Nonequilibrium dynamics of N-component Ginzburg-Landau fields in zero and one dimension

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Here we report on the use of the N^{-1} expansion in studying the time evolution of a nonequilibrium quenched state of a Ginzburg-Landau model in zero and one spatial dimension. Results obtained from the N^{-1} expansion are compared with those obtained using numerical simulations of the Langevin dynamics. We find that the N^{-1} approximation provides a useful qualitative picture which for $N \gtrsim 3$ becomes a reasonable quantitative representation.

I. INTRODUCTION

The nonequilibrium dynamics of an N-component Ginzburg-Landau model in zero and one spatial dimension has been studied by carrying out numerical simulations of the Langevin equations describing the order-parameter field. At time zero the system is quenched from an initial high temperature to a low temperature, and its subsequent evolution to a new equilibrium is determined. In particular, the time dependence of the order parameter, and the order-parameter correlation function and its dependence on N are explored. As is well known, the $1/N$ expansion generates an asymptotic expression for the static properties and provides a nonperturbative approach for calculating equilibrium phenomena.[1-3] Here we will be interested in examining the N dependence of the nonequilibrium dynamics.

In Sec. II we begin by considering the zero-dimensional case. Physically this limit with $N=2$ has been used to describe the transient onset of a single-mode laser.[4,5] It has also provided a useful model for small superconducting particles[6,7] where its predictions for the specific heat and diamagnetic susceptibility are in good agreement with experiment, although at present we know of no time-dependent experimental studies. For the zero-dimensional problem, Suzuki[8,9] has derived a scaling solution to describe the time evolution of the order parameter following an initial quench. This solution becomes exact in the low-temperature—long-time limit, and we will compare our results with it for various values of N. We also explore the $1/N$ expansion, calculating the leading $1/N$ correction to the $N=\infty$ dynamic mean-field result.

In Sec. III we turn to the one-dimensional case and present results for the time-dependent structure factor $S(k,t)$ for various values of N. Just as for the zero-dimensional case there is a great deal known about the static limit and its relationship to the $1/N$ expansion. We compare the long-time behavior of our calculations to these equilibrium results. Here we are particularly interested in the dynamics and in the way the system evolves towards its final equilibrium state after a quench. The time dependence of the structure factor for the kinetic one-dimensional Ising model has been recently evaluated by Mazenko and Widom[10] who found what they described as the propagation of a structure pulse. Similar behavior has been observed for the Ginzburg-Landau model by Petschek *et al.*[11,12] Here $S(k,t)$ for fixed k exhibited a peak which occurred at later times for smaller values of k. We find this behavior in $S(k,t)$, and various results for different values of N shown in Sec. III. We also compare our results with the $N\rightarrow\infty$, dynamic mean-field approximation.

There exists a variety of quasi-one-dimensional materials which in principle provide possible experimental systems for studying this behavior. However, one must stay above their three-dimensional ordering temperatures. It could be that thin superconducting whiskers or long one-dimensional over-damped Josephson junctions will provide the most experimentally accessible systems for studying the quenched nonequilibrium response for the one-dimensional systems discussed here. In Sec. IV we conclude with a summary of our findings and their possible extensions.

II. THE ZERO-DIMENSIONAL CASE

The stochastic equation of motion for an N-component Ginzburg-Landau field φ_i in zero spatial dimensions can be written in the form of a Langevin equation

$$\dot{\varphi}_i = -a\varphi_i - \frac{b}{N}\left[\sum_{j=1}^{N}\varphi_j^2\right]\varphi_i + f_i \ , \tag{1}$$

