Scattering observables in arbitrary dimension $n \geq 2$

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A rigorous theory for low-energy scattering in arbitrary dimension $n \geq 2$ with general, possibly long-range interactions is presented. Several applications are discussed. For $n=2$, the question about the existence of scattering parameters is resolved and the important role of a parametrization of scattering observables in the study of adsorbed systems is illustrated. For $n=3$, an analytic expansion of the phase shift and scattering length for charged-particle scattering around their neutral counterparts is derived, taking into account explicitly the finite extent of the charge.

I. INTRODUCTION

Recently there has been a lot of interest in two-dimensional scattering systems from different theoretical, e.g., Refs. 1—11 and references therein, as well as experimental points of view, e.g., Refs. 12 and 13. From the theoretical discussion of some of the phenomena occurring in these systems, e.g., recombination kinetics and spin relaxation in spin-polarized atomic hydrogen,[14] it is clear that one could take advantage of a simple parametrization of two-dimensional low-energy scattering observables. In the case of three-dimensional scattering, such a parametrization was introduced a long time ago[15—20] in terms of the scattering length and effective-range parameter. Obviously, these concepts have played a fundamental role in the discussion of low-energy phenomena in different branches of physics.

In this paper we obtain a parametrization of low-energy scattering observables in arbitrary dimension $n \geq 2$ from first principles, allowing general interactions including long-range ones. (For simplicity we assume the absence of zero-energy resonances and zero-energy bound states.) This study is based upon a detailed discussion of the Volterra integral equations satisfied by the regular and irregular solutions of the n-dimensional Schrödinger equation (see Refs. 21 and 22 in this connection) and a generalization of the three-dimensional results of Ref. 23.

In particular, looking at the asymptotic behavior of the regular solution at zero energy, we present a definition of the n-dimensional scattering length in Sec. II. Section III contains a treatment of the regular and irregular solutions at nonzero energy, leading, e.g., to a basic formula for the n-dimensional phase shift. On the basis of this formula we derive in Secs. IV and V a (Coulomb-modified) effective-range expansion for odd and even dimensions, respectively. We show that this expansion is analytic in the energy variable k^2 around $k^2=0$, if, roughly speaking, the (short-range) potential is exponentially decreasing at infinity. If we are only interested in asymptotic expansions, these conditions can be considerably relaxed in the short-range case, allowing, e.g., Lennard-Jones potentials. As an immediate application of these results, we recover the weak-coupling limit of two-dimensional ground states.[24]

Several other applications of this theory are discussed. For $n=3$ an analytic expansion of the phase shift and scattering length for charged-particle scattering around their neutral counterparts is derived, taking into account explicitly the finite extent of the charge, without making any approximations (besides the analytic expansion). Local as well as nonlocal interactions are allowed. This is done in Sec. VI. For the most recent results in connection with three-dimensional low-energy Coulomb plus short-range scattering, we refer to Refs. 25—33. (For earlier results see the references in these publications and Ref. 34.)

Section VII contains some applications in two dimensions, which is of special interest, as indicated in the beginning. First of all, we completely resolve the controversy around the existence of scattering parameters, as already explained briefly in Ref. 35. Secondly, we give a rule of thumb to estimate the effect of general potentials in low-energy phenomena, when knowing the hard-disk results. Finally, we discuss the low-energy behavior of two- (and n-) dimensional total cross sections and the possible occurrence of a Ramsauer-Townsend effect in adsorbed systems. (See, e.g., Ref. 36 for a recent discussion of this effect in three dimensions.) For proofs of the mathematical statements in the main text, we refer to Appendixes A—C.

II. "SCATTERING LENGTH" IN $n \geq 2$ DIMENSIONS

In this section we derive the analog of the three-dimensional scattering length in arbitrary dimensions $n \geq 2$. This is accomplished by a careful investigation of the asymptotic behavior of regular solutions of the zero-energy Schrödinger equation

$$\left[-\frac{d^2}{dr^2}+\frac{(n-1)(n-3)}{4r^2}+\frac{\gamma}{r}+V(r)\right]\psi(r)=0,\quad r>0,\quad \gamma\in\mathbb{R},\quad n=2,3,4,\ldots \tag{2.1}$$

where V is a real, measurable function which obeys the integrability conditions (A1)—(A6) of Appendix A (r and k are taken to be dimensionless).

As is well known,[18—20] one can choose regular and irregular solutions associated with Eq. (2.1), denoted by

$F_n(\gamma,r)$ and $G_n(\gamma,r)$, which obey the following Volterra integral equations:

$$F_n(\gamma,r)=F_n^{(0)}(\gamma,r)-\int_0^r dr'\, g_n^{(0)}(\gamma,r,r')V(r')F_n(\gamma,r') , \tag{2.2}$$

$$G_n(\gamma,r)=G_n^{(0)}(\gamma,r)+\int_r^\infty dr'\, g_n^{(0)}(\gamma,r,r')V(r')G_n(\gamma,r') ,\quad \gamma\in\mathbb{R},\ n=2,3,4,\ldots . \tag{2.3}$$

Here $F_n^{(0)}$ and $G_n^{(0)}$ are the "unperturbed" solutions corresponding to Eq. (2.1) with $V=0$. [They are listed explicitly in Eqs. (A7) and (A8) of Appendix A.] Furthermore,

$$g_n^{(0)}(\gamma,r,r')=G_n^{(0)}(\gamma,r)F_n^{(0)}(\gamma,r')-G_n^{(0)}(\gamma,r')F_n^{(0)}(\gamma,r) . \tag{2.4}$$

(In the short-range case $\gamma=0$ we adopt the convention to omit the variable γ in all quantities such as irregular and regular solutions, scattering lengths, phase shifts, etc.) We also introduce the Jost function

$$f_n(\gamma)=W(G_n(\gamma),F_n(\gamma)) =1+\int_0^\infty dr\, G_n^{(0)}(\gamma,r)V(r)F_n(\gamma,r) , \tag{2.5}$$

where

$$W(G,F)=G\frac{\partial}{\partial r}F-\left[\frac{\partial}{\partial r}G\right]F \tag{2.6}$$

denotes the Wronskian of G and F. We note that $W(G_n^{(0)}(\gamma),F_n^{(0)}(\gamma))=1$. By hypotheses (A1)—(A6), Eqs. (2.2) and (2.3) have unique solutions. (This has been proved, e.g., in Refs. 21 and 22 for $n\geq 3$. The proof given there easily extends to $n=2$.) Finally, from now on we always assume that

$$f_n(\gamma)\neq 0 . \tag{2.7}$$

Relation (2.7) corresponds to the absence of zero-energy resonances or zero-energy bound states associated with Eq. (2.1) [the case $f_n(\gamma)=0$ is discussed in Ref. 37 and its connection with point interactions is extensively treated in Ref. 38].

Based on assumptions (A1)—(A6) and (2.7), it is shown in Appendix A that

$$\begin{aligned}F_n(\gamma,r)&=f_n(\gamma)F_n^{(0)}(\gamma,r)-G_n^{(0)}(\gamma,r)\int_0^\infty dr'\, F_n^{(0)}(\gamma,r')V(r')F_n(\gamma,r')\\&\quad+G_n^{(0)}(\gamma,r)\int_r^\infty dr'\, F_n^{(0)}(\gamma,r')V(r')F_n(\gamma,r')-F_n^{(0)}(\gamma,r)\int_r^\infty dr'\, G_n^{(0)}(\gamma,r')V(r')F_n(\gamma,r')\\&=f_n(\gamma)F_n^{(0)}(\gamma,r)-\left[\int_0^\infty dr'\, F_n^{(0)}(\gamma,r')V(r')F_n(\gamma,r')\right]G_n^{(0)}(\gamma,r)+o(1),\quad \text{as } r\to\infty ,\\&\qquad\text{with } \gamma\in\mathbb{R},\ n=2,3,4,\ldots .\end{aligned} \tag{2.8}$$

In analogy with the case $n=3$ we are thus led to the following definition of the n-dimensional "scattering length" $a_n(\gamma)$:

$$a_n(\gamma)=[f_n(\gamma)]^{-1}\int_0^\infty dr\, F_n^{(0)}(\gamma,r)V(r)F_n(\gamma,r) =\frac{\int_0^\infty dr\, F_n^{(0)}(\gamma,r)V(r)F_n(\gamma,r)}{1+\int_0^\infty dr\, G_n^{(0)}(\gamma,r)V(r)F_n(\gamma,r)} ,\quad \gamma\in\mathbb{R},\ n=2,3,4,\ldots . \tag{2.9}$$

For $n=3$, definition (2.9) obviously coincides with the usual (Coulomb-modified) scattering length.[15–20,34] Despite the fact that, after inserting units, $a_n(\gamma)$ has the dimensions of length to the $(n-2)$th power, we call it scattering "length" since it shares additional similarities with its three-dimensional analog, as shown in the following sections. [Note that under a unitary scaling transformation $\psi(\vec{x})\to\epsilon^{-n/2}\psi(\vec{x}/\epsilon)$, $\psi\in L^2(\mathbb{R}^n)$, $\epsilon>0$, the short-range scattering length scales like $a\to(\epsilon^{n-2}a)$ for $n>3$ and $a^{-1}\to(a^{-1}-\ln\epsilon)$ for $n=2$, in agreement with its dimensional behavior.] To our knowledge Eq. (2.9) is new for $n\neq 3$. As explained in Ref. 35 it completely resolves earlier questions about the existence of a scattering length in two dimensions (cf. also the discussion in Sec. VII).

