(K-Z)\left[2|A_s|^2\Phi_A-2|A_i|^2\Phi_A+|A_s|^2A^*_i\frac{\partial\Phi_A}{\partial A^*_i}+\text{c.c.}-|A_i|^2A^*_s\frac{\partial\Phi_A}{\partial A^*_s}-\text{c.c.}\right]\\&-4(K+Z)\left[A^*_sA^*_i\frac{\partial^2\Phi_A}{\partial A^*_i\partial A^*_s}+\text{c.c.}\right]-4Z\left[-2|A_s|^2\frac{\partial^2\Phi_A}{\partial A^*_i\partial A_i}+A^*_i\frac{\partial\Phi_A}{\partial A^*_i}+\frac{\partial\Phi_A}{\partial A_i}A_i\right]\\&-4K\left[A^*_s\frac{\partial\Phi_A}{\partial A^*_s}+\frac{\partial\Phi_A}{\partial A_s}A_s-2|A_i|^2\frac{\partial^2\Phi_A}{\partial A^*_s\partial A_s}\right]\ ,\end{aligned} \qquad (11)$$

where

$$\sum_i\to 1i,2i\ ,\ \ \sum_s\to 1s,2s\ ,\ \ A_{i,s}=(\alpha_{1i,s}+\alpha_{2i,s})/\sqrt{2}\ .$$

Performing the Fourier transform

$$\Phi_A=\int C_A(\{\beta_j\},t)\frac{1}{\pi^4}\prod_j\exp(-\beta_j\alpha^*_j+\beta^*_j\alpha_j)d^2\beta_j\ , \qquad (12)$$

introducing a new variable $\theta_j=\beta_j\exp(i\omega_j t)$, and eliminating the fast oscillations $\exp(-i\omega_j t)$, we obtain

$$\begin{aligned}\frac{\partial C'_A(\{\theta_j\},t)}{\partial t}=&-(K+Z)\left[4C'_A+(\theta_{1i}+\theta_{2i})\left[\frac{\partial}{\partial\theta_{1i}}+\frac{\partial}{\partial\theta_{2i}}\right]C'_A+\text{c.c.}+(\theta_{1s}+\theta_{2s})\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]C'_A+\text{c.c.}\right.\\&\left.+(\theta_{1s}+\theta_{2s})(\theta_{1i}+\theta_{2i})\left[\frac{\partial}{\partial\theta_{1i}}+\frac{\partial}{\partial\theta_{2i}}\right]\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]C'_A+\text{c.c.}\right]\\&+K\left[2(\theta^*_{1i}+\theta^*_{2i})(\theta_{1i}+\theta_{2i})\left[\frac{\partial}{\partial\theta^*_{1s}}+\frac{\partial}{\partial\theta^*_{2s}}\right]\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]C'_A\right.\\&\left.+(\theta_{1i}+\theta_{2i})\left[\frac{\partial}{\partial\theta_{1i}}+\frac{\partial}{\partial\theta_{2i}}\right]C'_A+\text{c.c.}+4C'_A\right]\\&+Z\left[2(\theta^*_{1s}+\theta^*_{2s})(\theta_{1s}+\theta_{2s})\left[\frac{\partial}{\partial\theta^*_{1i}}+\frac{\partial}{\partial\theta^*_{2i}}\right]\left[\frac{\partial}{\partial\theta_{1i}}+\frac{\partial}{\partial\theta_{2i}}\right]C'_A\right.\\&\left.+(\theta_{1s}+\theta_{2s})\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]C'_A+\text{c.c.}+4C'_A\right]\\&+(K-Z)\left[\left[\frac{\partial}{\partial\theta^*_{1s}}+\frac{\partial}{\partial\theta^*_{2s}}\right]\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]C'_A-\left[\frac{\partial}{\partial\theta^*_{1i}}+\frac{\partial}{\partial\theta^*_{2i}}\right]\left[\frac{\partial}{\partial\theta_{1i}}+\frac{\partial}{\partial\theta_{2i}}\right]C'_A\right.\\&+(\theta_{1i}+\theta_{2i})\left[\frac{\partial}{\partial\theta^*_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]\left[\frac{\partial}{\partial\theta_{1i}}+\frac{\partial}{\partial\theta_{2i}}\right]C'_A+\text{c.c.}\\&\left.-(\theta_{1s}+\theta_{2s})\left[\frac{\partial}{\partial\theta^*_{1i}}+\frac{\partial}{\partial\theta^*_{2i}}\right]\left[\frac{\partial}{\partial\theta_{1i}}+\frac{\partial}{\partial\theta_{2i}}\right]\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]C'_A-\text{c.c.}\right]. \end{aligned}\tag{13}$$

We have used $C_A(\{\beta_j\},t)\to C'_A(\{\theta_j\},t)$ because in the problems involving interactions, these functions exist for all times, so that they are more convenient than the Glauber-Sudarshan quasidistribution $\Phi_N\to C_N$. Now we seek a solution of Eq. (13) in two cases: (a) small gain, i.e., $Z\sim K$ and (b) high gain, $Z\gg K$.