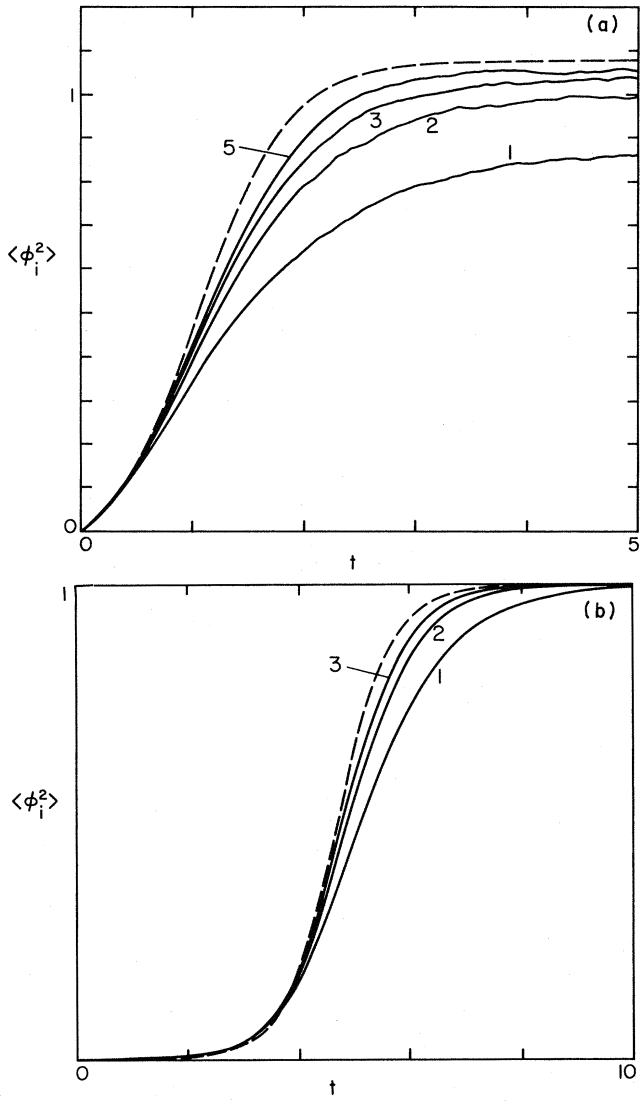

FIG. 1. (a) Square of one field component $\langle\phi_i^2\rangle$ vs t for various N-component Ginzburg-Landau models in zero spatial dimensions. Here the final quench temperature is $T=0.10$. Dashed line is the $N=\infty$ limit. (b) Here the system has been quenched to a final, low temperature, $T=10^{-4}$.

with f_i a Gaussian random force such that

$$\langle f_i(t)f_j(t')\rangle = 2T\delta(t-t')\delta_{ij} \ . \tag{2}$$

Here a is proportional to $(T-T_M)$ where T_M is a mean-field transition temperature and b is set equal to 1. In our analysis we assume that initially the system is well above T_M and set $\langle\varphi_i^2\rangle=0$ for $t<0$. At time $t=0$, we imagine that the system is quenched to a temperature T and explore how the resulting state evolves towards its final equilibrium. In particular we will be interested in the time dependence of $\langle\varphi_i^2(t)\rangle$.

We have obtained a numerical solution of Eq. (1) using the following procedure. Starting from a configuration $\phi_i(t)$, a configuration at time $t+\Delta t$ is generated by selecting a random number f_i consistent with Eq. (2) and setting

$$\phi_i(t+\Delta t)=\phi_i(t) + \left[-a\phi_i(t)-\left[\frac{1}{N}\sum_{j=1}^{N}\varphi_j^2(t)\right]\varphi_i(t)+f_i\right]\Delta t \ . \tag{3}$$

Proceeding iteratively from $\phi_i(0)=0$, a sequence of states $[\phi_i(n\Delta t)]$ are generated which we refer to as a trajectory. Since the $f_i(n)$ are stochastic variables, each time we repeat this procedure a different trajectory is generated. Averaging functions of φ_i over a set of these trajectories gives us the solution of Eq. (1). Typically 10^4 trajectories were used and simulations were also carried out using a first-order predictor-corrector (modified Euler) procedure in place of Eq. (3).

Figures 1(a) and 1(b) show some results for $\langle\phi_i^2(t)\rangle$ vs t for different values of N and two different quench conditions. In both cases, $a=-1$; however, Fig. 1(a) corresponds to a dimensionless final temperature $T=0.1$ while Fig. 1(b) corresponds to a much lower final temperature $T=10^{-4}$. As previously discussed, both start from a temperature well above T_M where, on the scale we are using, $\langle\phi_i^2\rangle=0$. The dashed line is the $N=\infty$ limit. The cases shown in Fig. 1 illustrate the evolution of a system from a "disordered" state with $\langle\phi_i^2\rangle=0$ to an "ordered" state with $\langle\phi_i^2\rangle\sim O(1)$. In Fig. 1(b), the thermal fluctuations associated with the quenched state are small so it takes more time before this switching occurs.

The equation governing the time development of $S=\langle\phi_i^2(t)\rangle$ follows from Eqs. (1) and (2)

$$\frac{d}{dt}S=-2aS-\frac{2}{N}\left\langle\left[\sum_{j=1}^{N}\varphi_j^2(t)\right]\phi_i^2\right\rangle+2T \ . \tag{4}$$

An expansion for S in powers of $1/N$ can be obtained by setting

$$S=S_0+\frac{1}{N}S_1+\cdots \ , \tag{5}$$

and writing Eq. (4) as

$$\frac{dS}{dt}=-2aS-2S^2+2T-\frac{2}{N}\Gamma \ , \tag{6}$$

with

$$\Gamma=\sum_{j=1}^{N}\langle(\varphi_j^2-\langle\varphi_j^2\rangle)(\varphi_i^2-\langle\varphi_i^2\rangle)\rangle \ . \tag{7}$$

Then to evaluate S to $O(1/N)$, we need only the leading

TABLE I. Comparison of the exact equilibrium value of $\langle\phi^2\rangle$ with the $O(1/N)$ expansion for $a=-1$, $T=0.1$, and various N values.