III. REGULAR AND IRREGULAR SOLUTIONS FOR $k\neq 0$

We now establish basic properties of regular and irregular solutions of the s-wave Schrödinger equation

$$\left[-\frac{d^2}{dr^2}+\frac{(n-1)(n-3)}{4r^2}+\frac{\gamma}{r}+V(r)-k^2\right]\psi(k,r)=0,\quad k,r>0,\ \gamma\in\mathbb{R},\ n=2,3,4,\ldots \tag{3.1}$$

which are needed in the following sections. We assume that V in Eq. (3.1) decreases exponentially at infinity, i.e.,

$$\int_0^{R'} dr\, r\,|\ln r|\,|V(r)|+\int_{R'}^\infty dr\, e^{br}|V(r)|<\infty \quad \text{for some } 0<R'<1 \text{ and some } b>0 \text{ if } n=2,\ \gamma\in\mathbb{R} \tag{3.2}$$

and

$$\int_0^R dr\, r\,|V(r)|+\int_R^\infty dr\, e^{br}|V(r)|<\infty \quad \text{for some } R,b>0 \text{ if } n=3,4,5,\ldots \text{ and } \gamma\in\mathbb{R} . \tag{3.3}$$

This is necessary in order to get an *analytic* effective-range expansion for the phase shift, as we will show in Secs. IV and V. (If one is only interested in asymptotic expansions, these conditions can be relaxed in the way explained there.)

Conditions (3.2) and (3.3) guarantee the existence of regular and irregular functions F_n and G_n, associated with Eq. (3.1), as the unique solutions of

$$F_n(\gamma,k,r)=F_n^{(0)}(\gamma,k,r)-\int_0^r dr'\, g_n^{(0)}(\gamma,k,r,r')V(r')F_n(\gamma,k,r'), \quad k\geq 0 \tag{3.4}$$

$$G_n(\gamma,k,r)=G_n^{(0)}(\gamma,k,r)+\int_r^\infty dr'\, g_n^{(0)}(\gamma,k,r,r')V(r')G_n(\gamma,k,r'), \quad k>0\ . \tag{3.5}$$

Here the unperturbed solutions $F_n^{(0)}$ and $G_n^{(0)}$ are given by[21,22]

$$F_n^{(0)}(\gamma,k,r)=r^{(n-1)/2}e^{-ikr}\,{}_1F_1(\tfrac{1}{2}(n-1)-\tfrac{1}{2}i\gamma/k;n-1;2ikr), \quad k\geq 0 \tag{3.6}$$

$$G_n^{(0)}(\gamma,k,r)=[\Gamma(n-1)]^{-1}\Gamma(\tfrac{1}{2}(n-1)-\tfrac{1}{2}i\gamma/k)(2ik)^{n-2}r^{(n-1)/2}e^{-ikr}U(\tfrac{1}{2}(n-1)-\tfrac{1}{2}i\gamma/k;n-1;2ikr), \quad k>0 \tag{3.7}$$

where ${}_1\mathrm{F}_1(\alpha;\beta;z)$ and $U(\alpha;\beta;z)$ denote the confluent hypergeometric functions.[39] Furthermore,

$$g_n^{(0)}(\gamma,k,r,r')=G_n^{(0)}(\gamma,k,r)F_n^{(0)}(\gamma,k,r')-G_n^{(0)}(\gamma,k,r')F_n^{(0)}(\gamma,k,r), \quad k\geq 0 \tag{3.8}$$

denotes the corresponding Green's function. Finally the Jost function $\mathscr{F}_n$ can be expressed by

$$\mathscr{F}_n(\gamma,k)=W(G_n(\gamma,k),F_n(\gamma,k))=1+\int_0^\infty dr\, G_n^{(0)}(\gamma,k,r)V(r)F_n(\gamma,k,r), \quad k>0\ . \tag{3.9}$$

Again $W(G_n^{(0)}(\gamma,k),F_n^{(0)}(\gamma,k))=1$. Also, all formulas in this section are valid for $\gamma\in\mathbf{R}$ and $n=2,3,4\ldots$.

Equation (3.6) implies that $F_n^{(0)}$ is real for $\gamma\in\mathbf{R}$ and $k\geq 0$ and entire with respect to k^2, giving

$$F_n^{(0)}(\gamma,k,r)=F_n^{(0)}(\gamma,-k,r)=F_n^{(0)*}(\gamma,k,r), \quad k\geq 0\ . \tag{3.10}$$

Analytic continuation in Eq. (3.7) leads to[39]

$$\begin{aligned}G_n^{(0)}(\gamma,e^{-i\pi}k,r)&=G_n^{(0)*}(\gamma,k,r)\\ &=G_n^{(0)}(\gamma,k,r)+i(2k)^{n-2}[\Gamma(n-1)]^{-2}\,|\,\Gamma(\tfrac{1}{2}(n-1)+\tfrac{1}{2}i\gamma/k)\,|^2 e^{-\pi\gamma/2k}F_n^{(0)}(\gamma,k,r), \quad k>0\ .\end{aligned} \tag{3.11}$$

As a consequence, also, $g_n^{(0)}$ is real for $\gamma\in\mathbf{R}$ and $k\geq 0$ and entire in k^2, giving

$$g_n^{(0)}(\gamma,k,r,r')=g_n^{(0)}(\gamma,-k,r,r')=g_n^{(0)*}(\gamma,k,r,r'), \quad k\geq 0\ . \tag{3.12}$$

This implies the same property for F_n, namely,

$$F_n(\gamma,k,r)=F_n(\gamma,-k,r)=F_n^*(\gamma,k,r), \quad k\geq 0 \tag{3.13}$$

i.e., F_n is real for $\gamma\in\mathbf{R}$, $k\geq 0$ and entire in k^2. [Of course Eqs. (3.10) and (3.13) could be inferred directly from Poincaré's theorem[40] since

$$\lim_{r\to 0_+} r^{(1-n)/2}F_n^{(0)}(\gamma,k,r)=\lim_{r\to 0_+} r^{(1-n)/2}F_n(\gamma,k,r)=1\ .$$

See also Refs. 41 and 42.] Defining the (Coulomb-modified) phase shift $\delta_n(\gamma,k)$ by

$$e^{2i\delta_n(\gamma,k)}=\mathscr{F}_n(\gamma,k)/\mathscr{F}_n(\gamma,e^{-i\pi}k), \quad k>0 \tag{3.14}$$

and choosing $\lim_{k\to\infty}\delta_n(\gamma,k)=0$ to get uniqueness (cf. Refs. 21 and 22), we finally obtain from Eqs. (3.11)–(3.13)

$$\begin{aligned}\cot\delta_n(\gamma,k)&=i\frac{\mathscr{F}_n(\gamma,k)+\mathscr{F}_n(\gamma,e^{-i\pi}k)}{\mathscr{F}_n(\gamma,k)-\mathscr{F}_n(\gamma,e^{-i\pi}k)}\\ &=-\frac{\int_0^\infty dr\,\mathrm{Re}[G_n^{(0)}(\gamma,k,r)]V(r)F_n(\gamma,k,r)}{\int_0^\infty dr\,\mathrm{Im}[G_n^{(0)}(\gamma,k,r)]V(r)F_n(\gamma,k,r)}, \quad k>0\ .\end{aligned} \tag{3.15}$$

Here Re $G_n^{(0)}$ and Im $G_n^{(0)}$ read

$$\mathrm{Re}[G_n^{(0)}(\gamma,k,r)]=G_n^{(0)}(\gamma,k,r)+i2^{-1}(2k)^{n-2}[\Gamma(n-1)]^{-2}\,|\,\Gamma(\tfrac{1}{2}(n-1)+\tfrac{1}{2}i\gamma/k)\,|^2e^{-\pi\gamma/2k}F_n^{(0)}(\gamma,k,r), \quad k>0 \tag{3.16}$$

$$\mathrm{Im}[G_n^{(0)}(\gamma,k,r)]=-2^{-1}(2k)^{n-2}[\Gamma(n-1)]^{-2}\,|\,\Gamma(\tfrac{1}{2}(n-1)+\tfrac{1}{2}i\gamma/k)\,|^2e^{-\pi\gamma/2k}F_n^{(0)}(\gamma,k,r), \quad k>0\ . \tag{3.17}$$

IV. SCATTERING PHASE SHIFTS IN ODD DIMENSIONS

Restricting ourselves first to odd dimensions $n=3,5,7,\ldots$ we derive the effective-range expansion associated with the phase shift $\delta_n(\gamma,k)$. Following Lambert,[23] the explicit series representation of the regular and irregular confluent hypergeometric functions ${}_1F_1(\alpha;\beta;z)$ and $U(\alpha;\beta;z)$ (Ref. 39) leads to

$$\mathrm{Re}[G_n^{(0)}(\gamma,k,r)]=\widetilde{G}_n^{(0)}(\gamma,k,r)+[-ike^{-\pi\gamma/2k}+h_n(\gamma,k)](2k)^{n-3}[\Gamma(n-1)]^{-2}\left|\Gamma\left[\frac{n-1}{2}+\frac{i\gamma}{2k}\right]\right|^2 \times F_n^{(0)}(\gamma,k,r),\quad \gamma\in\mathbf{R},\quad k>0,\quad n=3,5,7,\ldots \tag{4.1}$$

where

$$h_n(\gamma,k)=\gamma\left|\Gamma\left[1+\frac{i\gamma}{2k}\right]\right|^2\left[\frac{ik}{\gamma}+\ln\left[\frac{2k}{i\,|\gamma|}\right]+\Psi\left[1+\frac{i\gamma}{2k}\right]\right],\quad \gamma\in\mathbf{R},\quad k>0,\quad n=3,5,7,\ldots \tag{4.2}$$