(a) *Small gain* ($Z\sim K=p$). In this case Eq. (13) becomes

$$\begin{aligned}\frac{\partial C'_A}{\partial t}=-2p&\left[2\theta_i\frac{\partial}{\partial\theta_i}+\text{c.c.}+\frac{1}{2}(\theta_{1s}+\theta_{2s})\left[\frac{\partial}{\partial\theta_{2s}}+\frac{\partial}{\partial\theta_{1s}}\right]+\text{c.c.}+4\theta_i(\theta_{1s}+\theta_{2s})\left[\frac{\partial}{\partial\theta_i}\right]\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]+\text{c.c.}\right.\\&\left.-4|\theta_i|^2\left[\frac{\partial}{\partial\theta^*_{1s}}+\frac{\partial}{\partial\theta^*_{2s}}\right]\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]-4(\theta_{1s}+\theta_{2s})(\theta^*_{1s}+\theta^*_{2s})\frac{\partial^2}{\partial\theta^*_i\partial\theta_i}\right]C'_A\,,\end{aligned}\tag{14}$$

where, because the incident field is an interfering one, we have considered modes as indistinguishable: $\theta_{1i}=\theta_{2i}=\theta_i$. We seek the solution of Eq. (14) in the form

$$C'_A(\{\theta_j\},t)=\exp\left[\sum_{k=0}^{\infty}(2p)^k f_k(\{\theta_j\},t)\right],\tag{15}$$

where the field is assumed initially coherent. Substituting Eq. (15) into Eq. (14) and comparing the coefficients at the same power of p, we obtain a recursion system of equations for f_k:[4]

$$\frac{\partial f_k(\{\theta_j\},t)}{\partial t}=g_k(\{\theta_j\},t)\,,\tag{16}$$

where

$$\begin{aligned}g_k(\{\theta_j\},t)=-&\left[(A+B)f_{k-1}+B^*f_{k-1}\right.\\&+\sum_{m+r=k-1}(V_1f_mV_2f_r+C_1f_mC_2f_r)\\&\left.+\sum_{m+r=k-1}V^*_1f_mV^*_2f_r\right]\end{aligned}\tag{17}$$

with

$$A=-4|\theta_i|^2\left[\frac{\partial}{\partial\theta_{1s}^*}+\frac{\partial}{\partial\theta_{2s}^*}\right]\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]$$

$$-4(\theta_{1s}+\theta_{2s})(\theta_{1s}^*+\theta_{2s}^*)\frac{\partial^2}{\partial\theta_i^*\partial\theta_i},$$

$$B=2\theta_i\frac{\partial}{\partial\theta_i}+\frac{1}{2}(\theta_{1s}+\theta_{2s})\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]$$

$$+4\theta_i(\theta_{1s}+\theta_{2s})\left[\frac{\partial}{\partial\theta_i}\right]\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right],$$

$$C_1=4\theta_i\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right]e_1+4(\theta_{1s}+\theta_{2s})\left[\frac{\partial}{\partial\theta_i^*}\right]e_2,$$

$$C_2=-\theta_i^*\left[\frac{\partial}{\partial\theta_{1s}^*}+\frac{\partial}{\partial\theta_{2s}^*}\right]e_1-(\theta_{1s}^*+\theta_{2s})\left[\frac{\partial}{\partial\theta_i}\right]e_2, \tag{18}$$

$$e_1e_2=\delta_{1,2},$$

$$V_1=4\theta_i\frac{\partial}{\partial\theta_i},\quad V_2=(\theta_{1s}+\theta_{2s})\left[\frac{\partial}{\partial\theta_{1s}}+\frac{\partial}{\partial\theta_{2s}}\right].$$

The solution of Eq. (15) is given by

$$f_k(\{\theta_j\},t)=f_k(\{\theta_j\},0)+\int_0^t g_k(\{\theta_j\},t')dt'. \tag{19}$$