N	$\langle\phi^2\rangle_{eq}$	$S_{0eq}+S_{1eq}/N$
1	0.936	0.871
2	1.014	1.001
3	1.040	1.028
5	1.061	1.059
10	1.076	1.076
∞		1.092

behavior of Γ so that we arrive at the following set of equations:

$$\frac{dS_0}{dt} = -2aS_0 - 2S_0^2 + 2T \,, \tag{8a}$$

$$\frac{dS_1}{dt} = -2(a+2S_0)S_1 - 2\Gamma_0 \,, \tag{8b}$$

$$\frac{d\Gamma_0}{dt} = -4(a+2S_0)\Gamma_0 + 8TS_0 \,. \tag{8c}$$

From Eq. (8a) it follows that in equilibrium S_0 is given by the solution of the equation

$$S_0^2 + aS_0 = T \,, \tag{9}$$

so that

$$S_{0\text{eq}} = \frac{-a + (a^2+4T)^{1/2}}{2} \,. \tag{10}$$

Then from Eq. (8b) and (8c) one finds[3]

$$S_{1\text{eq}} = -2T\frac{S_{0\text{eq}}^3}{(T+S_{0\text{eq}}^2)^2} \,. \tag{11}$$

In Table I we compare the exact equilibrium results for $a=1.0$ and $T=0.1$ with the $S_{0\text{eq}}+S_{1\text{eq}}/N$ for various values of N.

Next we turn to the dynamical solution of Eq. (8). Starting the system from an initial state with $S(0)=0$ and quenching it to a low temperature, $T \ll 1$ final state, we integrate Eqs. (8) to obtain

$$\frac{S(\tau)}{S_{\text{eq}}} = \frac{\tau}{1+\tau} - \frac{2}{N}\frac{\tau^2}{(1+\tau)^3} \,. \tag{12}$$

Here τ is the scaled time introduced by Suzuki[8]

$$\tau = \frac{T}{S_{\text{eq}}^2} e^{2S_{\text{eq}}t} \,, \tag{13}$$

and $\tau \sim 1$ corresponds to the characteristic switching time for the system. Numerical results obtained from the Langevin equation are compared with Eq. (12) in Fig. 2 for various values of N. The solid line in these figures shows the numerical results while the dashed and long-dashed—short-dashed lines represent the $N=\infty$ solution $\tau/(1+\tau)$ and the $1/N$ result given by Eq. (12), respectively.

For $N=1$, a scaling solution of Eq. (4) was found by Suzuki[8] which becomes exact in the limit of small temperatures T with the time t long enough that $\tau \sim 1$,

$$\frac{S(\tau)}{S_{\text{eq}}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \frac{e^{-x^2/2} x^2 \tau}{x^2\tau + 1} \,. \tag{14}$$

Here τ is the scaling variable introduced in Eq. (13). For large values of τ, the asymptotic dependence of Eq. (14) is

$$\frac{S(\tau)}{S_{\text{eq}}} \simeq 1 - \left[\frac{\pi}{2\tau}\right]^{1/2} \,. \tag{15}$$

Note that the approximate $1/N$ expansion given by Eq. (12) has a different asymptotic behavior going as $1+(1+2/N)1/\tau$. In Fig. 3(a) we plot the asymptotic