and $\widetilde{G}_n^{(0)}$ is defined by

$$\begin{aligned}\widetilde{G}_n^{(0)}(\gamma,k,r)={}&[\Gamma(n-1)]^{-2}(2k)^{n-3}\left|\Gamma\left[\frac{n-1}{2}+\frac{i\gamma}{2}\right]\right|^2\left|\Gamma\left[1+\frac{i\gamma}{2k}\right]\right|^{-2}\gamma\ln(|\gamma|\,r)F_n^{(0)}(\gamma,k,r)\\
&-\gamma\,\mathrm{Re}\left\{[\Gamma(n-1)]^{-2}(2k)^{n-3}\left|\Gamma\left[\frac{n-1}{2}+\frac{i\gamma}{2k}\right]\right|^2\left|\Gamma\left[1+\frac{i\gamma}{2k}\right]\right|^{-2}r^{(n-1)/2}e^{-ikr}\right.\\
&\left.\times\sum_{p=0}^{\infty}\frac{\Gamma\left[\frac{n-1}{2}+p-\frac{i\gamma}{2k}\right]\Gamma(n-1)(2ikr)^p}{\Gamma\left[\frac{n-1}{2}-\frac{i\gamma}{2k}\right]\Gamma(n-1+p)\Gamma(p+1)}[\Psi(p+1)+\Psi(p+n-1)]\right\}\\
&+\gamma\,\mathrm{Re}\left\{[\Gamma(n-1)]^{-2}(2k)^{n-3}\left|\Gamma\left[\frac{n-1}{2}+\frac{i\gamma}{2k}\right]\right|^2\left|\Gamma\left[1+\frac{i\gamma}{2k}\right]\right|^{-2}r^{(n-1)/2}e^{-ikr}\right.\\
&\left.\times\sum_{p=0}^{\infty}\frac{\Gamma\left[\frac{n-1}{2}+p-\frac{i\gamma}{2k}\right]\Gamma(n-1)(2ikr)^p}{\Gamma\left[\frac{n-1}{2}-\frac{i\gamma}{2k}\right]\Gamma(n-1+p)\Gamma(p+1)}\sum_{s=1}^{\frac{n-1}{2}+p-1}\left[s-\frac{i\gamma}{2k}\right]^{-1}\right\}\\
&+\mathrm{Re}\left\{(n-2)^{-1}(2ik)^{n-2}r^{(n-1)/2}e^{-ikr}\sum_{q=0}^{n-3}\frac{\Gamma\left[\frac{3-n}{2}+q-\frac{i\gamma}{2k}\right]\Gamma(3-n)(2ikr)^{q+2-n}}{\Gamma\left[\frac{3-n}{2}-\frac{i\gamma}{2k}\right]\Gamma(q+3-n)(q+1)}\right\},\\
&\gamma\in\mathbb{R},\quad k\geq 0,\quad n=3,5,7,\ldots\end{aligned} \tag{4.3}$$

with $\Gamma(z)$ the gamma and $\Psi(z)$ the digamma functions, respectively.[39] [Of course, $\sum_{s=1}^{[(n-1)/2]+p-1}[s-(i\gamma/2k)]^{-1}$ is interpreted to be zero if $n=3$ and $p=0$.] By inspection $\widetilde{G}_n^{(0)}(\gamma,k,r)$ is real for $\gamma\in\mathbf{R}$, $k\geq 0$ and entire with respect to k^2.[23] Next we observe from Eqs. (3.4), (3.6), and (4.2) that

$$F_n^{(0)}(\gamma,k,r)=F_n^{(0)}(\gamma,r)+O(k^2),\quad \text{as } k^2\to 0,\quad \gamma\in\mathbf{R},\quad n=2,3,4,\ldots \tag{4.4}$$

$$F_n(\gamma,k,r)=F_n(\gamma,r)+O(k^2),\quad \text{as } k^2\to 0,\quad \gamma\in\mathbb{R},\quad n=2,3,4,\ldots \tag{4.5}$$

$$\widetilde{G}_n^{(0)}(\gamma,k,r)=G_n^{(0)}(\gamma,r)+O(k^2),\quad \text{as } k^2\to 0,\quad \gamma\in\mathbb{R},\quad n=3,5,7,\ldots \tag{4.6}$$

where $O(k^2)$ is uniform with respect to r if r varies in compact intervals (avoiding the origin in the case of $\widetilde{G}_n^{(0)}$). Inserting Eqs. (3.17) and (4.1) into (3.15), the bound (B6) mentioned in Appendix B, together with the Lebesgue-dominated convergence theorem, finally yield the main result of this section:

$$[\Gamma(n-1)]^{-2}(2k)^{n-3}\left|\Gamma\left(\frac{n-1}{2}+\frac{i\gamma}{2k}\right)\right|^2 e^{-\pi\gamma/2k}[k\cot\delta_n(\gamma,k)-ik+e^{\pi\gamma/2k}h_n(\gamma,k)]$$
$$=-\frac{1}{a_n(\gamma)}+O(k^2),\quad \gamma\in\mathbf{R},\quad k>0,\quad n=3,5,7,\ldots\,. \tag{4.7}$$

Here $a_n(\gamma)$ is the scattering length defined in Eq. (2.9). The right-hand side of Eq. (4.7) represents the effective-range expansion in odd dimensions. It is analytic in k^2 around $k^2=0$ for potentials satisfying conditions (3.3).

In the short-range case $\gamma=0$, the result (4.7) simplifies to

$$\left[\Gamma\left(\frac{n}{2}\right)\right]^{-2}\left[\frac{k}{2}\right]^{n-2}\frac{\pi}{2}\cot\delta_n(k)=-\frac{1}{a_n}+O(k^2),\quad \gamma=0,\quad k>0,\quad n=3,5,7,\ldots\,. \tag{4.8}$$

In particular Re $G_n^{(0)}$ simply becomes

$$\mathrm{Re}[G_n^{(0)}(k,r)]=-2^{-1}\pi\left[\Gamma\left(\frac{n}{2}\right)\right]^{-1}\left[\frac{k}{2}\right]^{(n-2)/2}r^{1/2}Y_{(n-2)/2}(kr)$$
$$=(n-2)^{-1}r^{(3-n)/2}+O(k^2),\quad \gamma=0,\quad k\geq 0,\quad n=3,5,7,\ldots \tag{4.9}$$

with $Y_\alpha(z)$ the Bessel function of second kind and order α.[39]

Of course the expansions (4.7) and (4.8) are classic results in the special case $n=3$.[15–20,34] However, we emphasize that even for this case the analyticity of the right-hand side of expansion (4.7) was only known in the literature for a rather restricted class of potentials,[43,44] whereas our proof works for the class of potentials described in (3.3) (cf. Appendix B).

In the short-range case ($\gamma=0$), the falloff conditions on V at infinity [see Eq. (3.3)] can be relaxed by merely demanding that $(1+r)^N V\in L^1((0,\infty))$ for suitable $N\in\mathbf{N}$. Then the above analytic expansions turn into asymptotic expansions, the order of which depends on N. As an example, we mention that potentials of the Lennard-Jones type can be treated in this way. (For $n=3$, cf. Refs. 45—48 and the literature cited therein.)

Next we remark that Eqs. (4.7) and (4.8) immediately generalize to higher partial waves $l=1,2,\ldots$ if the substitution

$$n\to 2+[(n-2)^2+4l(l+n-2)]^{1/2},\quad l=1,2,\ldots \tag{4.10}$$

is made throughout in Secs. II—IV. Finally we note that formulas similar to (2.9) for the higher coefficients in the right-hand sides of Eqs. (4.7) and (4.8) (effective-range parameter, etc.) can be derived in a straightforward manner. Also other scattering observables such as the S-matrix [see Eq. (3.14)] scattering amplitude, cross sections, etc., can be expanded in a similar way. The total cross sections, e.g., will be considered in Sec. VII.

V. SCATTERING PHASE SHIFTS IN EVEN DIMENSIONS

In this section we establish the effective-range expansion for even dimensions $n=2,4,6,\ldots$. Again we start from expression (3.16) for Re $G_n^{(0)}$ and obtain

$$\mathrm{Re}[G_n^{(0)}(\gamma,k,r)]=\widetilde{G}_n^{(0)}(\gamma,k,r)+[-ike^{-\pi\gamma/2k}+h_n(\gamma,k)](2k)^{n-3}[\Gamma(n-1)]^{-2}\left|\Gamma\left(\frac{n-1}{2}+\frac{i\gamma}{2k}\right)\right|^2 F_n^{(0)}(\gamma,k,r)\,,$$
$$\gamma\in\mathbf{R},\quad k>0,\quad n=2,4,6,\ldots \tag{5.1}$$

where

$$h_n(\gamma,k)=2k\left|\Gamma\left(\frac{1}{-2}+\frac{i\gamma}{2k}\right)\right|^{-2}\left[\ln\left(\frac{2i}{k}\right)+\Psi(1)+\Psi(\tfrac{1}{2})-\Psi\left(\frac{1}{2}+\frac{i\gamma}{2k}\right)\right],\quad \gamma\in\mathbf{R},\quad k>0,\quad n=2,4,6,\ldots \tag{5.2}$$

and $\widetilde{G}_n^{(0)}$ is given by

$$\tilde{G}_n^{(0)}(\gamma,k,r)=-[\Gamma(n-1)]^{-2}(2k)^{n-2}\left|\Gamma\left[\frac{n-1}{2}+\frac{i\gamma}{2k}\right]\right|^2\left|\Gamma\left[\frac{1}{2}+\frac{i\gamma}{2k}\right]\right|^{-2}[\ln(r)-2\mathscr{C}]F_n^{(0)}(\gamma,k,r,)$$
$$+\mathrm{Re}\left[[\Gamma(n-1)]^{-2}(2k)^{n-2}\left|\Gamma\left[\frac{n-1}{2}+\frac{i\gamma}{2k}\right]\right|^2\left|\Gamma\left[\frac{1}{2}+\frac{i\gamma}{2k}\right]\right|^{-2}r^{(n-1)/2}e^{-ikr}\right.$$
$$\left.\times\sum_{p=0}^{\infty}\frac{\Gamma\left[\frac{n-1}{2}+p-\frac{i\gamma}{2k}\right]\Gamma(n-1)(2ikr)^p}{\Gamma\left[\frac{n-1}{2}-\frac{i\gamma}{2k}\right]\Gamma(n-1+p)\Gamma(p+1)}[\Psi(p+1)+\Psi(p+n-1)]\right]$$
$$-\mathrm{Re}\left[[\Gamma(n-1)]^{-2}(2k)^{n-2}\left|\Gamma\left[\frac{n-1}{2}+\frac{i\gamma}{2k}\right]\right|^2\left|\Gamma\left[\frac{1}{2}+\frac{i\gamma}{2k}\right]\right|^{-2}r^{(n-1)/2}e^{-ikr}\right.$$
$$\left.\times\sum_{p=0}^{\infty}\frac{\Gamma\left[\frac{n-1}{2}+p-\frac{i\gamma}{2k}\right]\Gamma(n-1)(2ikr)^p}{\Gamma\left[\frac{n-1}{2}-\frac{i\gamma}{2k}\right]\Gamma(n-1+p)\Gamma(p+1)}\sum_{s=0}^{\frac{n}{2}+p-2}\left[s+\frac{1}{2}-\frac{i\gamma}{2k}\right]^{-1}\right]$$
$$+\mathrm{Re}\left[(n-2)^{-1}(2ik)^{n-2}r^{(n-1)/2}e^{-ikr}\sum_{q=0}^{n-3}\frac{\Gamma\left[\frac{3-n}{2}+q-\frac{i\gamma}{2k}\right]\Gamma(3-n)(2ik)^{q+2-n}}{\Gamma\left[\frac{3-n}{2}-\frac{i\gamma}{2k}\right]\Gamma(q+3-n)\Gamma(q+1)}\right],$$
$$\gamma\in\mathbb{R},\quad k\geq 0,\quad n=2,4,6,\ldots \qquad (5.3)$$

with $\mathscr{C}$ Euler's constant.[39] (If $n=2$ the last term in Eq. (5.3) should be omitted. If $n=2$ and $p=0$, $\sum_{s=0}^{(n/2)+p-2}[s+\frac{1}{2}-(i\gamma/2k)]^{-1}$ is meant to be zero.) As in Sec. IV, $\tilde{G}_n^{(0)}(\gamma,k,r)$ turns out to be real for $\gamma\in\mathbb{R}$ and $k\geq 0$ and entire with respect to k^2.