We assume as an initial condition for $C_A'(\{\theta_j\},t)$ that the field is in a coherent state $|\{\xi_j\}\rangle$ (at the time $t=0$); therefore, we have

$$f_0(\{\theta_j\},0)=\sum_{j=1}^{3}(-|\theta_j|^2+\theta_j\xi_j^*-\theta_j^*\xi_j), \tag{20}$$

$j=1\rightarrow i$, $j=2\rightarrow 1s$, $j=3\rightarrow 2s$, and $f_k(\{\theta_j\},0)=0$ for $k\geq 1$. With this condition, performing the calculation of Eq. (19), we obtain the following result, correct for $|p\xi t|<1$ ("short-time approximation," where t represents the interaction time):

$$\begin{aligned}C_N(\{\beta_j\},t)=\exp\Bigg[&\sum_{j=1}^{3}(\beta_je^{i\omega_jt}\xi_j^*-\beta_j^*e^{-i\omega_jt}\xi_s)\\&-2pt[+4|\beta_i|^2+3|\beta_{1s}|^2+3|\beta_{2s}|^2+3\beta_{1s}\beta_{2s}^*+3\beta_{1s}^*\beta_{2s}+2\beta_i\xi_i^*e^{i\omega_it}-2\beta_i^*\xi_ie^{-i\omega_it}\\&+\tfrac{1}{2}(\beta_{1s}+\beta_{2s})e^{i\omega_st}\xi_{1s}^*-\tfrac{1}{2}(\beta_{1s}^*+\beta_{2s}^*)\xi_{1s}e^{-i\omega_st}\\&+\tfrac{1}{2}\xi_{2s}^*(\beta_{1s}+\beta_{2s})e^{i\omega_st}-\tfrac{1}{2}(\beta_{1s}^*+\beta_{2s}^*)\xi_{2s}e^{-i\omega_st}\\&+4\beta_ie^{i\omega_it}e^{i\omega_st}(\beta_{1s}+\beta_{2s})(-\beta_i^*e^{-i\omega_it}+\xi_i^*)(-\beta_{1s}^*e^{-i\omega_st}+\xi_{1s}^*-\beta_{2s}^*e^{-i\omega_st}+\xi_{2s}^*)+\text{c.c.}\\&-4|\beta_i|^2(-\beta_{1s}^*e^{-i\omega_st}+\xi_{1s}^*-\beta_{2s}^*e^{-i\omega_st}+\xi_{2s}^*)(-\beta_{1s}e^{i\omega_st}-\xi_{1s}-\beta_{2s}e^{i\omega_st}-\xi_{2s})\\&-4(\beta_{1s}+\beta_{2s})(\beta_{1s}^*+\beta_{2s}^*)(-\beta_ie^{+i\omega_it}-\xi_i)(-\beta_i^*e^{-i\omega_it}+\xi_i^*)]\Bigg],\end{aligned} \tag{21}$$

where we have used the transformation $\theta_j\rightarrow\beta_j\exp(i\omega_jt)$ and assumed, in a first approximation, $\omega_{1s}\sim\omega_{2s}=\omega_s$ and

$$C_N(\{\beta_j\},t)=C_A(\{\beta_j\},t)\prod_j\exp(|\beta_j|^2).$$

To obtain information on the statistical properties of radiation emitted in modes $1s$ and $2s$ it is necessary to calculate the factorial moments. The first factorial moment gives some information on the mode intensity

$$\langle W_j\rangle=\left.\frac{\partial^2C_N(\{\beta_j\},t)}{\partial\beta_j\partial(-\beta_j^*)}\right|_{\{\beta_j\}=0}=\langle\hat{a}_j^\dagger(t)\hat{a}_j(t)\rangle, \tag{22}$$

then

$$\langle W_{1s}\rangle\simeq|\xi_{1s}|^2+pt[8|\xi_i|^2+6-2|\xi_{1s}|^2-(\xi_{1s}\xi_{2s}^*+\xi_{1s}^*\xi_{2s})], \tag{23}$$

$$\langle W_{2s}\rangle\simeq|\xi_{2s}|^2+pt[8|\xi_i|^2+6-2|\xi_{2s}|^2-(\xi_{1s}\xi_{2s}^*+\xi_{1s}^*\xi_{2s})], \tag{24}$$

where each ξ_j is proportional to the field amplitude (at the time $t=0$). We observe that if the scattered field is initially in a vacuum state, we have

$$\langle W_{1s}\rangle=\langle W_{2s}\rangle=8pt(|\xi_i|^2+\sigma)\ . \tag{25}$$