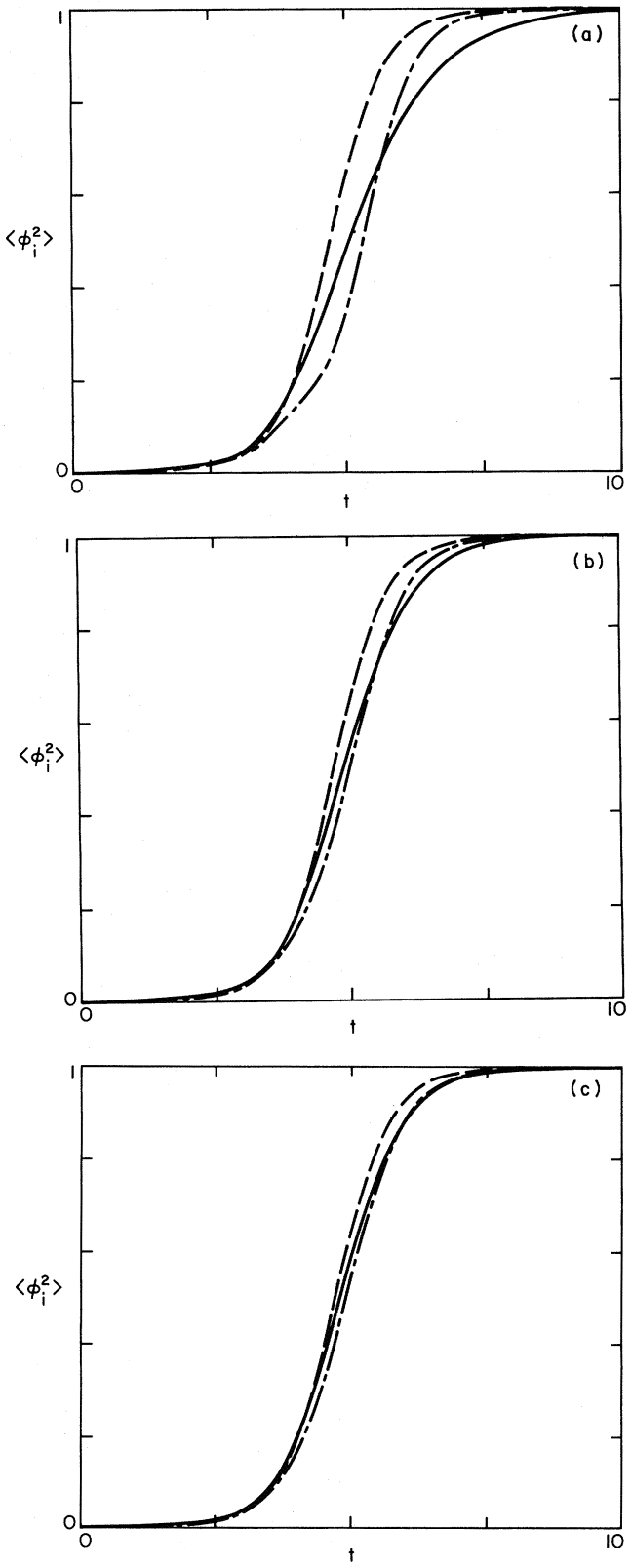

FIG. 2. (a) $\langle \phi_i^2 \rangle$ vs t for $N=1$. Solid line is the result of our numerical solution of the Langevin equation (1), the dashed line is the $N \to \infty$ part of Eq. (12) and the long-dashed—short-dashed curve includes the $1/N$ correction in Eq. (12). The final quench temperature $T=10^{-4}$. (b) Similar with $N=2$. (c) $N=3$.

