Statistical Mechanics of the XY Model. II. Spin-Correlation Functions

Eytan Barouch*
IBM Watson Laboratory, Columbia University, New York, New York 10025

and

Barry M. McCoy†
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790

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The spin-correlation functions of the one-dimensional XY model are studied in the presence of a constant magnetic field. We find that the asymptotic behavior of these correlation functions depends strongly on the various parameters of the Hamiltonian.

I. INTRODUCTION

In the first paper of this series[1] we studied the time-dependent magnetization $m_z(t)$ of the XY model in the presence of certain time-dependent magnetic fields. We found that this model has the most interesting property that as $t\rightarrow\infty$, $m_z(t)$ does not approach its thermal-equilibrium value. To understand this most peculiar property in more detail, we turn to investigation of the instantaneous spin-correlation functions as a function of time. However, in order to make a meaningful study of these time-dependent correlation functions, it is first necessary to understand the properties of the equilibrium correlation functions. These functions have been previously studied only in the following two special cases: (i) γ arbitrary in the absence of a magnetic field,[2,3] and (ii) $\gamma=1$ with an arbitrary magnetic field[4] (transverse Ising model). Therefore, the purpose of this paper is twofold. We first show in Sec. II how to compute the time-dependent correlation functions for a general $h(t)$ and then study in detail the behavior of the equilibrium correlations obtained by making $h(t)$ a constant independent of time. The behavior of the correlation functions as a function of time will be analyzed in detail in the third paper of this series.

Lieb, Schultz, and Mattis[2] (LSM) originally introduced the XY model to study the influence which symmetry, or lack of symmetry, has in a many-body system. The Hamiltonian they studied was (generalized to finite magnetic field[5])

$$H=J\sum_{j=1}^{N}\left[(1+\gamma)S_j^xS_{j+1}^x+(1-\gamma)S_j^yS_{j+1}^y\right]-h\mu\sum_{j=1}^{N}S_j^z\,, \tag{1.1}$$

where, without loss of generality, J may be taken positive; the symmetry they were interested in (when $h=0$) is the rotational symmetry in the XY plane which H possesses if

$$\gamma=0\,. \tag{1.2}$$

However, there is a second, somewhat less obvious, "symmetry" in (1.1) if

$$|h\mu|=J\,, \tag{1.3}$$

i.e., if the interaction energy with the magnetic field is equal (in some sense) to the arithmetic average of the interaction energies in the X and Y directions. The effects of these symmetries are most pronounced at $T=0$. The first symmetry then manifests itself in the fact that if $|\mu h|<J$, the ground-state energy is not an analytic function of γ at $\gamma=0$.[2,5] The second symmetry manifests itself in the fact that m_z fails to be an analytic function of h at $h\mu=J$ for any γ.[5]

A great deal more information about the influence of symmetry on the system described by (1.1) may be obtained by investigating the three spin-spin-correlation functions:

$$\rho_{vv}(R)=\langle S_0^vS_R^v\rangle\;,\qquad v=x,\,y,\,z\,. \tag{1.4}$$

When $h=0$ this study was started by LSM and continued by one of the present authors.[3] This study indicates that there are several qualitative differences between $\gamma=0$ and $\gamma\neq0$. When $\gamma=1$ a similar study made by Pfeuty[4] indicates that there are several qualitative differences between $h<1$, $h=1$, $h>1$. In this paper, we extend these calculations of M to general values of h. However, the actual computations are quite detailed. Therefore, in order to emphasize the qualitative effects of the symmetry conditions (1.2) and (1.3), we conclude this Introduction with a summary of the major results of this paper.

The asymptotic behavior of $\rho_{xx}(R)$ as $R\rightarrow\infty$ when $T>0$ is studied in Sec. III. A high-temperature expansion is given by (3.28), but the most interesting features are seen when $0<T\ll1$. These results are given in (3.63)–(3.68). In all cases $(-1)^R\times\rho_{xx}(R)$ vanishes exponentially rapidly as $R\rightarrow\infty$. However, the rate of this exponential vanishing depends on h, and this dependence is qualitatively

different if

$$0 \leq |h\mu| < J(1-\gamma^2), \tag{1.5a}$$

or

$$J(1-\gamma^2) < |h\mu| . \tag{1.5b}$$

Moreover, if $J < |h\mu|$, but

$$\gamma^2[(h\mu/J)^2 - (1-\gamma^2)] < (1-\gamma^2)(\pi/\beta)^2, \tag{1.6}$$

the approach of the leading term of $(-1)^R \rho_{xx}(R)$ to zero is not monotonic, but is oscillatory with a wavelength that depends on γ and h. Note in particular that qualitative restrictions similar to (1.5a) and (1.5b) have already been seen in Paper I. The asymptotic behavior of ρ_{yy} as $R\to\infty$ for $T>0$ is studied in Sec. V [see (5.11)] and that of ρ_{zz} in Sec. VI [see (6.3)–(6.7)]. They also vanish exponentially as $R\to\infty$ for all values of h and γ. Furthermore, $(-1)^R$ ρ_{yy} and ρ_{zz} will have some oscillatory behavior if (1.6) holds.

The effects of the symmetry conditions (1.2) and (1.3) are most sharply seen at $T=0$. This case is considered for ρ_{xx} in Sec. IV, for ρ_{yy} in Sec. V, and for ρ_{zz} in Sec. VI. The qualitative features of these expansions are summarized in Fig. 1. In particular we note (4.13) that for $\gamma > 0$

$$\lim_{R\to\infty} (-1)^R \rho_{xx}(R) = [1/2(1+\gamma)]\{\gamma^2[1-(h\mu/J)^2]\}^{1/4} \quad \text{if } |h\mu| < J$$

$$= 0 \quad \text{if } |h\mu| \geq J. \tag{1.7}$$

The boundary of the shaded region is the circle

$$(h\mu/J)^2 + \gamma^2 = 1. \tag{1.8}$$

Inside this circle the approaches of $(-1)^R\rho_{xx}$, $(-1)^R\rho_{yy}$ and ρ_{zz} to their $R\to\infty$ limits contain oscillatory terms, while outside the circle the approach is monotonic. On the circle, the correlation functions are known exactly and are independent of R. The parameter λ_2 is given by

$$\lambda_2 = \{h\mu/J - [(h\mu/J)^2 - (1-\gamma^2)]^{1/2}\}/(1-\gamma). \tag{1.9}$$

We also note that these asymptotic expansions are all valid for h and γ fixed and $R\to\infty$. Thus the expansions that are valid in the various regions of the γ-h plane all break down when one goes to the lines $\gamma=0$, $h\mu=J$, and $(h\mu/J)^2+\gamma^2=1$ bounding these regions. In Fig. 1, the effect of the symmetry conditions (1.2) and (1.3) is quite clear. If neither of the symmetry conditions holds, all correlations approach their $R\to\infty$ limits exponentially rapidly.

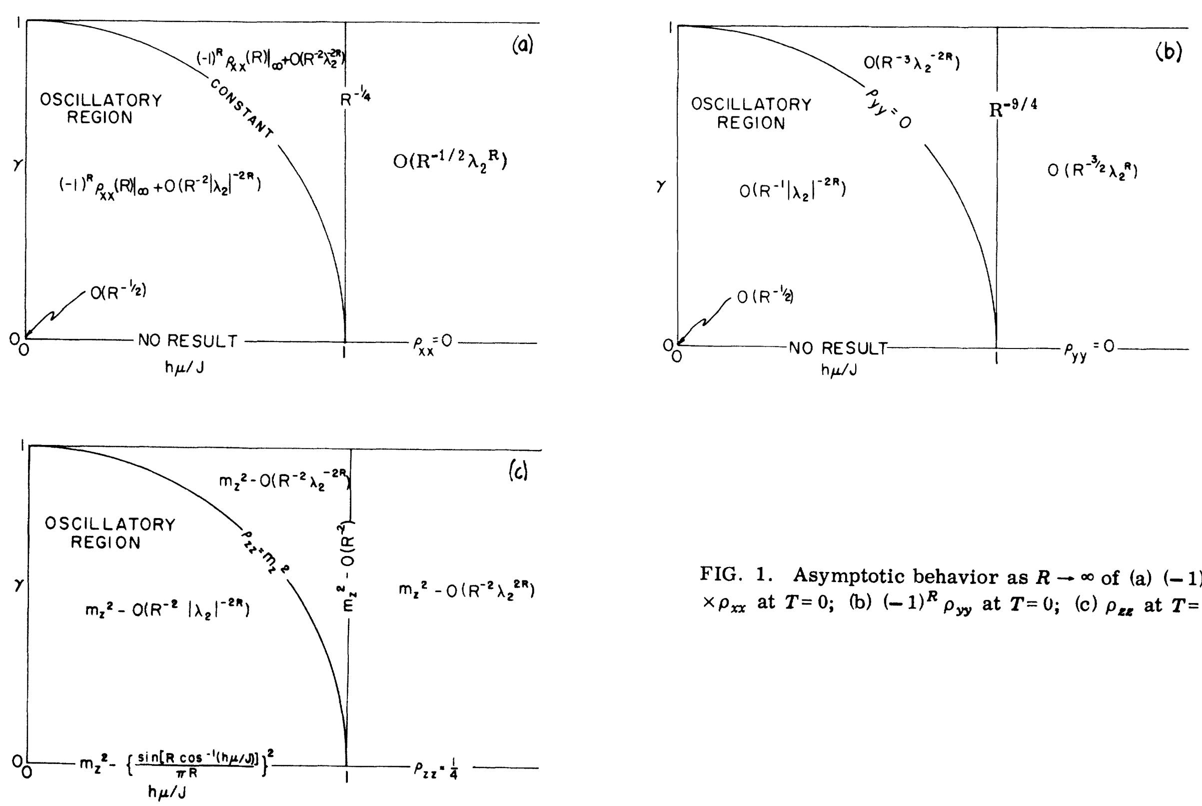

FIG. 1. Asymptotic behavior as $R\to\infty$ of (a) $(-1)^R \times\rho_{xx}$ at $T=0$; (b) $(-1)^R\rho_{yy}$ at $T=0$; (c) ρ_{zz} at $T=0$.