Given the result (5.1) one can follow the corresponding derivation in Sec. IV step by step. In fact, as we have already noted there, Eqs. (4.4) and (4.5) are valid for all $n=2,3,4,\ldots$. Furthermore, from definition (5.3) one obtains

$$\tilde{G}_n^{(0)}(\gamma,k,r)=G_n^{(0)}(\gamma,r)+O(k^2),\quad \text{as } k^2\to 0,\quad \gamma\in\mathbb{R},\quad n=2,4,6,\ldots \qquad (5.4)$$

Therefore, along the arguments employed in Sec. IV and Appendix B, Eqs. (3.17), (5.1), and (3.15) finally yield the effective-range expansion in even dimensions

$$[\Gamma(n-1)]^{-2}(2k)^{n-3}\left|\Gamma\left[\frac{n-1}{2}+\frac{i\gamma}{2k}\right]\right|^2 e^{-\pi\gamma/2k}[k\cot\delta_n(\gamma,k)-ik+e^{\pi\gamma/2k}h_n(\gamma,k)]$$
$$=-\frac{1}{a_n(\gamma)}+O(k^2),\quad \gamma\in\mathbb{R},\quad k>0,\quad n=2,4,6,\ldots \qquad (5.5)$$

Again $a_n(\gamma)$ is the scattering length introduced in Eq. (2.9) and the right-hand side of (5.5) is analytic in k^2 around $k^2=0$ if V satisfies conditions (3.2) ($n=2$) and (3.3) ($n=4,6,\ldots$), respectively.

If only the short-range interaction V is present, i.e., if $\gamma=0$, the expansion (5.5) becomes

$$\left[\Gamma\left[\frac{n}{2}\right]\right]^{-2}\left[\frac{k}{2}\right]^{n-2}\left[\frac{\pi}{2}\cot\delta_n(k)-\ln\left[\frac{k}{2}\right]-\mathscr{C}\right]=-\frac{1}{a_n}+O(k^2),\quad \gamma=0,\quad k>0,\quad n=2,4,6,\ldots \qquad (5.6)$$

In this special case $\mathrm{Re}\,G_n^{(0)}$ simplifies to

$$\mathrm{Re}[G_n^{(0)}(k,r)]=-2^{-1}\pi\left[\Gamma\left[\frac{n}{2}\right]\right]^{-1}\left[\frac{k}{2}\right]^{(n-2)/2}r^{1/2}Y_{(n-2)/2}(kr),\quad \gamma=0,\quad k>0,\quad n=2,4,6,\ldots \qquad (5.7)$$

Equations (5.5) and (5.6) are new. The particular importance of expansion (5.6) in two dimensions is illustrated in Sec. VII. Obviously the remarks at the end of Sec. IV apply to Eqs. (5.5) and (5.6). In particular the substitution (4.10) immediately generalizes the above results to higher partial waves $l=1,2,\ldots$ and weakening the falloff conditions on V at infinity to $(1+r)^N V \in L^1((0,\infty))$ for suitable N turns (5.5) and (5.6) into asymptotic expansions.

As a simple application of Eq. (5.6) we finally discuss the weak-coupling limit of ground-state energies in two dimensions. Starting from expansion (5.6) for $n=2$, replacing V by gV, $g \geq 0$, and setting $k=i(-E)^{1/2}$, $E<0$ and $\cot\delta_2(k)=i$, we get

$$\ln[(-E)^{1/2}]=[a_2(g)]^{-1}-\mathscr{C}+\ln 2+O(E) . \quad (5.8)$$

Here

$$a_2(g)=\frac{g\int_0^\infty dr\, r^{1/2}V(r)F_2(g,r)}{1-g\int_0^\infty dr\, r^{1/2}\ln(r)V(r)F_2(g,r)} \quad (5.9)$$

[cf. definition (2.9) where the dependence on the coupling constant g has been introduced explicitly]. From the integral equation (2.2) (with $\gamma=0$ and $V\to gV$) one proves that $F_2(g,r)$ is entire with respect to g, in particular

$$F_2(g,r)=r^{1/2}+O(g) \quad \text{as } g\to 0_+ . \quad (5.10)$$

This implies by a straightforward application of the Lebesgue-dominated convergence theorem

$$a_2(g)=g\int_0^\infty dr\, rV(r)+O(g^2) \quad \text{as } g\to 0_+ . \quad (5.11)$$

Consequently, if the Born approximation $a_{2,B}(g)$ is negative, viz.,

$$a_{2,B}(g)=g\int_0^\infty dr\, rV(r)<0 , \quad (5.12)$$

Eq. (5.8) has a unique solution $E_0(g)$ (the ground-state energy) as $g\to 0_+$ given by

$$E_0(g)=-\exp\left[\left[2^{-1}g\int_0^\infty dr\, rV(r)\right]^{-1}\times[1+O(g)]\right] \quad \text{as } g\to 0_+ . \quad (5.13)$$

(If $a_{2,B}(g)=0$, i.e., if $a_2(g)=O(g^2)$ as $g\to 0$, a more detailed analysis shows that $E_0(g)=-\exp[O(g^{-2})]$ as $g\to 0$.) If $a_{2,B}(g)>0$, Eq. (5.8) has no solution if g is small enough. A generalization of the above result, including nonspherically symmetric interactions V, was first proved by Simon[24] (cf. also Ref. 8) using modified Fredholm determinants. In the following sections we discuss further applications of the results derived so far to various problems in two and three dimensions.

VI. COULOMB-CORRECTED SCATTERING LENGTHS AND PHASE SHIFTS

As a first application of our results in Secs. II and IV we reconsider the old problem of how to take into account the effect of (extended) charges on the value of neutral scattering lengths and phase shifts in three dimensions. Our method is based on an analytic expansion of $a(\gamma)^{-1}$ and $\cot\delta(\gamma,k)$ in γ around their neutral counterparts a^{-1} and $\cot\delta(k)$, respectively. In contrast to earlier treatments of this problem (cf. e.g., Refs. 15—17, 26—31, 34, 44, and 49 and the literature cited therein), no approximations, besides the analytic expansion in γ, are made. In this section we systematically avoid mathematical technicalities and merely concentrate on a derivation of the desired results. Rigorous proofs based on the methods employed in Appendixes A—C can be given. Furthermore, since in this section we exclusively treat the three-dimensional case, we suppress the index 3 in all that follows.

In order to treat realistic interactions we explicitly take into account the finite size of the charge and replace γ/r in Eq. (2.1) by $\gamma\phi_c(r)/r$ where the form factor $\phi_c(r)$ fulfills

$$\int_0^R dr\, |\phi_c(r)| + \int_R^\infty dr\, e^{br}|\phi_c(r)-1| < \infty \quad \text{for some } R,b>0 \quad (6.1)$$

and V is assumed to obey condition (3.3). To indicate that ϕ_c has been introduced in Eq. (2.1) we add the index c in all quantities such as irregular and regular solutions, Jost functions, phase shifts, etc.

We start by discussing scattering lengths. If

$$f_c^{(0)}(\gamma)=W(G_c^{(0)}(\gamma),F_c^{(0)}(\gamma)) \quad (6.2)$$

denotes the Wronskian of $G_c^{(0)}$ and $F_c^{(0)}$, we obtain in analogy with Eq. (2.8) that

$$F_c(\gamma,r)=\left[1+[f_c^{(0)}(\gamma)]^{-1}\int_0^\infty dr'\, G_c^{(0)}(\gamma,r')V(r')F_c(\gamma,r')\right]F_c^{(0)}(\gamma,r) - \left[\int_0^\infty dr'\, F_c^{(0)}(\gamma,r')V(r')F_c(\gamma,r')\right][f_c^{(0)}(\gamma)]^{-1}G_c^{(0)}(\gamma,r)+o(1) \quad \text{as } r\to\infty \quad (6.3)$$

where $F_c^{(0)}$, $G_c^{(0)}$, and F_c are the unique solutions of the integral Eqs. (C3)—(C5) of Appendix C. As a consequence, the analog of Eq. (2.9) in the presence of extended charges reads

$$a_c(\gamma)=\frac{\int_0^\infty dr\, F_c^{(0)}(\gamma,r)V(r)F_c(\gamma,r)}{1+[f_c^{(0)}(\gamma)]^{-1}\int_0^\infty dr\, G_c^{(0)}(\gamma,r)V(r)F_c(\gamma,r)} . \quad (6.4)$$

[Note that $f_c^{(0)}(\gamma)\neq 0$ if $|\gamma|$ is small enough, as can be seen from Eq. (6.7).]