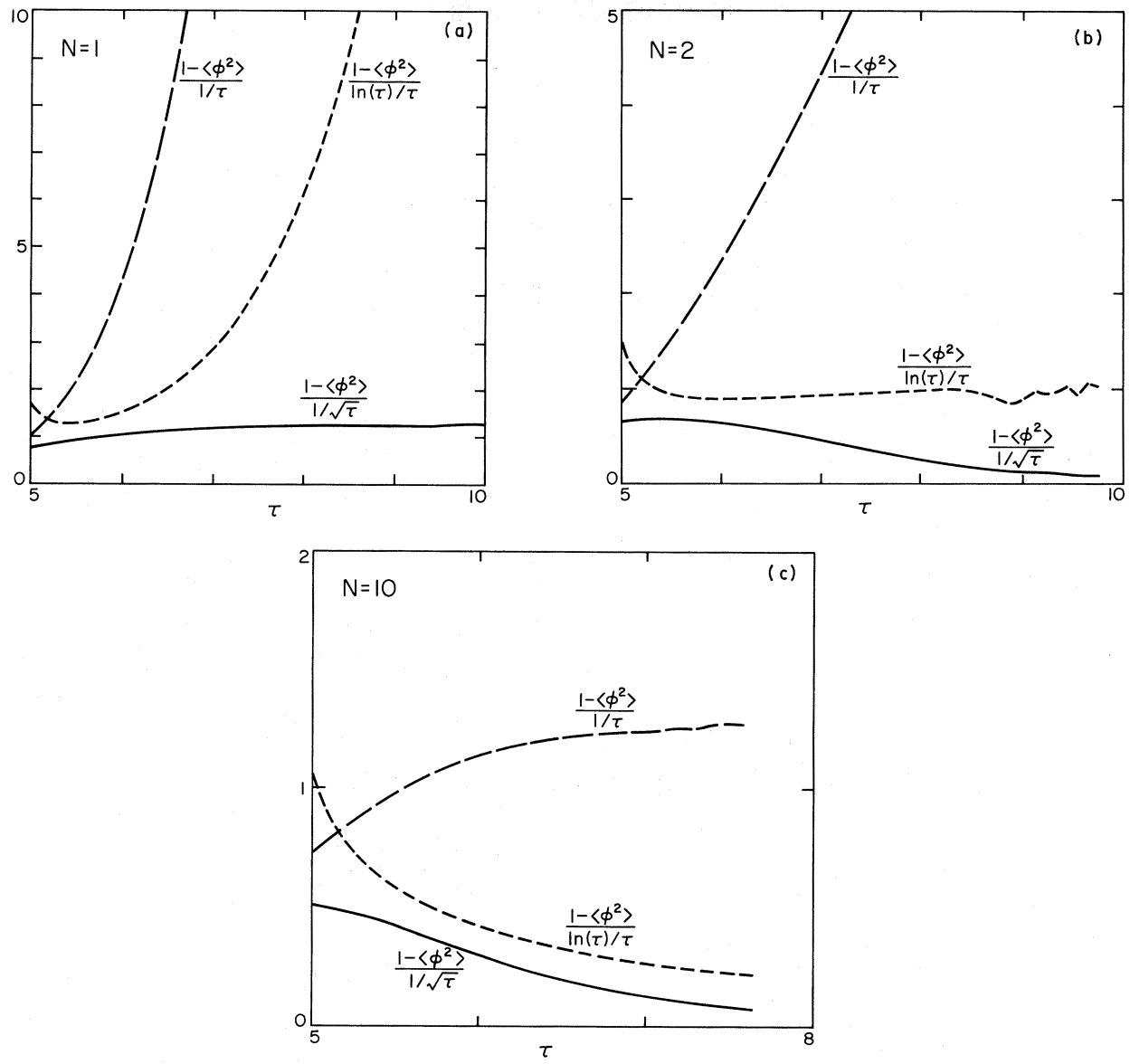

FIG. 3. (a) Long-time behavior of $1-\langle\phi^2(t)\rangle$ should vary as $\tau^{-1/2}$ according to Eq. (15). Here this is compared with other alternatives such as τ^{-1} and $\tau^{-1}[\ln(\tau)]$. Note that according to Eq. (15) the asymptotic value of $(1-\langle\phi^2\rangle)/(1/\sqrt{\tau})$ should be $(\pi/2)^{1/2}\simeq 1.25$. (b) For $N=2$, $1-\langle\phi^2(t)\rangle\simeq-(\ln\tau)/\tau$. (c) For $N\geq 3$, $1-\langle\phi^2(t)\rangle\simeq 1/\tau$; here we show results for $N=10$.

behavior of our numerical solution which clearly shows the $\tau^{-1/2}$ behavior. Note that the solid line is close to $1.25=(\pi/2)^{1/2}$ as it should be according to Eq. (15). For $N=2$ one can show that the long-time approach to equilibrium gives

$$S(t)/S_{\rm eq}-1\simeq(\ln\tau)/\tau\ , \tag{16}$$

while for $N\geq 3$ it varies as $1/\tau$. This agrees with our numerical results as shown in Figs. 3(b) and 3(c).

The failure of the $1/N$ expansion to yield the correct long-time approach to equilibrium for $N\leq 2$ may have to do with the fact that the $1/N$ expansion is not ordinarily summable. Note that for $N=1$, the power series expansion for $S(\tau)$ starts out as $\tau-3\tau^2+\cdots$. The $O(\tau^3)$ term from Eq. (12) cannot be relied upon as there will be contributions to it from $O(N^{-2})$ although none from the still higher terms. Looking at the small-τ behavior of the $O(N^{-2})$ term, it can be shown that

$$S(\tau)=\tau-3\tau^2+15\tau^3\ . \tag{17}$$

Now these are just the first three terms of the power-series expansion of Eq. (14), and Suzuki[8] has argued that Eq. (14) can be viewed as a Borel sum of a divergent series in the coupling. If so, this would indeed fit with the fact that the $1/N$ expansion is known to be asymptotic.

III. THE ONE-DIMENSIONAL PROBLEM

In this section we consider a one-dimensional N-component field which evolves according to

$$\frac{\partial\phi_i}{\partial t}=-\left[a-\frac{\partial^2}{\partial x^2}\phi_i-\frac{1}{N}\sum_{j=1}^{N}\phi_j^2\right]\phi_i+f_i\ , \tag{18}$$