The second factorial moment is

$$\langle(\Delta W_j)^2\rangle=\left.\frac{\partial^4 C_N(\{\beta_j\},t)}{\partial\beta_j^2(-\beta_j^*)^2}\right|_{\{\beta_j\}=0}-\left[\left.\frac{\partial^2 C_N(\{\beta_j\},t)}{\partial\beta_j\partial(-\beta_j^*)}\right|_{\{\beta_j\}=0}\right]^2$$

$$=\langle\hat a_j^{\dagger 2}(t)\hat a_j^2(t)\rangle-\langle\hat a_j^\dagger(t)\hat a_j(t)\rangle^2\ , \tag{26}$$

therefore

$$\langle(\Delta W_{1s})^2\rangle\simeq 16pt\,|\xi_i|^2\{|\xi_{1s}|^2-pt[2|\xi_{1s}|^2+(\xi_{1s}\xi_{2s}^*+\xi_{1s}^*\xi_{2s})]\} \tag{27}$$

and

$$\langle(\Delta W_{2s})^2\rangle\simeq 16pt\,|\xi_i|^2\{|\xi_{2s}|^2-pt[2|\xi_{2s}|^2+(\xi_{1s}\xi_{2s}^*+\xi_{1s}^*\xi_{2s})]\}\ , \tag{28}$$

again in the "short-time" approximation and neglecting the "second-order term" $(pt)^2$. The second factorial moment given by Eq. (26) corresponds to the quantum-mechanical normalized (second-order) intensity correlation function (normally ordered and time ordered) at $\tau=0$. If we define[5]

$$g^{(2)}(\tau)=\frac{\langle\hat I(t)\hat I(t+\tau)\rangle}{\langle\hat I\rangle^2}\ , \tag{29}$$

we have, in fact,

$$g^{(2)}(0)-1=\frac{\langle(\Delta W_j)^2\rangle}{\langle W_j\rangle^2}\ . \tag{30}$$

Usually the quantity $g^{(2)}-1<0$ is the basis for the definition of antibunching, though it reflects only instantaneous photon anticoincidence. In a more correct way the quantity $[g^{(2)}(0)-1]$ gives information on the corresponding photon counting distribution. In fact, some authors[5] use the Fano factor $F_n(T)=\langle(\Delta n)^2\rangle/\langle n\rangle$ to define the super-Poissonian $[F_n(T)>1]$ or sub-Poissonian $[F_n(T)<1]$ character of the photon counting distribution, and write

$$F_n(T)=1+\frac{\langle n\rangle}{M}[g^{(2)}(0)-1]\simeq 1+\eta\frac{\langle(\Delta W_j)^2\rangle}{\langle W_j\rangle}\ , \tag{31}$$

where M represents the number of degrees of freedom (number of modes). When $M>0$ we have a correspondence between antibunching-bunching and sub-Poissonian—super-Poissonian statistics. In the following we consider only the situation at $\tau=0$, i.e., we consider only the behavior of $g^{(2)}(0)$. From Eqs. (27) and (28) and from the short-time condition $(pt\xi<1)$, it is not possible to observe sub-Poissonian statistics $(g^{(2)}-1<0)$ and each mode exhibits a super-Poissonian statistics. In the case in which only one stimulating field is present ($|\xi_{1s}|\neq 0$, $|\xi_{2s}|=0$), we have for the integrated intensity

$$\langle W_{1s}\rangle=|\xi_{1s}|^2+pt(8|\xi_i|^2+6-2|\xi_{1s}|^2)\ , \tag{32}$$

and for the second factorial moment

$$\langle(W_{1s})^2\rangle=16pt\,|\xi_i|^2(|\xi_{1s}|^2-pt\,|\xi_{1s}|^2)\ . \tag{33}$$

The same result is obtained for the mode $2s$ when $|\xi_{2s}|\neq 0$ and $|\xi_{1s}|=0$. However, as is easy to verify, the statistics does not greatly change, with respect to the case of Eqs. (27) and (28).

The other second factorial moment is given by

$$\langle\Delta W_j\Delta W_k\rangle=\left.\frac{\partial^4 C_N(\{\beta_j\},t)}{\partial\beta_j\partial(-\beta_j^*)\partial\beta_k\partial(-\beta_k^*)}\right|_{\{\beta_j\}=0}-\left.\left[\frac{\partial^2 C_N(\{\beta_j\},t)}{\partial\beta_j\partial(-\beta_j^*)}\right]\left[\frac{\partial^2 C_N(\{\beta_j\},t)}{\partial\beta_k\partial(-\beta_k^*)}\right]\right|_{\{\beta_j\}=0}$$