However, in all cases where we have results, when one of these symmetry conditions does hold, all the correlation functions approach their limiting values as some power of R. It is most tempting to speculate that this distinction between the symmetric and asymmetric cases is a general property of spin Hamiltonians and is not just a special property of the XY model in one dimension.

II. FORMULATION

A. General

Consider the XY Hamiltonian[1] [(I 2. 1) with $J=1$, $\mu=1$]:

$$H=\sum_{j=1}^{N}\left[(1+\gamma)S_j^xS_{j+1}^x+(1-\gamma)S_j^yS_{j+1}^y-h(t)S_j^z\right]. \qquad (2.1)$$

The three instantaneous correlation functions are defined as

$$\rho_{vv}(R,t)=\langle S_j^v(t)S_{j+R}^v(t)\rangle,\quad v=x,\ y,\ z\ . \qquad (2.2)$$

(We assume as in Paper I that for $t\leq 0$ the system is in equilibrium at temperature T.)

LSM show that (LSM 2. 31)

$$\rho_{xx}(l-m)=\tfrac{1}{4}\langle B_lA_{l+1}B_{l+1}\cdots A_{m-1}B_{m-1}A_m\rangle, \qquad (2.3a)$$

$$\rho_{yy}(l-m)=\tfrac{1}{4}(-1)^{l-m}\langle A_lB_{l+1}A_{l+1}B_{l+2}\cdots B_{m-1}A_{m-1}B_m\rangle, \qquad (2.3b)$$

$$\rho_{zz}(l-m)=\tfrac{1}{4}\langle A_lB_lA_mB_m\rangle, \qquad (2.3c)$$

where A_i, B_i are given in terms of c_i, $c_i^\dagger$ (I 2. 3) as

$$A_i=c_i^\dagger+c_i, \qquad (2.4a)$$

$$B_i=c_i^\dagger-c_i. \qquad (2.4b)$$

The correlation functions $\rho^{vv}(R)$ are given as expectation values of products of fermion operators. Caianello and Fubini[6] show, by use of the Wick theorem in quantum field theory, that expressions of this nature can be expressed as Pfaffians. In particular, we have

$$\rho_{xx}(m-l)=\tfrac{1}{4}pf\left|\begin{array}{cccccccc}
S_{l,l+1} & S_{l,l+2} & \cdots & S_{l,m-1} & G_{l,l+1} & G_{l,l+2} & \cdots & G_{l,m}\\
 & \cdot & & \cdot & \cdot & & & \cdot\\
 & \cdot & & \cdot & \cdot & & & \cdot\\
 & \cdot & & \cdot & \cdot & & & \cdot\\
 & & & S_{m-2,m-1} & G_{m-2,l+1} & G_{m-2,l+2} & \cdots & G_{m-2,m}\\
 & & & & G_{m-1,l+1} & G_{m-1,l+2} & \cdots & G_{m-1,m}\\
 & & & & & Q_{l+1,l+2} & \cdots & Q_{l+1,m}\\
 & & & & & \cdot & & \cdot\\
 & & & & & & \cdot & \cdot\\
 & & & & & & \cdot & \cdot\\
 & & & & & & & Q_{m-1,m}
\end{array}\right|, \qquad (2.5)$$

where we use the definitions

$$S_{l,m}=\langle B_lB_m\rangle=S(m-l), \qquad (2.6a)$$

$$Q_{l,m}=\langle A_lA_m\rangle=Q(m-p), \qquad (2.6b)$$

$$G_{l,m}=\langle B_lA_m\rangle=G(m-l)\ . \qquad (2.6c)$$

An important simplification occurs in (2. 5) when we restrict our attention to the equilibrium case. Then one can show that

$$Q_{l,m}=S_{l,m}=0, \qquad (2.7)$$

and (2. 5) reduces to the Toeplitz determinant

$$\rho_{xx}(R)=\frac{1}{4}\left|\begin{array}{cccc}
G_{-1} & G_{-2} & \cdots & G_{-R}\\
G_0 & G_{-1} & \cdots & G_{-R+1}\\
\cdot & \cdot & \cdots & \cdot\\
\cdot & \cdot & \cdots & \cdot\\
\cdot & \cdot & \cdots & \cdot\\
G_{R-2} & G_{R-3} & \cdot\cdot & G_{-1}
\end{array}\right|. \qquad (2.8a)$$

Similarly, in thermal equilibrium, we have

$$\rho_{yy}=\frac{1}{4}\begin{vmatrix} G_1 & G_0 & \cdots & G_{-R+2} \\ G_2 & G_1 & \cdots & G_{-R+3} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ G_R & G_{R-1} & & G_1 \end{vmatrix}, \qquad (2.8\text{b})$$

$$\rho_{zz}=m_z^2-\tfrac{1}{4}G_R G_{-R}, \qquad (2.8\text{c})$$

where m_z is the z-direction magnetization, given in general by (I4.7).

One can immediately see that it is much easier to deal with ρ_{zz}, since evaluating ρ_{zz} involves simple products. On the other hand, ρ_{xx} and ρ_{yy} are Toeplitz determinants, whose asymptotic properties for large R were studied in general by Wu,[7] Szegö,[8] Kac,[9] Hartwig and Fisher,[10] and others.[11] These results have been applied to the zero-field case of ρ_{xx} and ρ_{yy} by one of the authors.[3]

B. Method

To study the asymptotic properties of ρ_{xx} and ρ_{yy}, we extensively use Szegö's theorem[7–10]: Let C_R be an $R\times R$ Toeplitz determinant

$$C_R=\begin{vmatrix} c_0 & c_{-1} & \cdots & c_{-R+1} \\ c_1 & c_0 & & c_{-R+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_{R-1} & c_{R-2} & \cdots & c_0 \end{vmatrix}, \qquad (2.9)$$

where

$$c_n=(1/2\pi)\int_{-\pi}^{\pi} e^{-in\phi}\,c(e^{i\phi})\,d\phi \qquad (2.10)$$

and $c(e^{i\phi})$ is the generating function.

If (i) $\sum_{n=-\infty}^{\infty}|c_n|<\infty$, (ii) $\sum_{n=-\infty}^{\infty}|n|\,|c_n|^2\infty$, (iii) $c(e^{i\phi})\neq 0$ on the unit circle, and (iv) $\ln c(e^{i\phi})$ is a periodic function with period 2π {this means that the winding index is zero, i.e., $\text{Ind}[c(e^{i\phi})]=0$}, then the leading asymptotic value of C_R is given as

$$C_R \doteq e^{k_0 R}\exp\Big(\sum_{n=1}^{\infty} n\,k_n k_{-n}\Big) \quad \text{as } R\sim\infty, \qquad (2.11)$$

where $\doteq$ means "asymptotically equivalent," as first introduced by Wu,[7] and k_m is the mth Fourier component of $\ln c(e^{i\phi})$, namely,

$$\ln c(e^{i\phi})=\sum_{n=-\infty}^{+\infty} k_n\,e^{in\phi}. \qquad (2.12)$$

In the process of investigation, we deal with determinants that violate condition (iv). When Ind $\times[c(e^{i\phi})]=\pm 1$, we use the Wiener-Hopf method introduced by Wu,[7] and when $\text{Ind}[c(e^{i\phi})]=-2$, its extension forms Theorem 4 of Hartwig and Fisher.[10]

C. Explicit Derivation of G_R (2.6c)

In this section we calculate G_R explicitly. We will confine our attention to a step-function magnetic field $h(t)$, namely,

$$h(t)=\begin{cases} a, & t\leq 0 \\ b, & t>0\,. \end{cases} \qquad (2.13)$$

It is clear that our particular $h(t)$ introduces no loss of generality in the following method of computing G_R.

G_R can be rewritten in a more convenient form, using (I2.5), as

$$\begin{aligned} G_R=\frac{1}{N^2}\sum_{p,q,j}\Big\langle &\exp\Big(\frac{2\pi i}{N}\,[j(p+q)+R_q]\Big)a_p^\dagger a_q^\dagger \\ &+\exp\Big(\frac{2\pi i}{N}\,[j(-p+q)+R_q]\Big)a_p a_q^\dagger \\ &-\exp\Big(\frac{2\pi i}{N}\,[j(p-q)-R_q]\Big)a_p^\dagger a_q \\ &-\exp\Big(\frac{2\pi i}{N}\,[j(-p-q)-R_q]\Big)a_p a_q\Big\rangle. \end{aligned} \qquad (2.14)$$

Elementary manipulation of (2.14) yields

$$G_R=\frac{1}{N}\sum_{p>0}^{N/2}\Big[-2\cos\Big(\frac{2\pi p}{N}R\Big)\langle a_p^\dagger a_p+a_{-p}^\dagger a_{-p}-1\rangle +2\sin\Big(\frac{2\pi p}{N}R\Big)\,\text{Re}[2i\rho_{21}^p(t)]\Big]. \qquad (2.15)$$