Assuming $\gamma>0$ for simplicity, we get from Eqs. (C1)—(C4) the expansions [analytic in γ, respectively, analytic in $(\gamma,\gamma\ln\gamma)$ around zero]

$$F_c^{(0)}(\gamma,r)=r+\frac{\gamma}{2}r^2-\gamma\int_0^r dr'(r'-r)[\phi_c(r')-1]+O(\gamma^2)\equiv r+\gamma F_{c,1}^{(0)}(r)+O(\gamma^2)\,, \tag{6.5}$$

$$\begin{aligned}G_c^{(0)}(\gamma,r)&=1+\gamma[\ln(\gamma r)+2\mathscr{C}-1]r+\gamma\int_r^\infty dr'(r'-r)(r')^{-1}[\phi_c(r')-1]+O(\gamma^2\ln(\gamma))\\&\equiv 1+\gamma[\ln(\gamma)+2\mathscr{C}-1]r+\gamma\hat{G}_{c,1}^{(0)}(r)+O(\gamma^2\ln(\gamma))\,,\end{aligned} \tag{6.6}$$

$$f_c^{(0)}(\gamma)=1+\gamma\int_0^\infty dr\,[\phi_c(r)-1]+O(\gamma^2\ln(\gamma))\,. \tag{6.7}$$

Finally, iterating the appropriate integral equation for $F_c(\gamma,r)$, one obtains the following expansion analytic in γ at $\gamma=0$:

$$\begin{aligned}F_c(\gamma,r)&=F(r)-\gamma f^{-1}\int_0^r dr'[G(r)F(r')-G(r')F(r)](r')^{-1}\phi_c(r')F_c(\gamma,r')\\&=F(r)-\gamma\left[f^{-1}\int_0^r dr'\,F(r')(r')^{-1}\phi_c(r')F(r')\right]G(r)+\gamma\left[f^{-1}\int_0^r dr'\,G(r')(r')^{-1}\phi_c(r')F(r')\right]F(r)+O(\gamma^2)\\&\equiv F(r)+\gamma F_{c,1}(r)+O(\gamma^2)\,.\end{aligned} \tag{6.8}$$

Insertion of the results (6.5)—(6.8) into (6.4) shows, after some elementary manipulations, that

$$\begin{aligned}[a_c(\gamma)]^{-1}&=a^{-1}+\gamma[\ln(\gamma)+2\mathscr{C}-1]+\gamma I^{-1}\int_0^\infty dr[G_{c,1}^{(0)}(r)-a^{-1}F_{c,1}^{(0)}(r)]V(r)F(r)\\&\quad+\gamma I^{-1}\int_0^\infty dr(1-a^{-1}r)V(r)F_{c,1}(r)+O(\gamma^2\ln(\gamma)),\quad \gamma\geq 0\end{aligned} \tag{6.9}$$

where $F_{c,1}^{(0)}$ and $F_{c,1}$ are defined in Eqs. (6.5) and (6.8),

$$G_{c,1}^{(0)}(r)=\hat{G}_{c,1}^{(0)}(r)-\int_0^\infty dr'[\phi_c(r')-1] \tag{6.10}$$

with $\hat{G}_{c,1}^{(0)}$ defined in Eq. (6.6), and

$$I=\int_0^\infty dr\,rV(r)F(r)\,. \tag{6.11}$$

In Eq. (6.9), a denotes the neutral scattering length (i.e, $\gamma=0$) and of course it is assumed that $a\neq 0$ implying $I\neq 0$. If $a=0$ one simply expands $a_c(\gamma)$ instead of $a_c(\gamma)^{-1}$ around its neutral value $a_c(0)=0$. As can be read off from the expansions (6.5)—(6.8), $[a_c(\gamma)]^{-1}$ [respectively, $a_c(\gamma)$] is analytic in the variables γ and $\gamma\ln(\gamma)$. Therefore higher-order corrections can systematically be obtained by iterating the corresponding integral equations. We finally again emphasize that in contrast to earlier treatments on this subject, no approximations in deriving the first-order terms in γ and $\gamma\ln(\gamma)$ in the right-hand side of Eq. (6.9) have been made.

Next we turn to the scattering phase shifts. We start from the expression

$$\begin{aligned}e^{2i\delta_c(\gamma,k)}&=\frac{\mathscr{F}_c(\gamma,k)}{\mathscr{F}_c(\gamma,e^{-i\pi}k)}\\&=\frac{1+[\mathscr{F}_c^{(0)}(\gamma,k)]^{-1}\int_0^\infty dr\,G_c^{(0)}(\gamma,k,r)V(r)F_c(\gamma,k,r)}{1+[\mathscr{F}_c^{(0)}(\gamma,e^{-i\pi}k)]^{-1}\int_0^\infty dr\,G_c^{(0)}(\gamma,e^{-i\pi}k,r)V(r)F_c(\gamma,k,r)},\quad \gamma\in\mathbb{R},\quad k>0\,.\end{aligned} \tag{6.12}$$

Here

$$\begin{aligned}\mathscr{F}_c^{(0)}(\gamma,k)&=W(G_c^{(0)}(\gamma,k),F_c^{(0)}(\gamma,k))\\&=1+\gamma\int_0^\infty dr\,G^{(0)}(\gamma,k,r)r^{-1}[\phi_c(r)-1]F_c^{(0)}(\gamma,k,r),\quad k>0\end{aligned} \tag{6.13}$$

and thus $\mathscr{F}_c^{(0)}(\gamma,k)\neq 0$ if $|\gamma|$ is small enough [cf. Eq. (6.19)]. Furthermore,

$$\mathscr{F}_c(\gamma,k)=W(G_c(\gamma,k),F_c(\gamma,k))+1+[\mathscr{F}_c^{(0)}(\gamma,k)]^{-1}\int_0^\infty dr\,G_c^{(0)}(\gamma,k,r)V(r)F_c(\gamma,k,r),\quad k>0\,. \tag{6.14}$$

We then get as in Sec. III

$$\begin{aligned}\cot\delta_c(\gamma,k)&=i\frac{\mathscr{F}_c(\gamma,k)+\mathscr{F}_c(\gamma,e^{-i\pi}k)}{\mathscr{F}_c(\gamma,k)-\mathscr{F}_c(\gamma,e^{-i\pi}k)}\\&=\frac{1+\int_0^\infty dr\,\mathrm{Re}[G_c^{(0)}(\gamma,k,r)/\mathscr{F}_c^{(0)}(\gamma,k)]V(r)F_c(\gamma,k,r)}{\int_0^\infty dr\,\mathrm{Im}[G_c^{(0)}(\gamma,k,r)/\mathscr{F}_c^{(0)}(\gamma,k)]V(r)F_c(\gamma,k,r)},\quad \gamma\in\mathbb{R},\quad k>0\end{aligned} \tag{6.15}$$

where $F_c^{(0)}$, $G_c^{(0)}$, and F_c uniquely solve the integral equations (C9)—(C11) stated in Appendix C. By means of Eqs. (C7)—(C10) one infers the following expansions analytic in γ:

$$F_c^{(0)}(\gamma,k,r)=\frac{\sin(kr)}{k}-\gamma\frac{\cos(kr)}{2k^2}[\ln(2kr)-\mathrm{ci}(2kr)+\mathscr{C}]+\gamma\frac{\sin(kr)}{2k^2}[\mathrm{si}(2kr)+\pi/2]$$
$$-\gamma\int_0^r dr'\frac{\sin[k(r'-r)]}{k}[\phi_c(r')-1]\frac{\sin(kr')}{kr'}+O(\gamma^2)$$
$$\equiv\frac{\sin(kr)}{k}+\gamma F_{c,1}^{(0)}(k,r)+O(\gamma^2),\quad \gamma\in\mathbf{R},\quad k\geq 0 \tag{6.16}$$

$$\mathrm{Re}[G_c^{(0)}(\gamma,k,r)]=\cos(kr)-\gamma\frac{\cos(kr)}{2k}[\mathrm{si}(2kr)+\pi/2]+\gamma\frac{\sin(kr)}{2k}[\ln(2kr)+\mathrm{ci}(2kr)+\mathscr{C}]$$
$$+\gamma\int_r^\infty dr'\frac{\sin[k(r'-r)]}{k}\frac{[\phi_c(r')-1]}{r'}\cos(kr')+O(\gamma^2)$$
$$\equiv\cos(kr)+\gamma\,\mathrm{Re}[\hat{G}_{c,1}^{(0)}(k,r)]+O(\gamma^2),\quad \gamma\in\mathbf{R},\quad k>0\,, \tag{6.17}$$

$$\mathrm{Im}[G_c^{(0)}(\gamma,k,r)]=-\sin(kr)-\gamma kF_{c,1}^{(0)}(k,r)+\frac{\gamma\pi}{2k}\sin(kr)+O(\gamma^2),\quad \gamma\in\mathbf{R},\quad k>0\,, \tag{6.18}$$

$$\mathscr{F}_c^{(0)}(\gamma,k)=1+\gamma\int_0^\infty dr\, e^{-ikr}[\phi_c(r)-1]\frac{\sin(kr)}{kr}+O(\gamma^2),\quad \gamma\in\mathbf{R},\quad k>0\,. \tag{6.19}$$

Here $\mathrm{si}(z)$ [$\mathrm{ci}(z)$] denotes the sine (cosine) integral as defined in Ref. 50. Similar to Eq. (6.8), we obtain by iterating the following integral equation for F_c,

$$F_c(\gamma,k,r)=F(k,r)-\gamma[\mathscr{F}(k)]^{-1}\int_0^r dr'[G(k,r)F(k,r')-G(k,r')F(k,r)](r')^{-1}\phi_c(r')F_c(\gamma,k,r')$$
$$=F(k,r)-\gamma\left[[\mathscr{F}(k)]^{-1}\int_0^r dr'\,F(k,r')(r')^{-1}\phi_c(r')F(k,r')\right]G(k,r)$$
$$+\gamma\left[[\mathscr{F}(k)]^{-1}\int_0^r dr'\,G(k,r')(r')^{-1}\phi_c(r')F(k,r')\right]F(k,r)+O(\gamma^2)$$
$$\equiv F(k,r)+\gamma F_{c,1}(k,r)+O(\gamma^2),\quad \gamma\in\mathbf{R},\quad k>0\,. \tag{6.20}$$

