Classical anharmonic oscillators: Rescaling the perturbation series

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A solution to the classical anharmonic-oscillator equation of motion $\ddot{x}=-x-\lambda x^{2n-1}$ is obtained by rescaling the perturbation series. The resulting series involves a coupling constant that remains finite for $\lambda >> 1$ and thus converges rapidly for all λ.

Classical anharmonic oscillators with Hamiltonian

$$H=\frac{p^2}{2}+\frac{x^2}{2}+\frac{\lambda}{2n}x^{2n} \text{ (for } n=2,3,4,\ldots)$$

exhibit "regiming" in that for $\lambda << 1$, the frequency ω of the motion can be perturbatively estimated as $\omega \simeq 1+O(\lambda)$, while for $\lambda >> 1$, $\omega \simeq \lambda^{1/2n}$. Perturbation theory for these oscillators becomes convergent after the removal of the secular terms[1,2] (using, e.g., the Lindstedt procedure) but the perturbation series in powers of λ is not capable of yielding the exact scaling behavior of $\omega(\lambda)\simeq\lambda^{1/2n}$ at large λ. This feature is common to a variety of problems in both classical and quantum mechanics.[3] We show that the effectiveness of the Lindstedt procedure can be greatly improved by rescaling the perturbation series by a function of the coupling constant λ which has the correct limiting $\lambda << 1$ and $\lambda >> 1$ forms. This involves the introduction of a rescaled coupling constant $\bar{\lambda}$ which remains finite as $\lambda \to \infty$. Perturbation theory in powers of $\bar{\lambda}$ yields good accuracy at low orders for all λ. First, we consider the quartic anharmonic oscillator for which the exact solution is known in terms of elliptic functions. This allows a direct numerical comparison of our global approximation with the exact answer. The general anharmonic oscillators are treated subsequently.

The equation of motion for the quartic $(n=2)$ anharmonic oscillator is

$$\ddot{x}+x=-\lambda x^3 \quad , \tag{1}$$

which has the exact solution

$$t=\frac{1}{(1+4\lambda E)^{1/4}}\left[K\left(\frac{a^2}{a^2+b^2}\right)-F\left(\cos^{-1}\frac{x}{a},\frac{a^2}{a^2+b^2}\right)\right] \quad . \tag{2}$$

Here $F(\theta,\phi)$ is the elliptic integral of the first kind, $K(\phi)=F(\pi/2,\phi)$; E is the energylike constant of motion:

$$E=\tfrac{1}{2}x^2+\tfrac{1}{2}\dot{x}^2+\frac{\lambda}{4}x^4 \quad , \tag{3}$$

and

$$b^2,a^2=\frac{1}{\lambda}(\sqrt{1+4\lambda E}\pm 1) \quad . \tag{4}$$

The frequency of the motion is

$$\omega=\frac{\pi}{2}\frac{(1+4\lambda E)^{1/4}}{K[a^2/(a^2+b^2)]} \quad . \tag{5}$$

As can be easily checked from the properties of $K(x)$, the frequency ω behaves as $1+O(\lambda)$ for $\lambda << 1$ and like $\lambda^{1/4}$ for $\lambda >> 1$.

To implement the rescaling procedure we introduce a renormalized frequency ω_0 and write the equation of motion as

$$\ddot{x}+\omega_0{}^2x=-\lambda(x^3-\tfrac{3}{2}\langle x^2\rangle x) \quad , \tag{6}$$

where

$$\omega_0^2=1+\tfrac{3}{2}\lambda\langle x^2\rangle \quad , \tag{7}$$

$$\langle\cdots\rangle=\frac{1}{T}\int_0^T(\cdots)dt \quad , \tag{8}$$

T being the period and $\langle x^2\rangle$ is to be determiend self-consistently. The factor 3/2 comes from the fact that $\langle x^4\rangle=3\langle x^2\rangle^2/2$ for a simple harmonic function. By construction the right-hand side of Eq. (6) is expected to be small for all values of λ. To find the renormalized frequency we first set the right-hand side of Eq. (6) equal to zero. This yields the zeroth order approximation to the solution

$$x_0=a\cos\omega_0 t \quad . \tag{9}$$

In this approximation Eq. (7) yields the frequency-amplitude relation

$$\omega_0^2=1+\tfrac{3}{4}\lambda a^2 \quad . \tag{10}$$

Besides Eq. (10), the frequency ω_0 and the amplitude a are also related through the first integral of motion E [Eq. (3)]:

$$E=\tfrac{1}{2}\omega_0^2a^2\left(1-\frac{3\lambda a^2}{16\omega_0^2}\right)=\tfrac{1}{2}a^2\omega_0^2(1-\tfrac{3}{16}\bar{\lambda}) \quad , \tag{11}$$

where $\bar{\lambda}=\lambda a^2/\omega_0^2$. To $O(\bar{\lambda})$ Eq. (11) can be inverted to yield

$$a^2=\frac{2E}{\omega_0^2}(1+\tfrac{3}{16}\bar{\lambda}) \quad . \tag{12}$$

The self-consistent frequency is now obtained from Eq. (10) by eliminating the amplitude a:

$$\omega_0^2=\tfrac{1}{2}\{1+[1+6\lambda E(1+\tfrac{3}{16}\bar{\lambda})]^{1/2}\} \quad , \tag{13}$$

where $\bar{\lambda}$, as a function of λ and E, to the lowest order is

$$\bar{\lambda}=\frac{\lambda a^2}{\omega_0^2}=\frac{8E\lambda}{(1+\sqrt{1+6\lambda E})^2} \quad . \tag{14}$$

Equations (13) and (14) completely specify the frequency ω_0. As expected $\omega_0\simeq 1+O(\lambda)$ for $\lambda << 1$ and $\simeq\lambda^{1/4}$ for $\lambda >> 1$. This approximation for ω is within 2% of the exact answer given in Eq. (5) for all values of λ.