Here $\rho_{21}^p(t)$ is the appropriate element of the pth sub-block of the density matrix, given explicitly by (I5.5), $\langle a_p^\dagger a_p+a_{-p}^\dagger a_{-p}-1\rangle$ is obtained by looking at (I4.3) and (I4.6). Therefore, we may take the thermodynamic limit $N\to\infty$ and obtain

$$G(R,t)=-\frac{1}{\pi}\int_0^{\pi} d\phi\,\cos(\phi R)\,\frac{\tanh(\frac{1}{2}\beta\Lambda(a))}{\Lambda(a)\Lambda^2(b)}\,\{[\gamma^2\sin^2\phi+(\cos\phi-a)(\cos\phi-b)]$$

$$\times(\cos\phi - b) - (a-b)\gamma^2\sin^2\phi\cos(2\Lambda(b)t)\} + (\gamma/\pi)\int_0^\pi d\phi\,\sin(\phi R)$$

$$\times\sin\phi\,\frac{\tanh(\frac{1}{2}\beta\Lambda(a))}{\Lambda(a)\Lambda^2(b)}\,[\gamma^2\sin^2\phi + (\cos\phi - a)(\cos\phi - b) + (a-b)(\cos\phi - b)\,\cos(2\Lambda(b)t)]\,. \qquad (2.16)$$

The equilibrium G_R can be obtained directly by setting $a=b$, namely,

$$G_R = -\frac{1}{\pi}\int_0^\pi d\phi\cos\phi R\,(\cos\phi - a)\,\frac{\tanh(\frac{1}{2}\beta\Lambda(a))}{\Lambda(a)} + \frac{\gamma}{\pi}\int_0^\pi d\phi\,\sin\phi R\sin\phi\,\frac{\tanh(\frac{1}{2}\beta\Lambda(a))}{\Lambda(a)}\,, \qquad (2.17)$$

where $\Lambda(a)$ is given by (I 3.14) as

$$\Lambda(a) = [\gamma^2\sin^2\phi + (a-\cos\phi)^2]^{1/2}. \qquad (2.18)$$

Note that (2.17) can also be obtained for (2.16) by setting $t=0$, as it must be.

From now on, G_R will be the equilibrium value of (2.17).

Using the methods of this section, one can easily obtain $Q_{l,m}$ and $S_{l,m}$. Both vanish for $t=0$ and $t=\infty$. They do not vanish in general for intermediate times. In other words, we get block Toeplitz determinants. The authors are unable to give a general treatment to these determinants. However, since (2.7) holds for $t=\infty$, we are able to study the ergodic properties of the correlation functions (2.2). This study will be published as the next paper in the present series.

III. EVALUATION OF ρ_{xx} FOR FINITE β AND $0<\gamma\le 1$

A. General

We devote this section to the evaluation of ρ_{xx} given by

$$\rho_{xx} = \frac{(-1)^R}{4}\begin{vmatrix} \bar G_{-1} & \bar G_{-2} & \cdots & \bar G_{-R} \\ \bar G_0 & \bar G_{-1} & \cdots & \bar G_{-R+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \bar G_{R-2} & \bar G_{R-3} & \cdots & \bar G_{-1}\end{vmatrix}\,, \qquad (3.1)$$

where from (2.8a) and (2.17) $\bar G_R$ is

$$\bar G_R = -(1/2\pi)\int_{-\pi}^{\pi}\tfrac{1}{2}\beta e^{-i\phi R}\,T(e^{i\phi}) \times(-\cos\phi + a + i\gamma\sin\phi)\,d\phi\,, \qquad (3.2)$$

and we have defined

$$T(e^{i\phi}) = \tanh(\tfrac{1}{2}\beta\Lambda(a))/\tfrac{1}{2}\beta\Lambda(a)\,. \qquad (3.3)$$

Let

$$\bar G_R = c_{R+1}\,, \qquad (3.4)$$

so that

$$c_R = -(1/2\pi)\int_{-\pi}^{\pi}\tfrac{1}{2}\beta e^{-i\phi R}\,T(e^{i\phi}) \times[e^{i\phi}(-\cos\phi + a + i\gamma\sin\phi)]\,d\phi. \qquad (3.5)$$

Then we obtain

$$\rho_{xx}(a,R) = (-1)^R\frac{1}{4}\begin{vmatrix} c_0 & c_{-1} & \cdots & c_{-R+1} \\ c_1 & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & c_0 c_{-1} \\ c_{R-1} & & & c_1 c_0\end{vmatrix}\,. \qquad (3.6)$$

We wish to evaluate (3.6) asymptotically for large R, by use of Szegö's theorem. Define

$$P_x(e^{i\phi}) = -e^{i\phi}(-\cos\phi + a + i\gamma\sin\phi) = \tfrac{1}{2}(1+\gamma)(1-\lambda_1^{-1}e^{i\phi})(1-\lambda_2^{-1}e^{i\phi})\,, \qquad (3.7)$$

where

$$\lambda_{1,2} = \{a \pm [a^2 - (1-\gamma^2)]^{1/2}\}/(1-\gamma)\,, \qquad (3.8)$$

$$\lambda_1 = \{a + [a^2 - (1-\gamma^2)]^{1/2}\}/(1-\gamma)\,, \qquad (3.8a)$$

$$\lambda_2 = \{a - [a^2 - (1-\gamma^2)]^{1/2}\}/(1-\gamma)\,, \qquad (3.8b)$$

and the square root is defined to be positive for $a^2 > 1-\gamma^2$. In Fig. 2 we show the motion of the two zeros of $P_x(e^{i\phi})$. These locations determine if condition (iv) of Szegö's theorem holds.

In Fig. 2, we distinguish three regions: (i) $a<1$, where $P_x(e^{i\phi}) \ne 0$ for $\phi\in[-\pi,\pi]$ and $\mathrm{Ind}[P_x(e^{i\phi})]=0$; and Szegö's theorem can be applied in a straightforward fashion. (ii) $a>1$, where $P_x(e^{i\phi})\ne 0$ for $\phi\in[-\pi,\pi]$ but $\mathrm{Ind}[P_x(e^{i\phi})]=1$. Therefore, to apply Szegö's theorem we "shift" the determinant once,

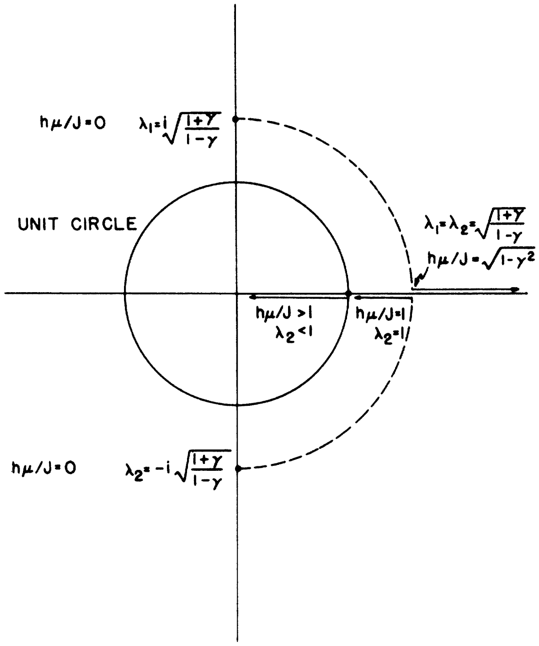

FIG. 2. Dependence of λ_1 and λ_2 on a.

as first studied by Wu.[7] (iii) $a=1$, pathological case, since $P_x(e^{i\phi})=0$ at $\phi=0$ and therefore Szegö's theorem cannot be used.

B. $a<1$ β Finite

It is convenient to write $T(e^{i\phi})$ (3.2) as an infinite product

$$T(e^{i\phi})=\prod_{k=1}^{\infty}\frac{[1+\beta^2\Lambda^2(2k\pi)^{-2}]}{[1+\beta^2\Lambda^2(2k-1)^{-2}\pi^{-2}]}=\prod_{k=1}^{\infty}\frac{S_{2k}}{S_{2k-1}}\;, \tag{3.9}$$

where

$$S_k(e^{i\phi})=(\tfrac{1}{2}\beta)^2(1-\gamma^2)(k\pi)^{-2}f_k^{-1}g_k^{-1}(1-e^{-i\phi}f_k)$$

$$\times(1-e^{i\phi}f_k)(1-e^{-i\phi}g_k)(1-e^{i\phi}g_k)\,, \tag{3.10}$$

$$W_k=\{\gamma^2a^2-(1-\gamma^2)[\gamma^2+(k\pi)^2\beta^{-2}]\}^{1/2}\,, \tag{3.11}$$

and f_k and g_k are two solutions of the quartic equation

$$(k\pi/\beta)^2+\Lambda^2=0, \tag{3.12}$$

where Λ is given by (2.18). More explicitly one obtains

$$f_k=\frac{a+W_k}{1-\gamma^2}-\left[\left(\frac{a+W_k}{1-\gamma^2}\right)^2-1\right]^{1/2}, \qquad |f_k|\le 1 \tag{3.13a}$$

$$g_k=\frac{a-W_k}{1-\gamma^2}-\left[\left(\frac{a-W_k}{1-\gamma^2}\right)^2-1\right]^{1/2}, \qquad |g_k|\le 1. \tag{3.13b}$$

Note that when $k=0$, (3.13) reduces to $f_0^{-1}=\lambda_1$ and

$$g_0^{-1}=\begin{cases}\lambda_2^{-1} & \text{if } a\ge 1,\\ \lambda_2 & \text{if } a\le 1.\end{cases} \tag{3.13c}$$

By use of (3.5) and (3.8), we may write (2.12) as

$$\ln[\tfrac{1}{4}\beta(1+\gamma)(1-\lambda_1^{-1}e^{i\phi})(1-\lambda_2^{-1}e^{i\phi})\,T(e^{i\phi})]$$

$$=\sum_{m=-\infty}^{+\infty}k_m e^{im\phi}\;. \tag{3.14}$$

Elementary algebra yields

$$k_0=\ln[\tfrac{1}{4}\beta(1+\gamma)]+\sum_{l=1}^{\infty}(-1)^l\ln[\tfrac{1}{4}\beta^2(1-\gamma^2)(k\pi)^{-2}f_l^{-1}g_l^{-1}]. \tag{3.15}$$