[We note that $\mathscr{F}(k_0)=0$ for some $k_0>0$ would imply that $F(k_0,r)$ vanishes identically by the arguments of Ref. 51, p. 141.] Inserting expansions (6.16)—(6.19) into (6.15) then implies our final result:

$$\cot\delta_c(\gamma,k)=\left[1+\frac{\gamma\pi}{2k}+\gamma[I(k)]^{-1}\int_0^\infty dr\,H_{c,1}(k,r)V(r)F(k,r)\right]\cot\delta(k)$$
$$-\gamma[I(k)]^{-1}\int_0^\infty dr\left[\mathrm{Re}[G_{c,1}^{(0)}(k,r)]+\cot\delta(k)\,kF_{c,1}^{(0)}(k,r)\right]V(r)F(k,r)$$
$$-\gamma[I(k)]^{-1}\int_0^\infty dr[\cos(kr)+\cot\delta(k)\sin(kr)]V(r)F_{c,1}(k,r)+O(\gamma^2),\quad \gamma\in\mathbf{R},\quad k>0\,. \tag{6.21}$$

Here $F_{c,1}^{(0)}$ and $F_{c,1}$ have been defined in Eqs. (6.16) and (6.20),

$$\mathrm{Re}[G_{c,1}^{(0)}(k,r)]=\mathrm{Re}[\hat{G}_{c,1}^{(0)}(k,r)]-\cos(kr)\int_0^\infty dr'\frac{\cos(kr')\sin(kr')}{kr'}[\phi_c(r')-1]+\sin(kr)\int_0^\infty dr'\frac{\sin^2(kr')}{kr'}[\phi_c(r')-1] \tag{6.22}$$

with $\mathrm{Re}\,\hat{G}_{c,1}^{(0)}$ given by Eq. (6.17) and

$$I(k)=\int_0^\infty dr\sin(kr)V(r)F(k,r)\,, \tag{6.23}$$

$$H_{c,1}(k,r)=\sin(kr)\int_0^\infty dr'\frac{\cos(kr')\sin(kr')}{kr'}[\phi_c(r')-1]+\cos(kr)\int_0^\infty dr'\frac{\sin^2(kr')}{kr'}[\phi_c(r')-1]\,. \tag{6.24}$$

In Eq. (6.21), $\delta(k)$ denotes the short-range ($\gamma=0$) phase shift. In contrast to $a_c(\gamma)$, the analytic expansions (6.16)—(6.20) imply that $\cot\delta_c(\gamma,k)$ is analytic in γ around $\gamma=0$. Consequently, higher-order corrections can be computed by iterating the appropriate integral equations (cf. Appendix C). Again we have assumed that $\cot\delta(k)\neq 0$, implying $I(k)\neq 0$, in expansion (6.21) [otherwise one simply expands $\tan\delta_c(\gamma,k)$].

The result (6.21) is new. In particular the first-order term in γ has been obtained in closed form without any approximations (for earlier approximate treatments of this problem we refer to Refs. 44 and 49).

Using the techniques presented above, it is trivial to expand other scattering quantities such as, e.g., the S matrix, $S_c(\gamma,k)=e^{2i\delta_c(\gamma,k)}$, with respect to γ such that we omit further details. We note that both expressions (6.9) and (6.21) considerably simplify in the point charge limit $\phi_c \to 1$.

Finally we remark that, under suitable conditions, nonlocal interactions can be incorporated in the above results in the following simple way. If V consists, e.g., out of a local and nonlocal part $V=V_l+V_{nl}$, all integrals of the type

$$\int_0^\infty dr\, \alpha(r)V(r)\beta(r)$$

in Eqs. (6.9) and (6.21) should be replaced by

$$\int_0^\infty dr\, \alpha(r)V_l(r)\beta(r)+\int_0^\infty dr\, \alpha(r)\int_0^\infty ds\, V_{nl}(r,s)\beta(s)\ .$$

The results of this section can be used as a starting point for new model calculations with realistic interactions. In particular they should lead to additional clarifications in the discussion about the charge (in)dependence of nuclear forces (cf., e.g., Refs. 15—17, 26, 27, 29—32, 34, and 52).

VII. LOW-ENERGY PHENOMENA IN TWO-DIMENSIONAL SYSTEMS

In this section we consider some further applications of the results obtained in Secs. II and V, which lead to new insights in some low-energy phenomena in two-dimensional systems. One of our main motivations thereby is to contribute to the discussion on spin-polarized quantum systems such as atomic hydrogen for which it is known that, at low temperatures, surface collisions are the dominant relaxation mechanism.[2,5,11,13]

The results in Secs. II and V, especially Eq. (2.9) for $\gamma=0$ and $n=2$ and Eq. (5.6) for $n=2$, completely solve the controversy around the existence of two-dimensional scattering parameters like scattering length and effective range. It is generally believed that their introduction is very difficult because of logarithmic effects. Some authors even doubt their existence in the usual sense (see, e.g., Ref. 6), while others[5,11] recently introduced related parameters in an, in our opinion, unsatisfactory way since the connection with the interactions in the system is either lacking[5] or incomplete.[11]

In the case of hard-disk scattering, the low-energy behavior of the two-dimensional phase shift $\delta_{\rm HD}$ is

$$\cot\delta_{\rm HD}(k)=\frac{2}{\pi}\left[\ln\left[\frac{kd_{\rm HD}}{2}\right]+\mathscr{C}\right]+O(k^2)\ , \qquad (7.1)$$

where $d_{\rm HD}$ is the hard-disk diameter. So formally, the result (5.6) ($n=2$) for a general potential V can be obtained by replacing the value of the hard-disk diameter in (7.1) by

$$d_{\rm HD}\to\exp(-1/a)\ , \qquad (7.2)$$

with a the scattering length for V, defined by (2.9) with $\gamma=0$ and $n=2$. This solves an interesting question posed, e.g., by Gibson[6] on how to estimate the effect of more general potentials in low-energy phenomena in two dimensions, when knowing the hard-disk results. In three dimensions, the analogous rule of thumb is to replace the diameter of the hard sphere by the scattering length for the general potential: $d_{\rm HS}\to a$. In this way we also understand the formulas written down in Refs. 5 and 11. These authors use $\exp(-1/a)$ as the analog of the three-dimensional scattering length without giving a definition from first principles in terms of V and F. As a consequence, their definition is unnatural when compared to other dimensions. [See foregoing sections, especially Eqs. (2.9) and (5.6) for $\gamma=0$ and $n=2$.]

The result (7.2) allows us to generalize immediately the existing results for hard-disk systems such as, e.g., the low-temperature expansion for the second virial coefficient of a two-dimensional quantum gas and its contribution to the specific heat in Ref. 6, the expression for the ground-state energy per particle, the chemical potential, and the condensate density of a system of hard-disk bosons in Ref. 53, etc. to more general potentials such as the Lennard-Jones.

In this respect we have made an independent calculation of the direct term, corresponding to Boltzmann statistics, in the second virial coefficient b_2, at low temperatures for a general potential V. Following closely the arguments of Ref. 6 (Sec. 3), taking into account the possibility of negative-energy bound states, we obtain on the basis of the results in Secs. III and V,

$$b_2(\beta)=\sum_m (e^{-\beta E_m}-1)-\frac{1}{\ln(4\pi\beta)}+\frac{2-a\,[\ln(\pi/2)+\mathscr{C}\,]}{a\,[\ln(4\pi\beta)]^2}+O[(\ln\beta)^{-3}]\ , \qquad (7.3)$$

where $\beta=1/k_BT, E_m$ are the possible s-wave bound-state energies and a is the scattering length for V. [Of course, employing the rule (7.2), we recover the hard-disk result.]

The next question we want to study is the possibility of a Ramsauer-Townsend effect in surface collisions. This is the phenomenon where the total cross section becomes anomalously small as a function of energy. The deep minimum usually occurs at low energy and arises because of the fact that the most significant partial-wave contribution becomes zero (and the dip is not filled up by contributions from higher partial waves). This is the s-wave contribution for Boltzmann particles; for spin-polarized fermions, such as ^{3}He↑-^{3}He↑, this would be the p-wave term. For recent discussions of this phenomenon in three dimensions, we refer to Ref. 36 and the literature cited therein.

So we have to investigate whether the $n=2$ total cross section can become small at a particular nonzero, but small value of the energy k^2. Starting from the following formula for the $n=2$ total cross section σ_2 for Boltzmann particles in s waves,

$$\sigma_2(k)=\frac{4}{k}\sin^2\delta_2(k)\ , \qquad (7.4)$$

we obtain, using Eq. (5.6) for $n=2$,

$$\sigma_2(k)=\frac{4}{k}\left\{1+\left[-\frac{2}{\pi a}+\frac{2}{\pi}\left[\ln\frac{k}{2}+\mathscr{C}\right]+O(k^2)\right]^2\right\}^{-1} \tag{7.5}$$

$$=\frac{\pi^2}{k[\ln(k/2)]^2}\left[1+\frac{2-2\mathscr{C}a}{a\ln(k/2)}+O((\ln k)^{-2})\right] \quad \text{as } k\to 0\ . \tag{7.6}$$

This result is new for general potentials and gives the known hard-disk formula when applying (7.2). At this point it is interesting to compare it with the low-energy behavior for the cross section in all other dimensions. Again, starting from the expression for s waves,

$$\sigma_n(k)=\frac{\Gamma(n/2)}{2\pi^{n/2}}(2\pi)^{n-1}\frac{1}{k^{n-1}}4\sin^2\delta_n(k)\ , \tag{7.7}$$

we get by employing Eq. (4.8) for n odd,

$$\sigma_n(k)=\frac{\Gamma(n/2)}{2\pi^{n/2}}(2\pi)^{n-1}\frac{4}{k^{n-1}}\left\{1+\left[-[\Gamma(n/2)]^2\frac{2}{\pi}\left[\frac{2}{k}\right]^{n-2}\frac{1}{a}+O(k^{4-n})\right]^2\right\}^{-1} \tag{7.8}$$

$$=[\Gamma(n/2)]^{-3}2^{2-n}\pi^{(n/2)+1}k^{n-3}a^2[1+O(k^2)] \quad \text{as } k\to 0,\ n=3,5,\ldots\ . \tag{7.9}$$

For n even we obtain by using Eq. (5.6),

$$\sigma_n(k)=\frac{\Gamma(n/2)}{2\pi^{n/2}}(2\pi)^{n-1}\frac{4}{k^{n-1}}\left\{1+\left[-[\Gamma(n/2)]^2\frac{2}{\pi}\left[\frac{2}{k}\right]^{n-2}\frac{1}{a}+\frac{2}{\pi}\left[\ln\frac{k}{2}+\mathscr{C}\right]+O(k^{4-n})\right]^2\right\}^{-1} \tag{7.10}$$

$$=[\Gamma(n/2)]^{-3}2^{2-n}\pi^{(n/2)+1}k^{n-3}a^2\begin{cases}[1+O(k^2\ln k)] & \text{as } k\to 0,\ n=4\\ [1+O(k^2)] & \text{as } k\to 0,\ n=6,8,\ldots\ .\end{cases} \tag{7.11}$$

These results agree with Ref. 54. Looking at the expressions (7.9) and (7.11), we see that they stay valid when the scattering length a becomes arbitrarily small. This immediately implies that in that case the cross section has the Ramsauer-Townsend minimum.