$$=\langle\hat a_j^\dagger(t)\hat a_k^\dagger(t)\hat a_j(t)\hat a_k(t)\rangle-\langle\hat a_j^\dagger(t)\hat a_j(t)\rangle\langle\hat a_k^\dagger(t)\hat a_k(t)\rangle\ , \tag{34}$$

which for scattered modes gives

$$\langle\Delta W_{1s}\Delta W_{2s}\rangle\simeq+8pt\,|\xi_i|^2(\xi_{1s}\xi_{2s}^*+\xi_{1s}\xi_{2s}^*) \tag{35}$$

in a short-time approximation and for small intensity values of stimulating fields $|\xi_{1s}|$ and $|\xi_{2s}|$. Because the terms ξ_j are given by $\xi_j=|\xi_j|e^{i\varphi_j}$, where φ_j is the phase factor, anticorrelation between the scattered modes is possible and it depends on the phases of the fields.

Finally we have between incident and scattered modes

$$\langle\Delta W_i\Delta W_{1,2s}\rangle\simeq-16pt\,|\xi_i|^2\mathrm{Re}[\xi_{1,2s}^*(\xi_{1s}+\xi_{2s})]\ , \tag{36}$$

which shows that positive correlation can exist between the incident and scattered modes when the phase difference between the two scattered modes is $\pi/2 \le \varphi_1 - \varphi_2 \le 3\pi/2$. In the case of spontaneous emission ($|\xi_{1,2s}|=0$) we have from Eqs. (27), (28), and (35),

$$\langle(\Delta W_{1s})^2\rangle = 0 ,\quad \langle(\Delta W_{2s})^2\rangle = 0 ,\quad \langle \Delta W_{1s}\Delta W_{2s}\rangle = 0 . \tag{37}$$

These relations correspond to having scattered modes with stabilized amplitudes. It is also interesting to inspect the behavior of the "normalized variance" $\langle(\Delta W_{1,2s})^2\rangle/\langle W_{1,2s}\rangle$ (corresponding to the $F_n - 1$ factor of Ref. 5) for each emitted mode. This quantity can assume the value zero for coherent field, -1 for "Fock state," or a value proportional to the mean number of photons in the mode, in the case of chaotic field.

In the case of small gain we have (when $|\xi_s|^2 \neq 0$)

$$\frac{\langle(\Delta W_{1s})^2\rangle}{\langle W_{1s}\rangle} = \frac{\langle(\Delta W_{2s})^2\rangle}{\langle W_{2s}\rangle} \simeq \frac{16pt\,|\xi_i|^2[\,|\xi_{1,2s}|^2 - 2pt(\,|\xi_{1,2s}|^2 + |\xi_{1s}||\xi_{2s}|\,)]}{|\xi_{1,2s}|^2 + pt(8\,|\xi_i|^2 + 6 - 2\,|\xi_{1,2s}|^2 - 2\,|\xi_{1s}||\xi_{2s}|\,)} . \tag{38}$$

With the hypothesis

$$|\xi_{1s}| = |\xi_{2s}| = |\xi_s| ,\quad \varphi_{1s} = \varphi_{2s} ,$$

$$16pt\,|\xi_i|^2 \ll |\xi_s|^2 ,\quad pt\xi < 1 ,$$

the previous equation becomes

$$F_n - 1 = \frac{\langle(\Delta W_{1s})^2\rangle}{\langle W_{1s}\rangle} = \frac{\langle(\Delta W_{2s})^2\rangle}{\langle W_{2s}\rangle} \simeq \frac{16pt\,|\xi_i|^2}{1 + 8pt\dfrac{|\xi_i|^2}{|\xi_s|^2}} . \tag{39}$$

Being that this quantity is greater than zero, we have in the used approximation and in the small-gain regime that the scattered modes tend to exhibit a super-Poissonian statistics.

(b) *High gain* ($Z > K$). Using the same considerations as in the case of small gain, we obtain a solution of Eq. (13) (see Appendix B). The factorial moments are now

$$\langle W_{1s}\rangle \simeq (\,|\xi_{1s}|^2 - 4zt\{2\,|\xi_{1s}|^2 + (\xi_{2s}^*\xi_{1s} + \xi_{2s}\xi_{1s}^*) + |\xi_i|^2[\xi_{1s}^*(\xi_{1s}+\xi_{2s}) + \xi_{1s}(\xi_{1s}^* + \xi_{2s}^*) - 2\,|\xi_i|^2]\}) \tag{40}$$