Define XY as

$$\exp\left(\sum_{m=1}^{\infty}mk_mk_{-m}\right)=XY\,, \tag{3.16}$$

where

$$X=\prod_{l=1}^{\infty}\frac{(1-\lambda_1^{-1}f_{2l-1})(1-\lambda_1^{-1}g_{2l-1})(1-\lambda_2^{-1}f_{2l-1})(1-\lambda_2^{-1}g_{2l-1})}{(1-\lambda_1^{-1}f_{2l})(1-\lambda_1^{-1}g_{2l})(1-\lambda_2^{-1}f_{2l})(1-\lambda_2^{-1}g_{2l})}\;, \tag{3.17}$$

$$Y=\prod_{l,l'}\frac{(1-f_{2l'}f_{2l-1})(1-f_{2l}f_{2l'-1})(1-g_{2l'}g_{2l-1})(1-g_{2l}g_{2l'-1})}{(1-f_{2l}f_{2l'})(1-f_{2l-1}f_{2l'-1})(1-g_{2l}g_{2l'})(1-g_{2l-1}g_{2l'-1})}\;\frac{(1-f_{2l'}g_{2l-1})(1-f_{2l}g_{2l'-1})(1-g_{2l'}f_{2l-1})(1-g_{2l}f_{2l'-1})}{(1-f_{2l}g_{2l'})(1-f_{2l'}g_{2l})(1-f_{2l-1}g_{2l'-1})(1-f_{2l-1}g_{2l-1})} \tag{3.18}$$

Then combining (3.15), (3.17), (3.18), and (2.11), we obtain asymptotically, for large R and $a<1$,

$$\rho_{xx}\doteq\tfrac{1}{4}(-1)^Re^{Rk_0}XY \qquad \text{as } R\to\infty\,. \tag{3.19}$$

C. High-T Expansion of ρ_{xx} $(a<1)$

The high-temperature asymptotic expansions of k_0 (3.15) and XY (3.16) are straightforward but tedious.

Let

$$x=k\pi/\beta, \quad (3.20)$$

then, $f(x)$ of (3.13) becomes (for large x)

$$f(x)\sim -i(1-\gamma^2)^{-1/2}x(z_2x^{-2}+iz_3x^{-3}+z_4x^{-4}), \quad (3.21)$$

where we use the definitions

$$\theta^2=\gamma^2[a^2-(1-\gamma^2)]/(1-\gamma^2), \quad (3.22a)$$

$$\eta=-2a(1-\gamma^2)^{-1/2}, \quad (3.22b)$$

$$\delta=-1+a^2(1+\gamma^2)/(1-\gamma^2), \quad (3.22c)$$

$$z_2=[\tfrac{1}{2}\theta^2-\tfrac{1}{2}\delta+\tfrac{1}{8}\eta^2], \quad (3.22d)$$

$$z_3=[-\tfrac{1}{4}\eta\theta^2-\tfrac{1}{4}\delta\eta+\tfrac{1}{16}\eta^3], \quad (3.22e)$$

$$z_4=\tfrac{1}{8}\theta^4-\tfrac{1}{8}\delta^2-\tfrac{1}{8}\theta^2\eta^2-\tfrac{5}{128}\eta^4+\tfrac{3}{16}\eta^2\delta. \quad (3.22f)$$

Using (3.21) and (3.22) in (3.15), we obtain

$$e^{k_0}\simeq\tfrac{1}{4}\beta(1+\gamma)\left(\left|\frac{z_3^2}{z_2^2}+2\frac{z_4}{z_2}\right|\beta\right)\cot\left(\left|\frac{z_3^2}{z_2^2}+2\frac{z_4}{z_2}\right|\beta\right)$$

$$\simeq\left[\frac{1}{4}\beta(1+\gamma)\left(1-\frac{1}{12}\beta^2\left|\frac{z_3^2}{z_2^2}+2\frac{z_4}{z_2}\right|+O(\beta^4)\right)\right]. \quad (3.23)$$

In the same manner we obtain

$$X_i\simeq y_i\cot y_i\simeq 1-\tfrac{1}{3}y_i^2, \quad i=1,2 \quad (3.24)$$

$$y_i=\tfrac{1}{2}\beta[2z_3(1-\gamma^2)^{-1/2}\lambda_i^{-1}-z_2^2(1-\gamma^2)^{-1}\lambda_i^{-2}]^{1/2}, \quad (3.25)$$

$$Y\simeq 1-Y_1^2, \quad (3.26)$$

where λ_i are given by (3.8) and Y_1 by

$$Y_1=\tfrac{1}{12}\beta^2[4(1-\gamma^2)z_3^2(\tfrac{1}{2}\beta)^2-(\tfrac{1}{2}\beta)^4z_3^4(1-\gamma^2)^{-2}]. \quad (3.27)$$

Combining (3.19), (3.23), (3.24), and (3.26), we obtain the desired expansion

$$\rho_{xx}\cong\tfrac{1}{4}(-1)^R\left[\frac{1}{4}\beta(1+\gamma)\left(1-\frac{1}{12}\left|\frac{z_3^2}{z_2^2}+\frac{z_4}{z_2}\right|\beta^2+O(\beta^4)\right)\right]^R$$

$$\times\left(1-\frac{1}{3}(y_1^2+y_2^2)+O(\beta^4)\right). \quad (3.28)$$

Taking the limit $a\to 0$ of (3.28) [and remembering that in M the ferromagnetic case was considered instead of the antiferromagnetic $\vec{\sigma}$ matrices, instead of $\vec{S}$ matrices, and $\beta_M=\tfrac{1}{4}\beta(1+\gamma)$], we obtain the known zero-field result (M 3.15).

D. Evaluation of k_0 for Low T

In this subsection we evaluate (3.15) for large β. Unfortunately, the method used for high temperature fails, and we proceed as in M (Sec. III A) by converting the sum into a contour integral as follows:

$$k_0=\ln\tfrac{1}{4}\beta(1+\gamma)+(1/2i)\int_c\csc(\pi k)$$

$$\times\ln[(\tfrac{1}{2}\beta\pi^{-1}k^{-1})^2f_k^{-1}g_k^{-1}(1-\gamma^2)]dk. \quad (3.29)$$

This is more conveniently written as

$$k_0=-(\beta/2\pi i)\int_{-\infty}^{\infty}dt\,\mathrm{csch}\beta t\ln(f(t)g(t)). \quad (3.30)$$

Equation (3.30) was obtained from (3.29) in the same way (M 3.20) was obtained from (M 3.8). It is clear that only the imaginary part of the logarithm in (3.30) will contribute a nonvanishing answer.

Detailed analysis of the analytical properties of $g(t)$ and $f(t)$ for $a<1$ yields division into two regions, namely,

$$a>1-\gamma^2, \quad (3.31a)$$

$$a<1-\gamma^2. \quad (3.31b)$$

This division to regions is the *same* as in the time evolution of the magnetization (Paper I). It should be noted here that (3.31a) is further divided into two regions. For $T=0$ we have

$$a^2>1-\gamma^2, \quad (3.32a)$$

$$a^2<1-\gamma^2. \quad (3.32b)$$

However for finite β we have

$$\gamma^2[a^2-(1-\gamma^2)]>(1-\gamma^2)\pi^2\beta^{-2}, \quad (3.32c)$$

$$\gamma^2[a^2-(1-\gamma^2)]<(1-\gamma^2)\pi^2\beta^{-2}. \quad (3.32d)$$

Let $1-\gamma^2<a<1$. By use of the dictionary given in the Appendix, (3.30) becomes

$$k_0=-\beta\pi^{-1}\int_{1-a}^{1+a}dt\,\mathrm{csch}\beta t\arctan\left(\frac{\{(1-\gamma^2)^2-(a-\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)t^2\}^{1/2})^2\}^{1/2}}{a-\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)t^2\}^{1/2}}\right)+\ln\tanh(\tfrac{1}{2}\beta(1+a)). \quad (3.33)$$

The second term is exponentially small compared to the first one when $\beta\to\infty$. Terms of this kind will be consistently neglected throughout this paper.

We obtain the asymptotic expansion of the integral in (3.33) by setting $\operatorname{csch}\beta t \sim 2e^{-\beta t}$, changing the variable $t=1-a+s$, and replacing the upper limit by ∞. One obtains

$$k_0 \doteq -2\beta\pi^{-1}e^{-\beta(1-a)}\int_0^\infty ds\, e^{-\beta s}\arctan\left(\frac{(1-\gamma^2)^2}{(a-\{\gamma^2[a^2-(1-\gamma^2)]+(1+\gamma^2)(1-a+s)^2\}^{1/2})^2}-1\right)^{1/2}. \quad (3.34)$$

There are many ways to obtain the asymptotic expansion of (3.34). All are tedious. Using

$$\arctan z = [z/(1+z^2)^{1/2}]\,{}_2F_1\{\tfrac{1}{2};\ \tfrac{1}{2};\ \tfrac{3}{2};\ [z^2/(1+z^2)]\} \quad (3.35)$$

and defining

$$u(s) \equiv \left[1-\left(\frac{a-\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)(1-a+s)^2\}^{1/2}}{1-\gamma^2}\right)^2\right]^{1/2}, \quad (3.36)$$

we can write, (using the first few terms of $u(s)\ {}_2F_1(\frac{1}{2};\frac{1}{2};\frac{3}{2};u^2(s))$,

$$k_0 \doteq -2\beta\pi^{-1}e^{-\beta(1-a)}\int_0^\infty ds\, e^{-\beta s} \times [u(s)+\tfrac{1}{6}u^3(s)+\tfrac{3}{40}u^5(s)+\cdots]. \quad (3.37)$$