However, for $n=2$, Eq. (7.6) is no longer valid for k small, but nonzero, and $a\to 0$. In the latter situation we obtain from Eq. (7.5)

$$\sigma_2(k)=\frac{\pi^2a^2}{k}\left[1+2a\left[\ln\frac{k}{2}\right]+\mathscr{C}+O(ak^2)\right] \quad \text{as } a\to 0,\ k\neq 0\ . \tag{7.12}$$

Unexpectedly, this behavior is similar to all other dimensions, in spite of the presence of the logarithm in (7.5). It then shows that also in the two dimensions a Ramsauer-Townsend effect can occur in s waves.

For higher partial waves l in two dimensions, obtained by replacing $n=2$ by $n=2+2l$ in the formulas above, one sees that the logarithmic effect is canceled by a factor k^{2l} [see Eqs. (5.6) and (7.10) for $n=2+2l$], so that, compared with other dimensions, nothing special does occur. As a consequence, one can expect that also here the effect is less likely since the vanishing of the higher particle-wave scattering length will occur at higher energies. In that case the minimum in the cross section may eventually be filled up by contributions from higher partial waves.[36] It would certainly be interesting to look for these two-dimensional effects in realistic systems.

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APPENDIX A

In order to prove Eq. (2.8) we first state the precise conditions on V:

If $\gamma=0$,

$$\int_0^{R'} dr\, r\,|\ln r||V(r)|+\int_{R'}^{\infty} dr\, r^{3/2}(1+|\ln r|^2)|V(r)|<\infty \quad \text{for some } 0<R'<1 \text{ if } n=2\ , \tag{A1}$$

$$\int_0^{R} dr\, r\,|V(r)|+\int_{R}^{\infty} dr\, r^{n-1}|V(r)|<\infty \quad \text{for some } R>0 \text{ if } n\geq 3\ . \tag{A2}$$

If $\gamma>0$,

$$\int_0^{R'} dr\, r\,|\ln r||V(r)|+\int_{R'}^{\infty} dr\, r^{1/2}e^{4(\gamma r)^{1/2}}|V(r)|<\infty \quad \text{for some } 0<R'<1 \text{ if } n=2\ , \tag{A3}$$

$$\int_0^{R} dr\, r\,|V(r)|+\int_{R}^{\infty} dr\, r^{1/2}e^{4(\gamma r)^{1/2}}|V(r)|<\infty \quad \text{for some } R>0 \text{ if } n\geq 3\ . \tag{A4}$$

If $\gamma < 0$,

$$\int_0^{R'} dr\, r\, |\ln r||V(r)| + \int_{R'}^{\infty} dr\, r^{3/4} |V(r)| < \infty$$

$$\text{for some } 0 < R' < 1 \text{ if } n = 2\,, \qquad \text{(A5)}$$

$$\int_0^{R} dr\, r\, |V(r)| + \int_{R}^{\infty} dr\, r^{3/4} |V(r)| < \infty$$

$$\text{for some } R > 0 \text{ if } n \geq 3\,. \qquad \text{(A6)}$$

Next we note the explicit expressions for the unperturbed solutions $F_n^{(0)}$ and $G_n^{(0)}$:

$$F_n^{(0)}(\gamma,r) = \begin{cases} r^{(n-1)/2}, & \gamma=0, \ n\geq 2 \\ \gamma^{(2-n)/2}\Gamma(n-1)r^{1/2}I_{n-2}((4\gamma r)^{1/2}), & \gamma \geq 0, \ n \geq 2 \\ (-\gamma)^{(2-n)/2}\Gamma(n-1)r^{1/2}J_{n-2}((-4\gamma r)^{1/2}), & \gamma \leq 0, \ n\geq 2 \end{cases} \qquad \text{(A7)}$$

$$G_n^{(0)}(\gamma,r) = \begin{cases} -r^{1/2}\ln(r), & \gamma=0, \ n=2 \\ (n-2)^{-1}r^{(3-n)/2}, & \gamma=0, \ n\geq 3 \\ 2\gamma^{(n-2)/2}[\Gamma(n-1)]^{-1}r^{1/2}K_{n-2}((4\gamma r)^{1/2}), & \gamma>0, \ n\geq 2 \\ -i\pi(-\gamma)^{(n-2)/2}[\Gamma(n-1)]^{-1}r^{1/2}H_{n-2}^{(2)}((-4\gamma r)^{1/2}), & \gamma<0, \ n\geq 2 \end{cases} \qquad \text{(A8)}$$

where $\Gamma(z)$ denotes the gamma function[39] and $J_\alpha(z), H_\alpha(z)[I_\alpha(z), K_\alpha(z)]$ denote the (modified) Bessel functions of order α.[39] Following Refs. 21 and 22 we have the bounds

$$|F_n^{(0)}(\gamma,r)| \leq C \begin{cases} r^{(n-1)/2}, & \gamma=0, \ n\geq 2 \\ r^{1/4}[r/(1+r)]^{(2n-3)/4}e^{(4\gamma r)^{1/2}}, & \gamma>0, \ n\geq 2 \\ r^{1/4}[r/(1+r)]^{(2n-3)/4}, & \gamma<0, \ n\geq 2 \end{cases} \qquad \text{(A9)}$$

and

$$|G_n^{(0)}(\gamma,r)| \leq C' \begin{cases} r^{1/2}|\ln(r)|, & \gamma=0, \ n=2 \\ r^{(3-n)/2}, & \gamma=0, \ n\geq 3 \\ e^{-(4\gamma r)^{1/2}}r^{1/2}[|\ln(r)|+1]/\{1+r^{1/4}[|\ln(r)|+1]\}, & \gamma>0, \ n=2 \\ r^{1/2}[|\ln(r)|+1]/\{1+r^{1/4}[|\ln(r)|+1]\}, & \gamma<0, \ n=2 \\ r^{1/4}[r/(1+r)]^{(5-2n)/4}e^{-(4\gamma r)^{1/2}}, & \gamma>0, \ n\geq 3 \\ r^{1/4}[r/(1+r)]^{(5-2n)/4}, & \gamma<0, \ n\geq 3\,. \end{cases} \qquad \text{(A10)}$$

Iterating Eq. (2.2) observing conditions (A1)—(A6) then immediately leads to

$$|F_n(\gamma,r)| \leq C'' \begin{cases} r^{1/2}[|\ln(r)|+1], & \gamma=0, \ n=2 \\ r^{(n-1)/2}, & \gamma=0, \ n\geq 3 \\ e^{(4\gamma r)^{1/2}}r^{1/2}[|\ln(r)|+1]/\{1+r^{1/4}[|\ln(r)|+1]\}, & \gamma>0, \ n=2 \\ r^{1/4}[r/(1+r)]^{(2n-3)/4}e^{(4\gamma r)^{1/2}}, & \gamma>0, \ n\geq 3 \\ r^{1/2}[|\ln(r)|+1]/\{1+r^{1/4}[|\ln(r)|+1]\}, & \gamma<0, \ n=2 \\ r^{1/4}[r/(1+r)]^{(2n-3)/4}, & \gamma<0, \ n\geq 3 \end{cases} \qquad \text{(A11)}$$

(here C, C', and C'' denote numerical constants depending on γ and n). In precisely the same way one proves that $|G_n(\gamma,r)|$ can be estimated in terms of the right-hand side of (A10). This finally proves Eq. (2.8).

APPENDIX B

In this appendix we sketch how to prove the analyticity of the expansions (4.7), (4.8), (5.5), and (5.6) with respect to k^2 around $k^2=0$. We first prove certain exponential bounds on $F_n^{(0)}(\gamma,k,r)$, $F_n(\gamma,k,r)$, and $\widetilde{G}_n^{(0)}(\gamma,k,r)$, valid uniformly in $k^2\in\mathbb{C}$ for k^2 small enough. Let V fulfill conditions (3.2) or (3.3) and let $P_n(\gamma,k^2,r)$ be a regular or irregular solution of

$$\left[-\frac{d^2}{dr^2}+\frac{(n-1)(n-3)}{4r^2}+\frac{\gamma}{r}+V(r)-k^2\right]\psi(k^2,r)=0,$$

$$r>0, \ k^2\in\mathbb{C}, \ \gamma\in\mathbf{R}, \ n=2,3,4,\ldots \qquad \text{(B1)}$$

which is entire with respect to k^2. Next we introduce

$$\rho_n(\gamma,k^2,r)=\kappa_0|P_n(\gamma,k^2,r)|^2+\kappa_0^{-1}\left|\frac{\partial}{\partial r}P_n(\gamma,k^2,r)\right|^2,$$

$$r\geq R_0>0, \ k^2\in\mathbb{C}, \ |k^2|\leq\kappa_0^2, \ \kappa_0>0 \qquad \text{(B2)}$$

and note that ρ_n is strictly positive since $P_n(\gamma,k,r)$ and $(\partial/\partial r)P_n(\gamma,k,r)$ cannot vanish simultaneously. From

$$\frac{\partial}{\partial r}\rho_n(\gamma,k^2,r)=\left[\kappa_0+\kappa_0^{-1}\left[\frac{(n-1)(n-3)}{4r^2}+\frac{\gamma}{2}+V(r)\right]\right]2\,\mathrm{Re}\left[P_n(\gamma,k^2,r)\frac{\partial}{\partial r}P_n^*(\gamma,k^2,r)\right]$$
$$-\kappa_0^{-1}2\,\mathrm{Re}\left[k^2P_n(\gamma,k^2,r)\frac{\partial}{\partial r}P_n^*(\gamma,k^2,r)\right] \tag{B3}$$

one infers

$$\frac{\partial}{\partial r}\rho_n(\gamma,k^2,r)\leq\left[2\kappa_0+\kappa_0^{-1}\left[\frac{(n-1)(n-3)}{4r^2}+\frac{|\gamma|}{r}+|V(r)|\right]\right]|\kappa_0^{1/2}P_n(\gamma,k^2,r)|\left|\kappa_0^{-1/2}\frac{\partial}{\partial r}P_n(\gamma,k^2,r)\right|$$
$$\leq\left[2\kappa_0+\kappa_0^{-1}\left[\frac{(n-1)(n-3)}{4r^2}+\frac{|\gamma|}{r}+|V(r)|\right]\right]\rho_n(\gamma,k^2,r)\,. \tag{B4}$$