Having determined the scale factors ω_0 and a, we now define scaled variables $y=x/a$ and $\tau=\omega_0 t$. Equation (7) now assumes the form

$$\frac{d^2y}{d\tau^2}+y=-\bar{\lambda}(y^3-\tfrac{3}{2}\langle y^2\rangle y)\ , \tag{15}$$

where $\bar{\lambda}$ defined by Eq. (14) is our *rescaled* coupling constant. Note that as $\lambda\to\infty$, $\bar{\lambda}$ remains finite and approaches the limiting value $\frac{4}{3}$. Our claim is that the standard Lindstedt perturbation technique on this equation converges very fast for all λ. To $O(\bar{\lambda})$, we obtain

$$\omega/\omega_0=1\ , \tag{16a}$$

$$x/a=\sin(\omega t)-\frac{\bar{\lambda}}{32}\sin(3\omega t)\ . \tag{16b}$$

Equation (16a) shows that the secular terms are automatically removed from Eq. (15) at $O(\bar{\lambda})$. The frequency to $O(\bar{\lambda})$ is simply the scaling frequency ω_0 and as stated above the accuracy is better than 2% for all λ. To test the accuracy of Eq. (16b) we have tabulated (Table I) x as a function of t as obtained from it as well as that obtained from the exact solution of Eq. (4). This is done for the value $\lambda=12$, $E=1$ which happens to lie in the "boundary layer" between the two regimes ($\lambda \ll 1$ and $\lambda \gg 1$) and is expected to be the most sensitive test of Eq. (16b). The close to 1% accuracy supports our claim about the fast convergence of the rescaled perturbation series.

We now turn to the general anharmonic oscillator

$$H=\frac{p^2}{2}+\frac{x^2}{2}+\frac{\lambda}{2n}x^{2n}$$

for which Eqs. (6) and (7) become

$$\ddot{x}+\omega_0{}^2x=-\lambda(x^{2n-1}-\beta\langle x^2\rangle^{n-1}x)\ , \tag{6'}$$

and

$$\omega_0^2=1+\beta\lambda\langle x^2\rangle^{n-1}\ , \tag{7'}$$

where

$$\beta=\frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})2^n}{\pi\Gamma(n+1)}\ . \tag{17}$$

As before, if we set the right-hand side to zero, the solution is $x=a\cos\omega_0 t$, leading to

$$\omega_0^2=1+\frac{\beta\lambda}{2^{n-1}}a^{2n-2}\ , \tag{10'}$$

TABLE I. Comparison of the approximant [Eq. (16b)] with the exact result of Eq. (2) for quartic anharmonicity.

t	0	0.0427	0.0862	0.172	0.332	0.679
x_{exact}	0	0.0616	0.123	0.242	0.454	0.707
x_{approx}	0	0.0596	0.120	0.239	0.449	0.703

which is the analog of Eq. (10). In terms of the total energy $E=\frac{1}{2}\dot{x}^2+\frac{1}{2}x^2+(\lambda/2n)x^{2n}$, we find [see Eq. (12)]

$$a^2=\frac{2E}{\omega_0^2}\left[1+\frac{\bar{\lambda}\beta(1-1/n)}{2^n}\right]\ , \tag{12'}$$

with $\bar{\lambda}=\lambda a^{2(n-1)}/\omega_0^2$. From Eqs. (10′) and (12′), we see that the frequency ω_0 satisfies the self-consistency condition

$$\omega_0^{2n}=\omega_0^{2n-2}+\beta\lambda\left\{E\left[1+\frac{\beta\bar{\lambda}}{2^n}\left(1-\frac{1}{n}\right)\right]\right\}^{n-1}$$

$$=\omega_0^{2n-2}+\lambda B\ , \tag{18}$$

with

$$B=\beta\left\{E\left[1+\frac{\beta\bar{\lambda}}{2^n}\left(1-\frac{1}{n}\right)\right]\right\}^{n-1}\ .$$

Equation (18) can be solved in closed form for $n=2$ (demonstrated above) and $n=3$. For other values of n it cannot be solved exactly but an analytic approximant to the root, obtained by Newton-Raphson technique, is

$$\omega_0^{2n}=\frac{\lambda B+1/n\,(1+\lambda B)^{1-1/n}}{1-(1-1/n)(1+\lambda B)^{-1/n}}\ . \tag{19}$$

The scaled coupling constant is obtained from

$$\bar{\lambda}=\lambda(2E)^{n-1}\left[\frac{1-(1-1/n)(1+\beta\lambda E^{n-1})^{-1/n}}{\beta\lambda E^{n-1}+1/n\,(1+\beta\lambda E^{n-1})^{1-1/n}}\right]\ . \tag{20}$$

Note that as $\lambda\to\infty$, the scaled coupling constant remains finite and reaches the value $2^{n-1}/\beta$. We can now scale the x in Eq. (6′) by a and t by ω_0^{-1} and get an equation of motion analogous to Eq. (15) on which the use of Lindstedt procedure yields accurate answers at low orders of perturbation theory.

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[1]For various perturbative techniques of the conservative single degree of freedom system see Chapter 2 in A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations* (Wiley Interscience, New York, 1979), Chap. 2.

[2]C. R. Eminhizer, R. H. G. Helleman, and E. W. Montroll, J. Math. Phys. 17, 121 (1976).

[3]For an application of the rescaled perturbation series to the quantum anharmonic oscillator, see K. Banerjee, Proc. R. Soc. London, Ser. A 368, 155 (1979); W. E. Caswell, Ann. Phys. 123, 153 (1979).