This is useful in obtaining the leading terms of the asymptotic series since the expansion of $u^2(s)$ is given by

$$u^2(s) = L_1 s + L_2 s + L_3 s^3 + \cdots, \quad (3.38)$$

where the first few constant L_i are explicitly given as

$$L_1 = 2(1-a)/[a-(1-\gamma^2)], \quad (3.39a)$$

$$L_2 = [\gamma^2a^2-(1-a)(1-\gamma^2)]/[a-(1-\gamma^2)]^2, \quad (3.39b)$$

$$L_3 = (1-a) - \frac{1}{[a-(1-\gamma^2)]^2} + \frac{(1-\gamma^2)(1-a)^2}{[a-(1-\gamma^2)]^4} - \frac{(1-\gamma^2)^2(1-a)^2}{[a-(1-\gamma^2)]^5}. \quad (3.39c)$$

For $L_1 \neq 0$ we readily obtain

$$u(s) \simeq L_1^{1/2}s^{1/2}[1+\tfrac{1}{2}L_2L_1^{-1}s + (\tfrac{1}{2}L_3L_1^{-1}-\tfrac{1}{8}L_2^2L_1^{-2})s^2+O(s^3)]. \quad (3.40)$$

Note that $L_1=0$ when $a=1$, but $L_2(a=1)\neq 0$. Therefore, we finally obtain the low-temperature expansion of k_0 for $1-\gamma^2<a<1$ to be

$$k_0 \doteq -2\pi^{-1}\beta^{-1/2}e^{-\beta(1-a)}\{L_1^{1/2}\Gamma(\tfrac{3}{2}) + (\tfrac{1}{2}L_1^{-1/2}L_2+\tfrac{1}{6}L_1^{3/2})\Gamma(\tfrac{5}{2})\beta^{-1} + [\tfrac{1}{2}L_3L_1^{-1/2}-\tfrac{1}{8}L_2^2L_1^{-3/2}+\tfrac{1}{4}L_1^{1/2}L_2+\tfrac{3}{40}L_1^{5/2}] \times \Gamma(\tfrac{7}{2})\beta^{-2}+O(\beta^{-3})\}, \quad (3.41)$$

and the limiting value of k_0 for $a=1$ is

$$\lim_{a\to 1}k_0(a) = -4\pi\gamma^{-2}\beta^{-1}. \quad (3.42)$$

Note that (3.41) breaks for $a=1-\gamma^2$. For the case $a>1$, k_0 is immediately obtained from (3.41) by replacing $1-a$ by $a-1$.

We now turn to the region $a<1-\gamma^2$. In this case k_0 is given as a sum of two integrals,

$$k_0 = M(|a-(1-\gamma^2)|) - \beta\pi^{-1}\int_\theta^{1-a} dt\,\operatorname{csch}\beta t \times \operatorname{Im}\ln\{[x(t^2-\theta^2)-i(1-x^2(t^2-\theta^2))^{1/2}] \times [y(t^2-\theta^2)+i(1-y^2(t^2-\theta^2))^{1/2}]\}, \quad (3.43)$$

where

$$\theta = \gamma[1-a^2/(1-\gamma^2)]^{1/2}, \quad (3.44)$$

$$d = a/(1-\gamma^2), \quad (3.45)$$

$$x(t^2-\theta^2) = d + (1-\gamma^2)^{-1/2}(t^2-\theta^2)^{1/2}, \quad (3.46a)$$

$$y(t^2-\theta^2) = d - (1-\gamma^2)^{-1/2}(t^2-\theta^2)^{1/2}. \quad (3.46b)$$

The square root is defined positive for $\mathrm{Re}\,t>\theta$ and $M(|a-(1-\gamma^2)|)$ is given by (3.41), where $|a-(1-\gamma^2)|$ is substituted instead of $a-(1-\gamma^2)$. Since $1-a>\theta$ in the region $a<1-\gamma^2$, it is clear that $M(|a-(1-\gamma^2)|)$ is exponentially small compared with the integral in (3.43), and may be neglected.

We make the change of variable in the integral (3.43) $s = t-\theta$ and obtain

$$k_0 \doteq -2\beta\pi^{-1}e^{-\beta\theta}\int_0^\infty ds\, e^{-\beta s} \times \arctan\left(\frac{x(1-y^2)^{1/2}-y(1-x^2)^{1/2}}{xy+(1-y^2)^{1/2}(1-x^2)^{1/2}}\right), \tag{3.47}$$

where $x(s(s+2\theta))$ and $y(s(s+2\theta))$ are given by (3.46).

Finally, after sufficient labor one obtains the result

$$k_0 \doteq -2\pi^{-1}\beta^{-1/2}(2\theta)^{1/2}d^{-1}(1-\gamma^2)^{-1/2}e^{-\beta\theta} \times \{\Gamma(\tfrac{1}{2})+[\tfrac{3}{4}\theta^{-1}+(1-\lambda^2)^{-1}d^{-2}2\theta]\Gamma(\tfrac{3}{2})\beta^{-1} + [\tfrac{5}{2}(1-\gamma^2)^{-1}d^{-2}-\tfrac{5}{32}\theta^{-2}(1-\gamma^2)^{-2}d^{-4}] \times \Gamma(\tfrac{5}{2})\beta^{-2}+O(\beta^{-2})\}. \tag{3.48}$$

This completes the evaluation of k_0 for all a.

E. Evaluation of XY

To evaluate XY given by (3.17) and (3.18), we proceed in the same method given by M, Sec. (3.B). We convert the double product into a double contour integral. The single product X cancels the contributions to Y from the poles at $l=0$, $l'=0$. We therefore obtain[12]

$$XY \doteq \frac{2}{1+\gamma}[\gamma^2(1-a^2)]^{1/4}\exp\left[\beta^2(2\pi)^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dt\,dt'\,\operatorname{csch}\beta t\,\operatorname{csch}\beta t'\ln\left(\frac{G(t,t')G(0,0)}{G(t,0)G(0,t')}\right)\right], \tag{3.49}$$

where

$$G(t,t') = [1-f(t)f(t')][1-g(t)g(t')][1-f(t)g(t')][1-f(t')g(t)]. \tag{3.50}$$

Define the function $\mu(z)$ to be

$$\mu(z) = \begin{cases} 1 & \text{if } z>0 \\ 0 & \text{if } z\le 0. \end{cases} \tag{3.51}$$

Equation (3.49) becomes

$$XY \doteq \frac{2}{1+\gamma}[\gamma^2(1-a^2)]^{1/4}\exp\left(\beta^2\pi^{-2}\int_{1-a}^{1+a}\int_{1-a}^{1+a} dt\,dt'\,[\operatorname{csch}\beta t\,\operatorname{csch}\beta t'\ln A(t,t')]\,\mu(1-\gamma^2-a)\beta^2\pi^2 \times\int_\theta^{1-a}\int_\theta^{1-a} dt\,dt'\,[\operatorname{csch}\beta t\,\operatorname{csch}\beta t'\,\ln B(t,t')]\right). \tag{3.52}$$

$A(t,t')$ and $B(t,t')$ have the same formal expression in terms of $f(t)$. However, it should be emphasized that $f(t)$ in each case is taken from the appropriate range (see Appendix). The function $A(t,t')$ is given formally as

$$A(t,t') = \frac{|[1-f(t)f(t')][1-f^*(-t)f^*(-t')][1-f(t)f^*(-t')][1-f(t')f^*(-t)]|^2}{|[1-f(-t)f(t')][1-f(t)f(-t')][1-f(t)f^*(t')][1-f(-t)f^*(-t')]|^2}. \tag{3.53}$$

If $a>1-\gamma^2$, the second term vanishes identically, and if $a<1-\gamma^2$, the first integral is exponentially small and does not contribute to an asymptotic expansion.

The final expansion of (3.52) is performed in the same method used before: (i) making the appropriate change of variables so the lower limit of integration becomes 0, and replacing the upper limit by ∞; (ii) replacing $\operatorname{csch}\beta t$ by $2e^{-\beta t}$, expanding the rest of the integrand, and integrating term by term. Using the expansions

$$\ln A(s,s') = \ln\left(\frac{s^{1/2}+s'^{1/2}}{s^{1/2}-s'^{1/2}}\right)^2 + s^{1/2}s'^{1/2}\sum_{m,m'}A_{mm'}s^m s'^{m'}, \tag{3.54a}$$

$$\ln B(s,s') = \ln\left(\frac{s^{1/2}+s'^{1/2}}{s^{1/2}-s'^{1/2}}\right)^2 + s^{1/2}s'^{1/2}\sum_{mm'}B_{mm'}s^m s'^{m'}, \tag{3.54b}$$

we obtain the desired results for $1-\gamma^2 < a < 1$:

$$XY \doteq [2/(1+\gamma)][\gamma^2(1-a^2)]^{1/4}\exp\{e^{-2\beta(1-a)}[4\pi^{-1}+\beta^{-1}\pi^{-2}\sum_{mm'}A_{mm'}\tfrac{1}{2}\Gamma(2m+1)\ \tfrac{1}{2}\Gamma(2m'+1)\beta^{-(m+m')}]\} \quad (3.55a)$$

and $0 \le a < -\gamma^2$:

$$XY \doteq [2/(1+\gamma)][\gamma^2(1-a^2)]^{1/4}\exp\{e^{-2\beta\theta}[8\pi^{-1}+2\beta^{-1}\pi^{-2}\sum_{mm'}B_{mm'}\tfrac{1}{2}\Gamma(2m+1)\tfrac{1}{2}\Gamma(2m'+1)\beta^{-(m+m')}]\}. \quad (3.55b)$$

F. $\rho_{xx}(a>1)$ for Finite β

In order to evaluate $\rho_{xx}(R)$ by Wu's procedure we transpose the determinant for ρ_{xx}. The generating function for this transposed matrix is $C(\xi^{-1})$, when $C(\xi)$ is the generating function of (3.5). The index of $C(\xi^{-1})$ is -1. The correlation function then becomes

$$\rho_{xx} \doteq (-1)^R \tfrac{1}{4} e^{k_0(R+1)}(XY)\,x_R\ , \quad (3.56)$$

where k_0 is given by (3.41) with $|1-a|$, XY by (3.55a) with λ_2^{-1} replaced by λ_2 and a multiplicative factor of $(1-\lambda_2\lambda_1^{-1})$, and x_R is given in general by (Wu 2.27).