and

$$\langle W_{2s}\rangle \simeq (\,|\xi_{2s}|^2 - 4zt\{2\,|\xi_{2s}|^2 + (\xi_{2s}^*\xi_{1s} + \xi_{2s}\xi_{1s}^*) + |\xi_i|^2[\xi_{2s}^*(\xi_{1s}+\xi_{2s}) + \xi_{2s}(\xi_{1s}^* + \xi_{2s}^*)] - 2\,|\xi_i|^2\}) , \tag{41}$$

$$\langle(\Delta W_{1s})^2\rangle = 16zt\,|\xi_i|^2(\,|\xi_{1s}|^2 - 4zt\{\,|\xi_{1s}|^2 + \xi_{1s}^*\xi_{2s} + \xi_{1s}\xi_{2s}^* + |\xi_i|^2[\xi_{1s}^*(\xi_{1s}+\xi_{2s}) + \xi_{1s}(\xi_{1s}^*+\xi_{2s}^*)] - 2\,|\xi_i|^2\}) , \tag{42}$$

$$\langle(\Delta W_{2s})^2\rangle = 16zt\,|\xi_i|^2(\,|\xi_{2s}|^2 - 4zt\{\,|\xi_{2s}|^2 + \xi_{1s}^*\xi_{2s} + \xi_{1s}\xi_{2s}^* + |\xi_i|^2[\xi_{2s}^*(\xi_{1s}+\xi_{2s}) + \xi_{2s}(\xi_{1s}^*+\xi_{2s}^*)] - 2\,|\xi_i|^2\}) . \tag{43}$$

In this case also, from Eqs. (42) and (43) and from the short-time condition ($zt\,|\xi_j| < 1$), it is possible to verify that scattered modes exhibit a super-Poissonian statistics.

In the case in which only one stimulating field is present we have the following expression for the integrated intensity and for the second factorial moment ($|\xi_{1s}| \neq 0$, $|\xi_{2s}| = 0$):

$$\langle W_{1s}\rangle = [\,|\xi_{1s}|^2 - 8zt(\,|\xi_{1s}|^2 - |\xi_i|^2)] , \tag{44}$$

$$\langle(\Delta W_{1s})^2\rangle \simeq 16zt\,|\xi_i|^2(\,|\xi_{1s}|^2 - 8zt\,|\xi_{1s}|^2) . \tag{45}$$

We observe also that in the case of high gain, the characteristics of the super-Poissonian statistics do not greatly change with respect to the case in which two stimulating fields are present. Moreover, again in the short-time approximation, the correlation between modes 1s and 2s is ($|\xi_{1s}| \neq 0$ and $|\xi_{2s}| = 0$),

$$\langle \Delta W_{1s}\Delta W_{2s}\rangle \simeq zt\,16\,|\xi_i|^2(\xi_{1s}^*\xi_{2s} + \xi_{1s}\xi_{2s}^*) , \tag{46}$$

and

$$\langle \Delta W_i \Delta W_{1,2s}\rangle \simeq -8zt\,|\xi_i|^2\,\mathrm{Re}[\xi_{1,2s}^*(\xi_{1s} + \xi_{1,2s})] , \tag{47}$$

where the incident and the scattered modes exhibit anticorrelation depending on the field phase. In the case in which $|\xi_{1s}| = |\xi_{2s}| = |\xi_s| = 0$, $\varphi_{1s} = \varphi_{2s}$, and $16zt\,|\xi_i|^2 \ll |\xi_s|^2$, $zt\,|\xi_j| < 1$, we have the following expression for $F_n - 1$:

$$F_n - 1 = \frac{\langle (\Delta W_{1,2s})^2 \rangle}{\langle W_{1,2s} \rangle} = \frac{16zt \, |\xi_i|^2}{1 - 4zt \dfrac{(4|\xi_{1,2s}|^2 + 4|\xi_i|^2 |\xi_{1,2s}|^2 - 2|\xi_i|^2)}{|\xi_{1,2s}|^2}} . \tag{48}$$

Being this quantity greater than zero, the field exhibits a super-Poissonian statistics. In the case of high intensity of the stimulating field, Eq. (48) also becomes proportional to $16zt$, and each of the scattered modes exhibits a super-Poissonian statistics. When a spontaneous scattering occurs ($|\xi_{1s}| = |\xi_{2s}| = 0$), from Eqs. (45) and (47) we have

$$\langle (W_{1s})^2 \rangle = 0 , \quad \langle (\Delta W_{2s})^2 \rangle = 0 , \quad \langle (\Delta W_{1s} \Delta W_{2s}) \rangle = 0 , \tag{49}$$

corresponding to a stabilized amplitude of scattered modes.