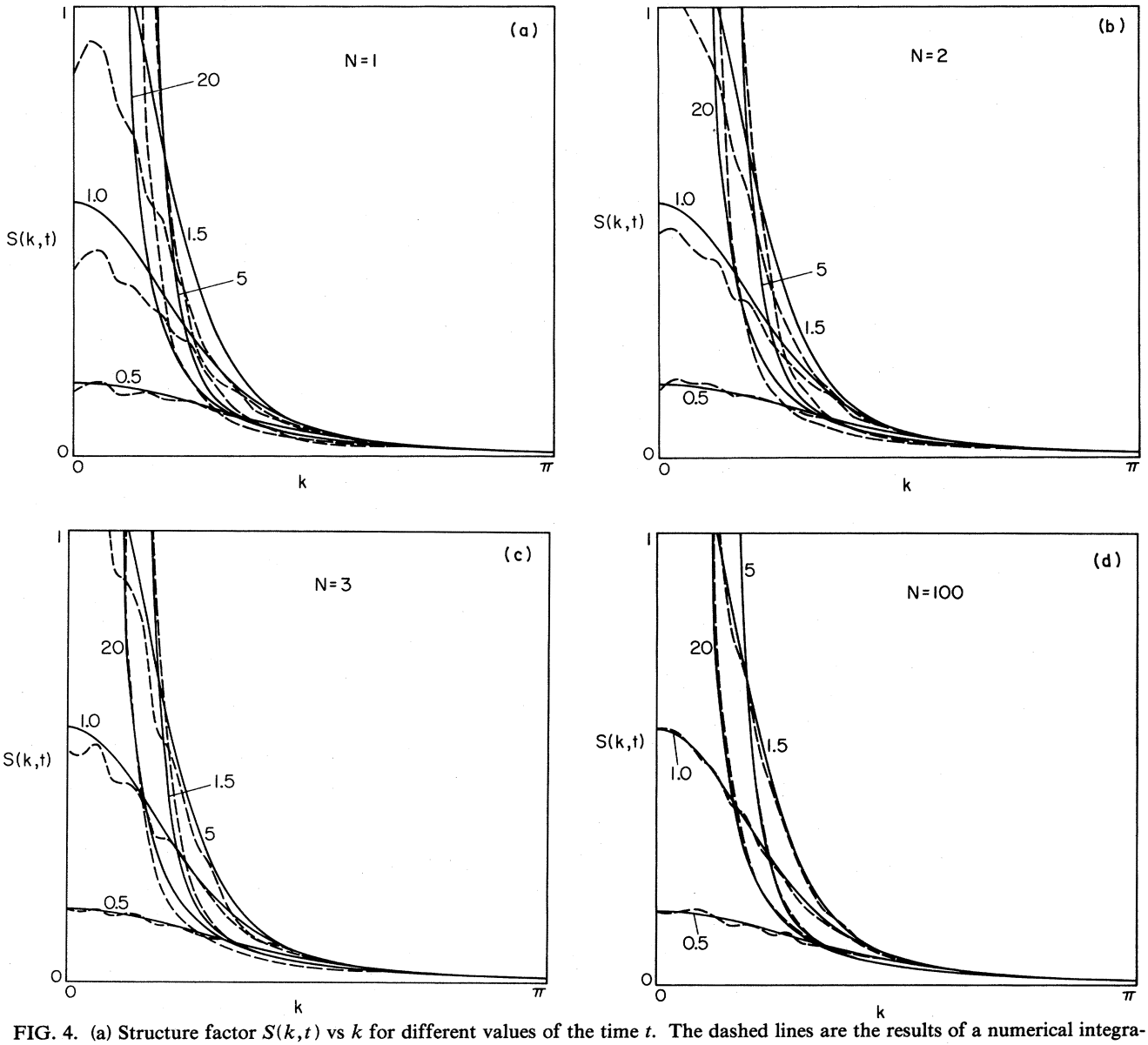

FIG. 4. (a) Structure factor $S(k,t)$ vs k for different values of the time t. The dashed lines are the results of a numerical integration of the Langevin equation and the solid lines are the leading large-N behavior obtained from Eq. (25); (b) $N=2$; (c) $N=3$; (d) $N=10$.

with

$$\langle f_i(x,t)f_j(x',t')\rangle = 2T\delta(t-t')\delta(x-x')\delta_{ij} \ . \tag{19}$$

Just as before, we are interested in the equal-time order parameter, order-parameter correlation function

$$\langle \phi_i(x,t)\phi_i(x',t)\rangle \tag{20}$$

and its spatial Fourier transform $S(k,t)$. We will assume that initially $S(k,0)=0$ and that at time $t=0$, the system is quenched to a low temperature. For the purposes of obtaining a numerical solution, Eq. (18) will be put on a spatial grid. Periodic grids of 64 and 128 sites were used with the 128 site grid providing a check that the quantities which were calculated were not affected by the grid size. The integration procedure was similar to that described in Sec. II except that now a random field $f_i(n,x_l)$ must be generated for each site and a finite difference representation for d^2/dx^2 was used. A first-order predictor-corrector method was used to integrate the Langevin dynamics of the one-dimensional systems.