To evaluate x_R we make a Wiener-Hopf factorization of

$$[\xi^{-1}C(\xi^{-1})]^{-1} = P(\xi)\,Q(\xi^{-1})\ , \quad (3.57)$$

where P and Q are explicitly given for $a>1$ and β finite as

$$P(\xi) = e^{-k_0}(1-\lambda_2\xi)^{-1}\prod_{l=1}^{\infty}\frac{(1-\xi f_{2l-1})(1-\xi g_{2l-1})}{(1-\xi f_{2l})(1-\xi g_{2l})}\ , \quad (3.58a)$$

$$Q(\xi) = (1-\lambda_1^{-1}\xi)^{-1}\prod_{l=1}^{\infty}\frac{(1-\xi f_{2l-1})(1-\xi g_{2l-1})}{(1-\xi f_{2l})(1-\xi g_{2l})}\ . \quad (3.58b)$$

Wu shows

$$x_R \doteq (1/2\pi i)\oint_{|\xi|=1}\xi^{R-1}P(\xi^{-1})[Q(\xi)]^{-1}d\xi\ . \quad (3.59)$$

After some manipulations, including use of the infinite-product representation of x ctnh$x$, and conversion of the remaining product into a contour integral, we obtain

$$x_R \doteq (1/2\pi i)\oint \xi^{R-1}[(1-\lambda_2\xi^{-1})(1-\lambda_2\xi)]^{-1} \times(\tfrac{1}{2}\beta)^{-1}(\tfrac{1}{2}\beta\Lambda)\coth(\tfrac{1}{2}\beta\Lambda)\,d\xi$$
$$\times\exp -2\pi^{-1}\beta\int_{a-1}^{a+1}dt\,\mathrm{csch}\beta t \times\arctan\frac{\mathrm{Im}([1-\xi f(t)][1-\xi g(t)])}{\mathrm{Re}([1-\xi f(t)][1-\xi g(t)])}\ . \quad (3.60)$$

After some labor we finally obtain the desired low-temperature expansion of x_R for fixed β to be

$$x_R \doteq \lambda_2^R\,4\pi(2/\beta)f_2g_2[(1-\lambda_2^2)(1-\lambda_2^{-1}f_2) \times(1-\gamma^2)(1-\lambda_2g_2)(1-\lambda_2f_2)]^{-1} \times[\beta^{-2}(1-\lambda_2^{-1}g_2)^{-1}] \times\exp[4\pi^{-1}\beta^{-1/2}e^{-\beta(a-1)}\textstyle\sum_m\beta^{-m}A_m\Gamma(m+\tfrac{3}{2})]\ . \quad (3.61a)$$

where A_m are given by

$$\sum_{m=0}^{\infty}A_mt^{m+1/2} = \arctan\left(\frac{\mathrm{Im}\{[1-\lambda_2f(t)][1-\lambda_2g(t)]\}}{\mathrm{Re}\{[1-\lambda_2f(t)][1-\lambda_2g(t)]\}}\right)\ . \quad (3.61b)$$

Note that (3.61) is restricted by (3.32c). Since $g_2\sim\lambda_2+\mathrm{const}\ \beta^{-2}$ the term $[\beta^{-2}(1-\lambda_2^{-1}g_2)^{-1}]$ behaved as a constant so $x_R\sim\beta^{-1}$ for large β.

However, when (3.32d) holds for finite β, we have $g_2=f_2^*$, both terms contribute equally to an asymptotic series, and their sum oscillates with R. The result can be expressed as

$$x_R \doteq (\tfrac{1}{2}\beta)^{-1}\,2\mathrm{Re}\ g_2^{R-1}f_2g_2[(1-\lambda_2^2)(1-\lambda_2^{-1}f_2) \times(1-\gamma^2)(1-\lambda_2g_2)(1-\lambda_2f_2)]^{-1}[\beta^{-2}(1-\lambda_2^{-1}g_2)^{-1}] \times\exp[-4\pi^{-1}\beta^{-1/2}\textstyle\sum_m\beta^{-m}\bar{A}_m\Gamma(m+\tfrac{3}{2})]\ , \quad (3.62a)$$

where $\bar{A}_m$ are given by

$$\sum_{m=0}^{\infty}t^{m+1/2}\bar{A}_m = \arctan\left(\frac{\mathrm{Im}\{[1-g_2g(t)][1-g_2f(t)]\}}{\mathrm{Re}\{[1-g_2g(t)][1-g_2f(t)]\}}\right)\ . \quad (3.62b)$$

G. Results for ρ_{xx} for Finite Large β

For convenience and easy reference we compile the asymptotic expansions of ρ_{xx} for large but fixed β and $\gamma\ne 0$.

(i) $0<a<1-\gamma^2$. Combine (2.11), (3.48), and (3.55b),

$$\rho_{xx}(R) \doteq \tfrac{1}{4}(-1)^R[2/(1+\gamma)][\gamma^2(1-a^2)]^{1/4}\ \exp(-2\pi^{-1}R\beta^{-1/2}(2\theta)^{1/2}d^{-1}(1-\gamma^2)^{-1/2}e^{-\beta\theta}$$

$$\times \{\Gamma(\tfrac{1}{2}) + [\tfrac{3}{4}\theta^{-1} + (1-\gamma^2)^{-1} d^{-2} 2\theta]\Gamma(\tfrac{3}{2})\beta^{-1} + O(\beta^{-2})\}\}$$

$$\times \exp\{e^{-2\beta\theta}[8\pi^{-1} + 2\beta^{-1}\pi^{-2}\sum_{mm'} B_{mm'}\tfrac{1}{2}\Gamma(2m+1)\tfrac{1}{2}\Gamma(2m'+1)\beta^{-m+m'}]\}. \quad (3.63)$$

(ii) $1-\gamma^2 < a < 1$. Combine (2.11), (3.41) and (3.55a):

$$\rho_{xx}(R) \doteq \tfrac{1}{4}(-1)^R[2/(1+\gamma)][\gamma^2(1-a^2)]^{1/4}\exp\{-2\pi^{-1}R\beta^{-1/2}e^{-\beta(1-a)}[L_1^{1/2}\Gamma(\tfrac{3}{2}) + (\tfrac{1}{2}L_1^{-1/2}L_2 + \tfrac{1}{6}L_1^{3/2})\Gamma(\tfrac{5}{2})\beta^{-1} + O(\beta^{-2})]\}$$

$$\times \exp\{e^{-2\beta(1-a)}[4\pi^{-1} + \beta^{-1}\pi^{-2}\sum_{mm'} A_{mm'}\tfrac{1}{2}\Gamma(2m+1)\tfrac{1}{2}\Gamma(2m'+1)\beta^{-(m+m')}]\}\ . \quad (3.64)$$

(iii) $a > 1$ and $\gamma^2[a^2 - (1-\gamma^2)] > (1-\gamma^2)\pi^2\beta^{-2}$. Combine (3.56), (3.41), (3.61), and (3.55a):

$$\rho_{xx}(R) \doteq \tfrac{1}{4}(-1)^R C\lambda_2^R\beta^{-1}\exp\{-2\pi^{-1}R\beta^{-1/2}e^{-\beta(a-1)}[\bar{L}_1^{1/2}\Gamma(\tfrac{3}{2}) + (\tfrac{1}{2}\bar{L}_1^{-1/2}\bar{L}_2 + \tfrac{1}{6}\bar{L}_1^{3/2})\Gamma(\tfrac{5}{2})\beta^{-1} + O(\beta^{-2})]\}$$

$$\times \exp(-4\pi^{-1}\beta^{-1/2}e^{-\beta(a-1)}\textstyle\sum_m \beta^{-m}A_m\Gamma(m+\tfrac{3}{2}))$$

$$\times \exp\{e^{-2\beta(a-1)}[4\pi^{-1} + \beta^{-1}\pi^{-2}\sum_{mm'} A_{mm'}\tfrac{1}{2}\Gamma(2m+1)\tfrac{1}{2}\Gamma(2m'+1)\beta^{-(m+m')}]\}\ , \quad (3.65)$$

where C is given by

$$C = 4\pi^{-1}[\lambda_2\lambda_1^{-1}(1-\gamma^2)^{-1}(1-\lambda_2^2)^{-2}(1-\lambda_1^{-1}\lambda_2^{-1})]\,[(1-\lambda_1^{-2})(1-\lambda_2^2)(1-\lambda_1^{-1}\lambda_2)^2]^{1/4}$$

$$\times \{[\gamma^2(a^2-1+\gamma^2)]^{1/2} + [a-\gamma(a^2-1+\gamma^2)^{1/2}][(1-\gamma^2)\gamma(a^2-1+\gamma^2)]^{-1/2}$$

$$\times \frac{[a-\gamma(a^2-1+\gamma^2)^{1/2}]^2}{(1-\gamma^2)^2} - 1 \quad {}^{-1/2}\ {}^{-1}\ , \quad (3.66)$$

$$\lambda_1^{-1} = \frac{a - [a^2 - (1-\gamma^2)]^{1/2}}{1+\gamma}\ , \quad (3.67a)$$

$$\lambda_2 = \frac{a - [a^2 - (1-\gamma^2)]^{1/2}}{1-\gamma}\ . \quad (3.67b)$$

$\bar{L}_i$ is obtained from L_i of (3.34) by replacing $1-a$ by $|1-a|$.