This implies the bound

$$\rho_n(\gamma,k^2,r)\leq\rho_n(\gamma,k^2,R_0)\exp\left[2\kappa_0(r-R_0)+4^{-1}\kappa_0^{-1}(n-1)(n-3)(R_0^{-1}-r^{-1})\right.$$
$$\left.+\kappa_0^{-1}|\gamma|\ln(r/R_0)+\int_{R_0}^{r}dr'\,|V(r')|\right]. \tag{B5}$$

Since $\rho_n(\gamma,k^2,R_0)$ is uniformly bounded with respect to k^2 if $|k^2|\leq\kappa_0^2$, one finally obtains

$$\rho_n(\gamma,k^2,r)\leq\mathrm{const}(\kappa_0,R_0)\exp[2\kappa_0(r-R_0)+\kappa_0^{-1}|\gamma|\ln(r/R_0)],\quad \gamma\in\mathbf{R},\quad |k^2|\leq\kappa_0^2,\quad r\geq R_0,\quad n=2,3,4,\ldots\,. \tag{B6}$$

Similarly, introducing

$$\sigma_n(\gamma,k^2,r)=\rho_n(\gamma,k^2,r)+\kappa_0^5\left|\frac{\partial}{\partial k^2}P_n(\gamma,k^2,r)\right|^2+\kappa_0^3\left|\frac{\partial^2}{\partial k^2\partial r}P_n(\gamma,k^2,r)\right|^2,\quad r\geq R_0>0,\quad k^2\in\mathbf{C}\,,\quad |k^2|\leq\kappa_0^2,\quad \kappa_0>0 \tag{B7}$$

one obtains

$$\frac{\partial}{\partial r}\sigma_n(\gamma,k^2,r)\leq\left[3\kappa_0+\kappa_0^{-1}\left[\frac{(n-1)(n-3)}{4r^2}+\frac{|\gamma|}{r}+|V(r)|\right]\right]\sigma_n(\gamma,k^2,r)\,. \tag{B8}$$

This implies

$$\sigma_n(\gamma,k^2,r)\leq\mathrm{const}(\kappa_0,R_0)\exp[3\kappa_0(r-R_0)+\kappa_0^{-1}|\gamma|\ln(r/R_0)],\quad \gamma\in\mathbf{R},\quad |k^2|\leq\kappa_0^2,\quad r\geq R_0,\quad n=2,3,4,\ldots\,. \tag{B9}$$

It remains to show that integrals of the type

$$\int_{R_0}^{\infty}dr\,F_n^{(0)}(\gamma,k,r)V(r)F_n(\gamma,k,r)\,,$$
$$\int_{R_0}^{\infty}dr\,\widetilde{G}_n^{(0)}(\gamma,k,r)V(r)F_n(\gamma,k,r) \tag{B10}$$

are analytic with respect to k^2 in an open neighborhood of $k^2=0$ [since $|F_n^{(0)}(\gamma,k,r)F_n(\gamma,k,r)|$ and $|\widetilde{G}_n^{(0)}(\gamma,k,r)F_n(\gamma,k,r)|$ are uniformly bounded if $|k|<b/2$, $0\leq r\leq R_0$, integrals over compact intervals in r are obviously analytic around $k^2=0$ by Lebesgue-dominated convergence theorem]. This is realized by noting that the bound (B9) proves by the Lebesgue-dominated convergence theorem and conditions (3.2) and (3.3) on V, that both integrals in (B10) are (continuously) complex differentiable with respect to k^2 around $k^2=0$.

APPENDIX C

We sketch a derivation of the expansions (6.5)—(6.7) and (6.16)—(6.20), valid in three dimensions. (We omit the index 3 in what follows.) We start with the zero-energy expansions. From Eqs. (A7) and (A8) we get for $n=3$,

$$F^{(0)}(\gamma,r)=r+\gamma\frac{r^2}{2}+O(\gamma^2),\quad \text{as } \gamma\to 0,\quad \gamma\in\mathbf{R} \tag{C1}$$

$$G^{(0)}(\gamma,r)=1+\gamma r[\ln(\gamma r)+2\mathscr{C}-1]+O(\gamma^2\ln(\gamma))\quad \text{as } \gamma\to 0,\quad \gamma>0\,. \tag{C2}$$

Using this, one directly obtains the expansions (6.5)—(6.7) by iterating the integral equations

$$F_c^{(0)}(\gamma,r)=F^{(0)}(\gamma,r)-\gamma\int_0^r dr'\,g^{(0)}(\gamma,r,r')(r')^{-1}\times[\phi_c(r')-1]F_c^{(0)}(\gamma,r')\,, \tag{C3}$$

$$G_c^{(0)}(\gamma,r)=G^{(0)}(\gamma,r)+\gamma\int_r^{\infty} dr'\,g^{(0)}(\gamma,r,r')(r')^{-1}\times[\phi_c(r')-1]G_c^{(0)}(\gamma,r')\,. \tag{C4}$$

For completeness we also note the integral equation for F_c,

$$F_c(\gamma,r)=F_c^{(0)}(\gamma,r)-\int_0^r dr'\,g_c^{(0)}(\gamma,r,r')V(r')\times F_c(\gamma,r')\,, \tag{C5}$$

where

$$g_c^{(0)}(\gamma,r,r')=[f_c^{(0)}(\gamma)]^{-1}[G_c^{(0)}(\gamma,r)F_c^{(0)}(\gamma,r')-G_c^{(0)}(\gamma,r')F_c^{(0)}(\gamma,r)]\,. \tag{C6}$$

Now we turn to $k>0$. By iterating the integral equation for $F_c^{(0)}$ below, one obtains[50]

$$\begin{aligned}F^{(0)}(\gamma,k,r)&=\frac{\sin(kr)}{k}-\gamma\int_0^r dr'\,\frac{\sin[k(r'-r)]}{kr'}F^{(0)}(\gamma,k,r')\\&=\frac{\sin(kr)}{k}-\gamma\frac{\cos(kr)}{2k^2}[\ln(2kr)-\mathrm{ci}(2kr)+\mathscr{C}]+\gamma\frac{\sin(kr)}{2k^2}[\mathrm{si}(2kr)+\pi/2]+O(\gamma^2),\quad \gamma\in\mathbf{R},\ \ k\geq 0\,.\end{aligned} \tag{C7}$$

For the expansion of $G_c^{(0)}$ we proceed differently. Using definition (3.7) for $n=3$ and the explicit series representation of $U(\alpha;\beta;z)$,[39] one obtains

$$\begin{aligned}G^{(0)}(\gamma,k,r)&=e^{-ikr}+\gamma\frac{\sin(kr)}{k}\ln(2ikr)-\gamma re^{-ikr}\sum_{p=0}^{\infty}\frac{\Psi(p+2)}{\Gamma(p+2)}(2ikr)^p+O(\gamma^2)\\&=e^{-ikr}+i\gamma\frac{e^{-ikr}}{k}\ln(2ikr)+\gamma\frac{e^{-ikr}}{2ik}\Psi(1)-\gamma(2ik)^{-3/2}r^{-1/2}W_{-1/2;0}(2ikr)+O(\gamma^2)\\&=e^{-ikr}-\gamma\frac{e^{-ikr}}{2ik}[\ln(2ikr)+\mathscr{C}]+\gamma\frac{e^{ikr}}{2ik}[\mathrm{ci}(2kr)-i\,\mathrm{si}(2kr)]+O(\gamma^2)\quad \gamma\in\mathbf{R},\ \ k>0\end{aligned} \tag{C8}$$

where $W_{\alpha;\beta}(z)$ denotes the Whittaker function[50] and pages 1063 and 929 of Ref. 50 have been used in the second and third steps, respectively.

Equations (C7) and (C8) together with

$$F_c^{(0)}(\gamma,k,r)=F^{(0)}(\gamma,k,r)-\gamma\int_0^r dr'\,g^{(0)}(\gamma,k,r,r')(r')^{-1}[\phi_c(r')-1]F_c^{(0)}(\gamma,k,r')\,, \tag{C9}$$

$$G_c^{(0)}(\gamma,k,r)=G^{(0)}(\gamma,k,r)+\gamma\int_r^\infty dr'\,g^{(0)}(\gamma,k,r,r')(r')^{-1}[\phi_c(r')-1]G_c^{(0)}(\gamma,k,r') \tag{C10}$$

then immediately yield expansions (6.16)—(6.19). We finally note the integral equation for F_c,

$$F_c(\gamma,k,r)=F_c^{(0)}(\gamma,k,r)-\int_0^r dr'\,g_c^{(0)}(\gamma,k,r,r')V(r')F_c(\gamma,k,r')\,, \tag{C11}$$

where

$$g_c^{(0)}(\gamma,k,r,r')=[\mathscr{F}_c^{(0)}(\gamma,k)]^{-1}[G_c^{(0)}(\gamma,k,r)F_c^{(0)}(\gamma,k,r')-G_c^{(0)}(\gamma,k,r')F_c^{(0)}(\gamma,k,r)]\,. \tag{C12}$$

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