Some numerical results showing the time evolution of $S(k,t)$ for different values of N are given in Figs. 4(a)–4(d) for $N=1$, 2, 3, and 100, respectively. The dashed lines are the results of the numerical integration of Eq. (18) and the solid lines are the results of a calculation of the leading large-N behavior discussed below. In all these cases $a=-1$ and $T=0.1$. Note that the same characteristic behavior is evident in all of these: Following the quench, the evolving $S(k,t)$ overshoots its final equilibrium value and relaxes down towards it from above.

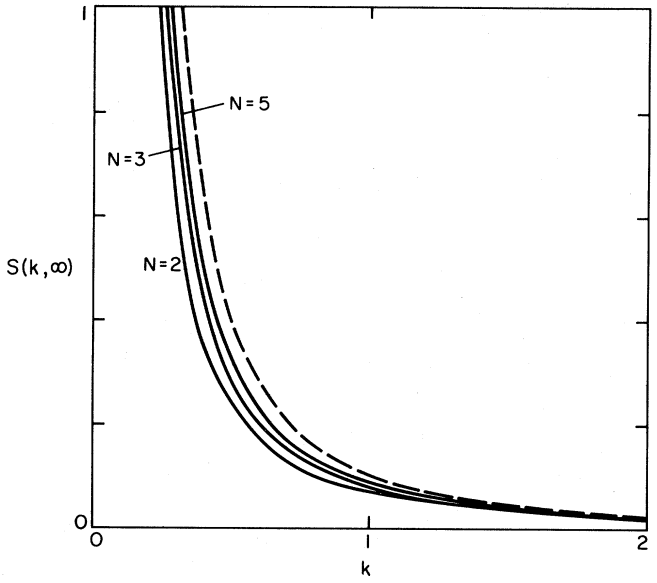

FIG. 5. Equilibrium structure factor $S(k)$ for various N values. The dashed line shows the $N\rightarrow\infty$ result $S_{0\text{eq}}(k)$ given by Eqs. (21) and (22).

Just as for the zero-dimensional case it is instructive to begin by exploring the final equilibrium state. In this case, the $1/N$ expansion for $S_{\text{eq}}(k)$ can be obtained in a variety of different ways. Using the diagrammatic formulation the leading behavior is[1]

$$S_{0\text{eq}}(k)=\frac{T}{k^2+k^2}\ , \tag{21}$$

with

$$K^2=a+\sum_k \frac{T}{k^2+K^2}\ . \tag{22}$$

The $1/N$ correction is usually written as a self-energy correction so that through this order

$$S_{\text{eq}}(k)=\frac{T}{k^2+K^2+\Sigma(k)}\ , \tag{23}$$

with

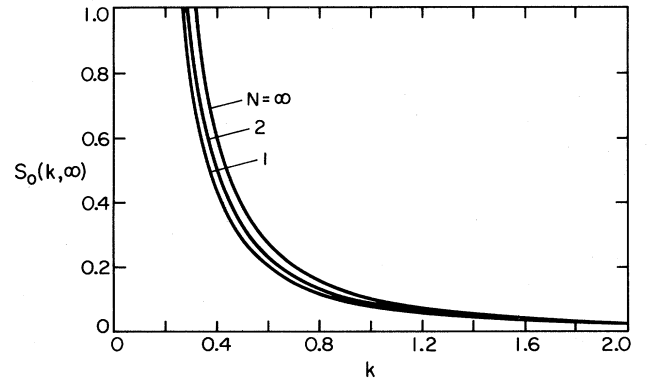

FIG. 6. Approximate structure factor given by Eqs. (23) and (24) for $N=1$, 2, and ∞.