(iv) $a > 1$, $\gamma^2[a^2 - (1-\gamma^2)] < (1-\gamma^2)\pi^2\beta^{-2}$. Combine (3.56), (3.41), (3.62), and (3.55a):

$$\rho_{xx}(R) \doteq \tfrac{1}{4}(-1)^R[(1-\lambda_1^{-2})(1-\lambda_2^2)(1-\lambda_1^{-1}\lambda_2)^2]^{1/4}(\tfrac{1}{2}\beta)^{-1}[1+O(\beta^{-2})]\ 2\,\mathrm{Re}\,[g_2^{R-1}C^1\exp(-4\pi^{-1}\beta^{-1/2}\sum_m \bar{A}_m\Gamma(m+\tfrac{3}{2})\beta^{-m})]$$

$$\times \exp\{-2\pi^{-1}R\beta^{-1/2}e^{-\beta(a-1)}[\bar{L}_1^{1/2}\Gamma(\tfrac{3}{2}) + O(\beta^{-1})]\}$$

$$\times \exp\{e^{-2\beta(a-1)}[4\pi^{-1} + \beta^{-1}\pi^{-2}\sum_{mm'} A_{mm'}\tfrac{1}{2}\Gamma(2m+1)\tfrac{1}{2}\Gamma(2m'+1)\beta^{-(m+m')}]\}\ , \quad (3.68)$$

where C^1 is obtained from C by replacing $\gamma[a^2-1+\gamma^2]^{1/2}$ by $i\gamma[a^2-1+\gamma^2]^{1/2}$.

(v) $a = 1$. In this pathological case one might consider studying the limit $a \to 1$ in (3.63) or (3.64). However both limits vanish. We return to this case in Sec. IV for zero temperature.

IV. ρ_{xx} FOR $T = 0$

This section is devoted to the ground-state correlation function $\rho_{xx}(R)$. Some of the results can be obtained from the finite-β cases by the limiting procedure $\beta \to \infty$. For the other cases, one can obtain in the zero-temperature correlations by holding $R\beta^{-2}$ fixed (as was done in M Sec. 4) or substituting $\beta = \infty$ in (3.2), (3.5), etc. In this paper we use the latter procedure.

A. Long-Range Order for $a<1$, $\gamma>0$

By looking at (3.62) and (3.63), and letting R and β be independently large, we obtain the long-range order for $a<1$:

$$\lim_{R\to\infty}\rho_{xx}(R)=(-1)^R[2(1+\gamma)]^{-1}[\gamma^2(1-a^2)]^{1/4} . \quad (4.1)$$

This long-range order vanishes at $a=1$, which is where Szegö's theorem is invalid. Setting $a=0$, one obtains the long-range order found in (M 3.36) for zero field.

Similarly, in the Ising limit $\gamma\to 1$, (4.1) specializes to[4]

$$\lim_{R\to\infty}\rho_{xx}(R)=\tfrac{1}{4}(-1)^R(1-a^2)^{1/4} . \quad (4.1')$$

B. Special Cases

For the case $a^2=1-\gamma^2$ we are able to calculate $\rho_{xx}(R)$ exactly for all R, and we find that the exact result is the long-range order (4.1):

$$\rho_{xx}=\tfrac{1}{4}(-1)^R\begin{vmatrix} 1-\alpha^2 & \alpha(1-\alpha^2) & \cdots & \alpha^{R-1}(1-\alpha^2) \\ -\alpha & 1-\alpha^2 & \cdots & \alpha^{R-2}(1-\alpha^2) \\ 0 & -\alpha & \cdots & \alpha^{R-3}(1-\alpha^2) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \alpha(1-\alpha^2) \\ 0 & 0 & \cdots & (1-\alpha^2) \end{vmatrix} , \quad (4.2)$$

where

$$\alpha=[(1-\gamma)/(1+\gamma)]^{1/2} , \quad (4.3)$$

so

$$\rho_{xx}=\tfrac{1}{4}(-1)^R 2\gamma/(1+\gamma) . \quad (4.4)$$

The determinant also gives the result for special case $a=1$ $\gamma=0$:

$$\rho_{xx}(R)=0 . \quad (4.5)$$

By the same method we find that for $a>1$ and $\gamma=0$

$$\rho_{xx}(R)=\rho_{yy}(R)=0 . \quad (4.6)$$

The case $a=1$, $\gamma\neq 0$ is very closely related to Wu's $T=T_c$, so by the use of (Wu 5.31), we obtain

$$\rho_{xx}(R)\doteq\tfrac{1}{4}(-1)^R[2\gamma/(1+\gamma)](\gamma R)^{-1/4}e^{1/4}2^{1/12}A^{-3}$$
$$\times[1+O(R^2)\cdots] , \quad (4.7)$$

where $A=1.282\,427\,130$ is Glaisher's constant.[13]

Note that when $\gamma\to 0$ and $R\to\infty$ such that γR is fixed, (4.7) vanishes, in agreement with (4.5). Note further that (4.7) is valid in the limit $\gamma\to 1$. Indeed, in this case ρ_{xx} reduces to the determinant which was evaluated exactly in Sec. 4 of Wu.

When $a<1$ and $\gamma=0$, if $T\neq 0$, the asymptotic expansion of ρ_{xx} can be studied by taking the limit $\gamma\to 0$ directly in (3.19). However it is not possible to recover the $T=0$ expansion from the resulting expression. Indeed, we have not been able to obtain the $T=0$ result at all.

C. Asymptotic Approach to (4.1) for $a<1$

When $a<1$, we have $\lambda_1>\lambda_2>1$ for $a^2>1-\gamma^2$ and $\lambda_1=\lambda_2^*$ for $a^2<1-\gamma^2$. Accordingly, the generating function $C(\xi)$ of (3.5) is conveniently written as

$$C(\xi)=\left(\frac{(1-\lambda_1^{-1}\xi)(1-\lambda_2^{-1}\xi)}{(1-\lambda_1^{-1}\xi^{-1})(1-\lambda_2^{-1}\xi^{-1})}\right)^{1/2} , \quad (4.8)$$

whose Wiener-Hopf factorization is given as

$$C(\xi)=P(\xi)^{-1}Q(\xi^{-1})^{-1} , \quad (4.9a)$$

$$P(\xi)=[(1-\lambda_1^{-1}\xi)(1-\lambda_2^{-1}\xi)]^{-1/2} , \quad (4.9b)$$

$$Q(\xi)=[(1-\lambda_1^{-1}\xi)(1-\lambda_2^{-1}\xi)]^{1/2} . \quad (4.9c)$$

Wu defines $x_0(m)$ such that $\rho_{xx}(T=0,R)$ is given as

$$\rho_{xx}\doteq\tfrac{1}{4}(-1)^R\frac{2}{1+\gamma}[\gamma^2(1-a^2)]^{1/4}\left(1+\sum_{m=R}^{\infty}[x_0(m)-1]\right) . \quad (4.10)$$

By substitution of (4.9) in (Wu 3.13) we obtain

$$\rho_{xx}\doteq\tfrac{1}{4}(-1)^R[2/(1+\gamma)][\gamma^2(1-a^2)]^{1/4}$$
$$\times[1+(2\pi)^{-2}\oint\oint d\xi\,d\eta\,\xi^R\eta^{-R}(\eta-\xi)^{-2}M(\eta,\xi)] , \quad (4.11)$$

where

$$M(\eta,\xi)=\left(\frac{(1-\lambda_1^{-1}\xi^{-1})(1-\lambda_2^{-1}\xi^{-1})(1-\lambda_1^{-1}\xi)(1-\lambda_2^{-1}\xi)}{(1-\lambda_1^{-1}\eta^{-1})(1-\lambda_2^{-1}\eta^{-1})(1-\lambda_1^{-1}\eta)(1-\lambda_2^{-1}\eta)}\right)^{1/2} . \quad (4.12)$$

Brute force (Wu's most accurate choice of words) is the only way to expand (4.11) for large R. We do not wish to tire the reader with algebraic details and feel that it is sufficient to state that (4.11) was expanded in the method used to expand (Wu 3.15).

The result for $1-\gamma^2<a^2<1$ is

$$\rho_{xx}\doteq\tfrac{1}{2}(-1)^R(1+\gamma)^{-1}[\gamma^2(1-a^2)]^{1/4}\{1+(2\pi)^{-1}R^{-2}\lambda_2^{-2R}(\lambda_2-\lambda_2^{-1})^{-2}\,[1+\tfrac{1}{2}(\lambda_1^{-1}\lambda_2^{-1}(1-\lambda_1^{-1}\lambda_2^{-1})^{-1}-7\lambda_2^{-2}(1-\lambda_2^{-2})^{-1}$$

$$- (1-\lambda_1^{-1}\lambda_2)^{-1} - \tfrac{5}{2})R^{-1} + O(R^{-2})]\} \ . \tag{4.13}$$

This expansion breaks when $\lambda_1 = \lambda_2$, i.e., $a^2 = 1-\gamma^2$, and when $a=1$. For $a^2 < 1-\gamma^2$, we obtain

$$\rho_{xx} \doteq \tfrac{1}{2}(-1)^R(1+\gamma)^{-1}[\gamma^2(1-a^2)]^{1/4}\Big(1+\pi^{-1}R^{-2}\alpha^{2R}\,\mathrm{Re}(e^{i\psi R}(\alpha^{-1}e^{-i\psi}-\alpha e^{i\psi})^{-2}$$
$$\times \{1+\tfrac{1}{2}[\alpha^2(1-\alpha^2)^{-1} - 7\alpha^2 e^{2i\psi}(1-\alpha^2 e^{2i\psi})^{-1} - (1-e^{-2i\psi})^{-1} - \tfrac{5}{2}]R^{-1} + O(R^{-2})\})\Big) \ , \tag{4.14}$$

where α is given by (4.3), $\lambda_{1,2}$ by (3.8), and ψ by

$$\psi = \arctan[(1-\gamma^2-a^2)^{1/2}/a] \tag{4.15a}$$

or

$$\cos\psi = a(1-\gamma^2)^{-1/2} \ . \tag{4.15b}$$

It is clear from (4.14) that for $a^2 < 1-\gamma^2$, $(-1)^R\rho_{xx}(R)$ is an oscillating function of R with a period of oscillation that decreases as a^2 increases.