An idea of the coupling parameter can be obtained from Ref. 2. In the laboratory frame the interaction time is $t \sim L/c$, L being the interaction length. In the ERF it is $t_e = \gamma t$ ($\gamma = E/mc^2$). If $\gamma = 10^3$ and $L \sim 1$ cm, then $t_e \sim 3\times 10^3 \times 10^{-9} \sim 3\times 10^{-6}$ sec. The coefficient $p(\sim Z)$ (in the ERF) is

$$p \sim \bar{n}_e N_0 N^2 r_0^2 \frac{\lambda_i \lambda_s}{\gamma L} I_1 , \tag{50}$$

where

$$I_1 = \int \int d\omega_l d\omega_m f(\omega_l) f(\omega_m) \sum_n |M_{pi}(x_n)|^2 \tag{51}$$

(see Appendix A) and $M_{pi}(x_n)$ is a matrix element; $\bar{n}_e$ is the total electron density in the beam (10^{10} cm^{-3}); N_0 is the number of incident photons per cm^{2}sec (10^{20} cm^{-2}sec^{-1}); $\lambda_i = \lambda_s$ (in the ERF) are the incident and scattered wavelengths, respectively; and N is the fringe number.

With the values, in the laboratory frame, $\lambda_i = 0.5\mu$ and $N \sim 10^3$ (fringe number) (for interfering angles $\theta = 5°$), in the ERF we have

$$p \sim 1 \text{ sec}^{-1} \text{ and } pt_e(<zt_e) \sim 10^{-6} . \tag{52}$$

The short-time approximation is $|pt\xi_j| < 1$ and therefore we must have $|\xi_j|^2 < 10^{12}$, corresponding to a photon flux of about 10^{22} photons per second. Because we have an incident photon flux of (in photons per second)

$$\dot{n} \sim 10^{20} \tag{53}$$

the term $|\xi_i|^2$ is

$$|\xi|_i^2 \sim 3\times 10^9 .$$

Therefore, in this case, the short-time approximation is a good approximation, and some of the predicted effects can possibly be observed.

IV. CONCLUSIONS

We have shown how a method usually employed in the statistical study of optical processes can be applied to a system involving free electrons. We remember that this is possible if the energy spread of radiation is smaller than the energy spread of the electron beam.

It is interesting to observe that the statistical properties of emitted radiation, as expressed by the value of the second factorial moment, are connected to the strength of the interaction and its time duration. The technique used allows one to discuss sub- or super-Poissonian statistics.

APPENDIX A

We have a radiation field coupled to a reservoir (free-electron system), and we wish to study the statistical properties of the radiation field. By using a standard technique, we can write the master equation (to second order in the perturbation theory) (in the interaction picture) for the reduced density operator

$$\frac{\partial \hat{\rho}_f}{\partial t} = -\sum_{i,j} \delta(\omega_i, -\omega_j)[(Q_i Q_j \hat{\rho}_f - Q_j \hat{\rho}_f Q_i) W_{ij}^{+} - (Q_i \hat{\rho}_f Q_j - \hat{\rho}_f Q_j Q_i) W_{ji}^{-}] , \tag{A1}$$

where the Q terms contain the field operators, $i = 1,2$, $j = 1,2$, and $W_{i,j}^{\pm}$ is the reservoir spectral density.

Equation (A1) is obtained in the Markov approximation: the interaction time $(t - t_0)$ is

$$t_c \ll t - t_0 \ll \gamma^{-1} ,$$

where γ^{-1} is the damping time of radiation field and t_c is the "reservoir correlation time." The terms $W_{i,j}^{\pm}$ are given by

$$W_{1,2}^{\pm} = \int_0^\infty dt \, e^{i\Delta\omega t} \sum_{l,m} \sum_n |K^{(2)}(\vec{x}_n)|^2 \langle \hat{c}_{ln}^{\dagger} \hat{c}_{mn} \hat{c}_{ln} \hat{c}_{mn}^{\dagger} \rangle , \tag{A2}$$

where $\Delta\omega = \omega_l - \omega_m$ and $\sum_{l,m}$ is the energy sum, and

$$W_{2,1}^{\pm} = \int_0^\infty dt \, e^{i\Delta\omega t} \sum_{l,m} \sum_n |K^{(2)}(\vec{x}_n)|^2 \langle \hat{c}_{ln} \hat{c}_{mn}^{\dagger} \hat{c}_{ln}^{\dagger} \hat{c}_{mn} \rangle ; \tag{A3}$$