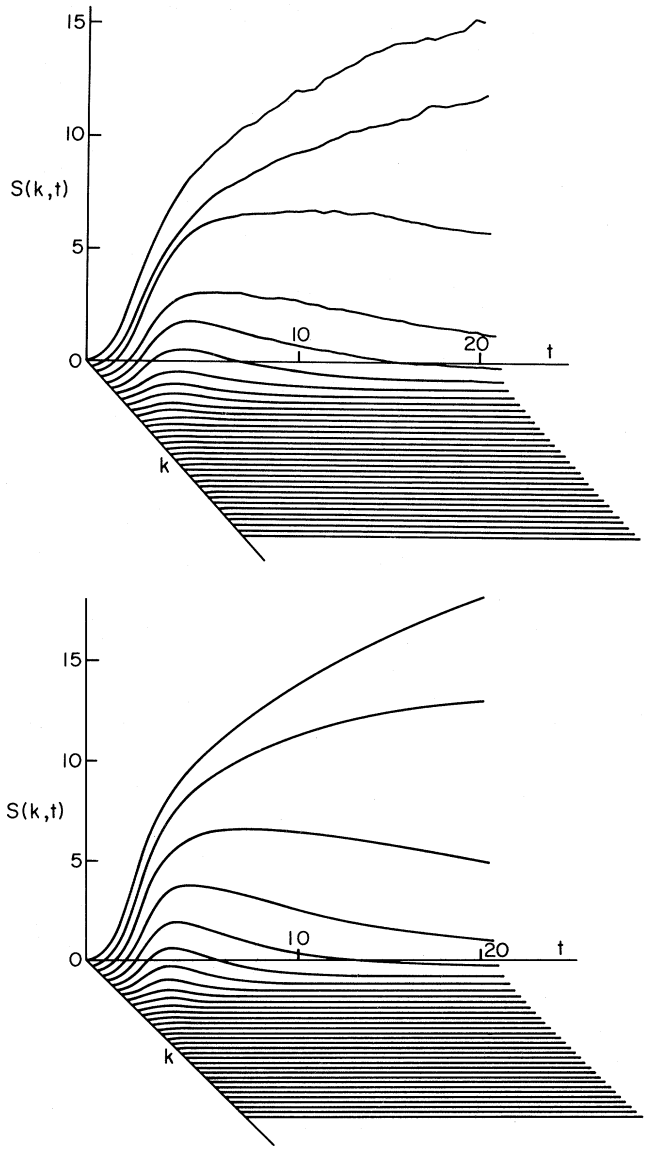

FIG. 7. (a) Structure factor $S(k,t)$ obtained by numerically integrating the Langevin equation vs t for various values of k. Here $N=3$. (b) $S_0(k,t)$ obtained from the leading large-N dynamic mean-field approximation given by Eq. (25).

$$\Sigma(k)=\frac{T}{N}\left[\frac{1}{2K}-\frac{T}{2K^2K_2}\frac{K+K_2}{(K+K_2)^2+k^2}\right]\ , \tag{24}$$

and $K_2{}^2=4K^2+T/K$. In Fig. 5 we have plotted $S_{\text{eq}}(k)$ obtained from Monte Carlo calculations of a Ginzburg-Landau one-dimensional field theory with various number of components N. The dashed line is the leading large-N result $S_{0\text{eq}}(k)$, given by Eq. (21). The effect of the $1/N$ correction to Eq. (21) given by Eq. (23) and (24) is shown in Fig. 6.

Finally, we turn to the dynamics and study the leading large-N behavior. In this limit the one-dimensional equation is a simple generalization of Eq. (8a)

$$\frac{d}{dt}S_0(k,t)=-2\left[a+k^2+\frac{1}{N}\sum_p S_0(p,t)\right]S_0(k,t)\ . \qquad (25)$$

This dynamic mean-field approximation gives the leading $O(1)$ behavior for $S(k,t)$ with the next correction being of order N^{-1}. The equilibrium limit of $S_0(k,t)$ is just Eq. (21).

Numerically integrating Eq. (25) with $S_0(k,t)$ computed self-consistently, we obtained the results shown as the solid lines in Figs. 4(a)–4(d) for $a=-1$ and $T=0.1$. In order to obtain a further comparison of $S_0(k,t)$ with the Langevin results, we plot sections of $S(k,t)$ for fixed k vs t for $N=3$. Figure 7(a) shows the full Langevin results and Fig. 7(b) shows $S_0(k,t)$. Note the structure pulse previously discussed by Mazenko and Widom[10] for the one-dimensional Ising model and by Petscheck[11] *et al.* for the three-dimensional dynamic mean-field calculation.

IV. CONCLUSIONS

From the numerical solutions of the Langevin equation for a Ginzburg-Landau field with N components we have been able to explore the approach to the large-N behavior. It is possible to construct expansions for large N so that it is useful to be able to see how well these approximations represent the actual behavior of such systems.

What we have seen is that much of the qualitative behavior is contained in the $1/N$ solutions and that for $N\gtrsim 3$ a reasonable overall fit is obtained. However, as clearly shown from the figures, there are some important discrepancies. In particular, the asymptotic τ dependence of S for the zero-dimensional problem with $N=1$ and 2 varies as $\tau^{-1/2}$ and $(\ln\tau)/\tau$ rather than τ^{-1}. Furthermore, the time evolution of $S(k,t)$ shows clear differences from $S_0(k,t)$ for small-N values. Theoretically, in one dimension the large-N limit is clearly suspect for the case $N=1$, since the existence of well-defined domain walls for $N=1$ clearly affects its dynamics.

It will be of interest to extend these calculations to higher dimension and explore the N dependence of systems in three-dimensional space where the quench can lead to an actual phase transition. In addition this approach can be extended to explore systems with a conserved order parameter.

ACKNOWLEDGMENTS

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