because the energy distribution of the electron system is a continuous one

$$\sum_{l,m} \to \int d\omega_l f(\omega_l) \int d\omega_m f(\omega_m) ,$$
$$W_{1,2}^{\pm} = \int \int I \, d\omega_l d\omega_m f(\omega_l) f(\omega_m) \times \sum_n |K^{(2)}(\vec{x}_m)|^2 \langle \hat{c}_{ln}^{\dagger} \hat{c}_{mn} \hat{c}_{ln} \hat{c}_{mn}^{\dagger} \rangle , \tag{A4}$$

where

$$I=\int_0^\infty e^{i\Delta\omega\tau}d\tau\,,$$

$$W^{\pm}_{2,1}=\int\int I\,d\omega_l d\omega_m f(\omega_l)f(\omega_m)\times\sum_n |K^{(2)}(\vec{x}_n)|^2\langle \hat{c}_{ln}\hat{c}^{\dagger}_{mn}\hat{c}^{\dagger}_{ln}\hat{c}_{mn}\rangle\,. \tag{A5}$$

If we consider the fermion operators in Eqs. (A4) and (A5), we observe that

$$\langle \hat{c}^{\dagger}_{ln}\hat{c}_{mn}\hat{c}_{ln}\hat{c}^{\dagger}_{mn}\rangle=\bar{n}_e(1-\bar{n}_m)\,,$$
$$\langle \hat{c}_{ln}\hat{c}^{\dagger}_{mn}\hat{c}^{\dagger}_{ln}\hat{c}_{mn}\rangle=\bar{n}_m(1-\bar{n}_l)\,,$$

so if l states are higher (in energy) than m states we can think of the electron system as a two-level system. In this way the $W^{\pm}_{1,2}$ terms represent the emission processes and $W^{\pm}_{2,1}$ the absorption processes.

APPENDIX B

Using the same considerations as in the case of small gain, we obtain as a solution of Eq. (13), with $C_N(\{\theta_j\},t)\to C_N(\{\xi\beta_j\},t)$,

$$\begin{aligned}
C_N(\{\beta_j\},t)=\exp\Bigg[&\sum_{j=1}^{3}(\beta_j\xi_j^* e^{i\omega_j t}-\beta_j^*\xi_j e^{-i\omega_j t})\\
&-zt[(\beta_{1s}+\beta_{2s})e^{+i\omega_s t}(-\beta_{1s}^*e^{-i\omega_s t}+\xi_{1s}^*-\beta_{2s}^*e^{-i\omega_s t}+\xi_{2s}^*)+\text{c.c.}\\
&+4(-\beta_i e^{+i\omega_i t}-\xi_i)(-\beta_i^* e^{-i\omega_i t}+\xi_i^*)\\
&-8|\beta_i|^2(-\beta_{1s}^*e^{-i\omega_s t}-\beta_{2s}^*e^{-i\omega_s t}+\xi_{1s}^*+\xi_{2s}^*)(-\beta_{1s}e^{+i\omega_s t}-\beta_{2s}e^{+i\omega_s t}-\xi_{1s}-\xi_{2s})\\
&+(-\beta_{1s}e^{+i\omega_s t}-\xi_{1s}-\beta_{2s}e^{+i\omega_s t}-\xi_{2s})(-\beta_{1s}^*e^{-i\omega_s t}-\xi_{1s}-\beta_{2s}^*e^{-i\omega_s t}-\xi_{2s})\\
&+4\beta_i e^{i\omega_i t}(\beta_{1s}+\beta_{2s})(-\beta_i^*e^{-i\omega_i t}+\xi_i^*)(-\beta_{1s}^*e^{-i\omega_s t}-\beta_{2s}^*e^{-i\omega_s t}+\xi_{1s}^*+\xi_{2s}^*)+\text{c.c.}\\
&+4\beta_i e^{+i\omega_i t}(-\beta_i^*e^{-i\omega_i t}+\xi_i^*)(-\beta_{1s}e^{+i\omega_s t}-\xi_{1s}-\beta_{2s}e^{+i\omega_s t}-\xi_{2s})(-\beta_{1s}^*e^{-i\omega_s t}+\xi_{1s}^*-\beta_{2s}^*e^{-i\omega_s t}+\xi_{2s}^*)\\
&+\text{c.c.}-4(\beta_{1s}+\beta_{2s})e^{+i\omega_s t}(-\beta_{1s}^*e^{-i\omega_s t}+\xi_{1s}^*+\xi_{2s}^*-\beta_{2s}^*e^{-i\omega_s t})\\
&\times(-\beta_i^*e^{-i\omega_i t}+\xi_i^*)(-\beta_i e^{+i\omega_i t}-\xi_i)+\text{c.c.}]\Bigg]
\end{aligned} \tag{B1}$$

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