D. ρ_{xx} $(T=0, a>1)$

To evaluate ρ_{xx} when $T=0$ and $a>1$ we consider the transpose of the determinant (3.6) which has the generating function:

$$\bar{C}(\xi) = C(\xi^{-1}) = \left[\frac{(1-\lambda_1^{-1}\xi^{-1})(1-\lambda_2^{-1}\xi^{-1})}{(1-\lambda_1^{-1}\xi)(1-\lambda_2^{-1}\xi)}\right]^{1/2} \ . \tag{4.16}$$

Since $\mathrm{Ind}\,\bar{C}(\xi) = -1$ we may instantly apply the results of Sec. 2 of Wu. In particular (using Wu's R_N)

$$\lim_{N\to\infty}(-1)^N R_{N+1} = [(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2)^{-2}]^{1/4} \tag{4.17}$$

and in the factorization

$$-\xi^{-1}\bar{C}(\xi) = P^{-1}(\xi)Q^{-1}(\xi^{-1}) \ , \tag{4.18}$$

$$P(\xi) = \left(\frac{1-\lambda_1^{-1}\xi}{1-\lambda_2\xi}\right)^{1/2} \ , \tag{4.19a}$$

$$Q(\xi) = \left(\frac{1-\lambda_2\xi}{1-\lambda_1^{-1}\xi}\right)^{1/2} = P^{-1}(\xi) \ . \tag{4.19b}$$

Then

$$\rho_{xx}(R) \doteq (-1)^R \tfrac{1}{4}\,[\lim_{N\to\infty}(-1)^N R_{N+1}]x_R \ , \tag{4.20}$$

where

$$x_R \doteq \frac{1}{2\pi i}\oint_{|\xi|=1} d\xi\, \xi^{R-1}P(\xi^{-1})Q(\xi)^{-1}$$
$$= \frac{1}{2\pi i}\oint_{|\xi|=1} d\xi\, \xi^{R-1}\left[\frac{(1-\lambda_1^{-1}\xi^{-1})(1-\lambda_1^{-1}\xi)}{(1-\lambda_2\xi^{-1})(1-\lambda_2\xi)}\right]^{1/2} \ . \tag{4.21}$$

For large R this is easily expanded as

$$x_R = \pi^{-1}\lambda_2^R \sum_{m=0}^{\infty} \frac{\Gamma(R)\Gamma(\frac{1}{2}+m)}{\Gamma(R+\frac{1}{2}+m)}\hat{A}_m \ , \tag{4.22}$$

where $\hat{A}_m$ is defined by

$$\left[\frac{(1-\lambda_1^{-1}\lambda_2^{-1}-y)(1-\lambda_1^{-1}\lambda_2+\lambda_1^{-1}\lambda_2 y)}{1-\lambda_2^2+\lambda_2^2 y}\right]^{1/2} = \sum_{m=0}^{\infty}\hat{A}_m y^m \ . \tag{4.23}$$

In particular

$$\hat{A}_0 = \left[\frac{(1-\lambda_1^{-1}\lambda_2^{-1})(1-\lambda_1^{-1}\lambda_2)}{1-\lambda_2^2}\right]^{1/2} \ , \tag{4.24a}$$

$$\hat{A}_1 = \tfrac{1}{2}\hat{A}_0\left[\frac{\lambda_1^{-1}\lambda_2}{1-\lambda_1^{-1}\lambda_2} - \frac{1}{1-\lambda_1^{-1}\lambda_2^{-1}} - \frac{\lambda_2^2}{1-\lambda_2^2}\right]. \tag{4.24b}$$

Therefore we explicitly find that

$$\rho_{xx} \doteq (-1)^R \tfrac{1}{4}\pi^{-1/2}R^{-1/2}\lambda_2^R[(1-\lambda_2^2)^{-1}(1-\lambda_1^{-2})$$
$$\times(1-\lambda_1^{-1}\lambda_2^{-1})^2]^{1/4}[1+\tfrac{1}{2}R^{-1}(\tfrac{1}{4}+\hat{A}_1/\hat{A}_0)+O(R^{-2})] \ . \tag{4.25}$$

Note in particular that when $\gamma=1$ then $\lambda_1^{-1}=0$, $\lambda_2=a^{-1}$, and (4.25) specializes to

$$(-1)^R\tfrac{1}{4}\pi^{-1/2}R^{-1/2}a^{-R}(1-a^{-2})^{-1/4}$$
$$\times\{1-\tfrac{1}{8}R^{-1}(1+a^{-2})/(1-a^{-2})+O(R^{-2})\} \ . \tag{4.26}$$

This has been previously obtained by Pfeuty.[4]

V. EVALUATION OF ρ_{yy}

A. General

Define

$$b_n = (1/2\pi)\int_{-\pi}^{\pi}(\tfrac{1}{2}\beta)T(e^{i\phi})e^{-i\phi n}P_y(e^{i\phi})d\phi \ , \tag{5.1}$$

where $T(e^{i\phi})$ is given by (3.3) and

$$P_y(e^{i\phi}) = \tfrac{1}{2}(1-\gamma)\left(1-\frac{2a}{1-\gamma}e^{-i\phi}+\frac{1+\gamma}{1-\gamma}e^{-2i\phi}\right) \ . \tag{5.2}$$

Then ρ_{yy} can be written as

$$\rho_{yy}=\tfrac{1}{4}(-1)^R\begin{vmatrix} b_0 & b_{-1} & \cdots & b_{-R+1}\\ b_1 & b_0 & \cdots & b_{-R+2}\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ b_{R-1} & \cdot & \cdots & b_0\end{vmatrix}\,. \qquad (5.3)$$

Once again we obtain three regions:

(i) $a<1$. $\mathrm{Ind}[P_y]=-2$. We "shift" twice[10] in order to apply Szegö's theorem.

(ii) $a>1$. $\mathrm{Ind}[P_y(e^{i\phi})]=-1$.

(iii) $a=1$. $P_y(e^{i\phi})$ vanishes on the unit circle, and Szegö's theorem is invalid.

B. $\rho_{yy}(a<1)$ for β Finite

By inspection one obtains

$$c_{n+2}=b_n\ , \qquad (5.4)$$

where c_n is given by (3.5) and b_n by (5.1).

Let U_{R+2} be the $(R+2)\times(R+2)$ Toeplitz determinant

$$U_{R+2}=\tfrac{1}{4}(-1)^{R+2}\begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{R+1}\\ c_{-1} & c_0 & c_1 & \cdots & c_R\\ c_{-2} & c_{-1} & c_0 & \cdots & c_{R-1}\\ \cdot & \cdot & & \cdots & \cdot\\ \cdot & \cdot & & \cdots & \cdot\\ \cdot & \cdot & & \cdots & \cdot\\ c_{-R+1} & c_{-R+2} & c_{-R+3} & \cdots & c_2\\ c_{-R} & c_{-R+1} & c_{-R+2} & \cdots & c_1\\ c_{-R-1} & c_{-R} & c_{-R+1} & \cdots & c_0\end{vmatrix}\,. \qquad (5.5)$$

The determinant U_{R+2} is exactly $\rho_{xx}(R+2)$, and its asymptotic properties were studied in Secs. III and IV.

Theorem 4 of Hartwig and Fisher[10] yields

$$\rho_{yy}(R)\doteq\rho_{xx}(R+2)\begin{vmatrix} Y_R & Y_{R+1}\\ Y_{R-1} & Y_R\end{vmatrix}\ , \qquad (5.6)$$

where

$$Y_R\doteq(1/2\pi i)\oint \xi^{R-1}P(\xi^{-1})\,Q^{-1}(\xi)\,d\xi\ , \qquad (5.7)$$

$$P(\xi)=e^{-k_0}(1-\lambda_1^{-1}\xi)^{-1}(1-\lambda_2^{-1}\xi)^{-1}$$
$$\times\prod_{k=1}^{\infty}\frac{(1-\xi f_{2k-1})(1-\xi g_{2k-1})}{(1-\xi f_{2k})(1-\xi g_{2k})}\ , \qquad (5.8)$$

$$Q(\xi)=\prod_{k=1}^{\infty}\frac{(1-\xi f_{2k-1})(1-\xi g_{2k-1})}{(1-\xi f_{2k})(1-\xi g_{2k})}\ . \qquad (5.9)$$

Therefore (5.7) can be rewritten as

$$Y_R\doteq(1/2\pi i)\oint d\xi\,\xi^{R-1}(\tfrac{1}{2}\beta)^{-1}[\tfrac{1}{2}\beta\Lambda\coth(\tfrac{1}{2}\beta\Lambda)]$$
$$\times[(1-\lambda_1^{-1}\xi)(1-\lambda_1^{-1}\xi^{-1})(1-\lambda_2^{-1}\xi)$$
$$\times(1-\lambda_2^{-1}\xi^{-1})]^{-1}\exp\{-2\beta\pi^{-1}$$
$$\times\int_{-\infty}^{+\infty}dt\,\mathrm{csch}\beta t\,\mathrm{Im}\ln[(1-\xi f(t))(1-\xi g(t))]\}\ . \qquad (5.10)$$

The asymptotic values of Y_R for $a<1$ and fixed large β are (i) $(1-\gamma^2)^{1/2}<a<1$:

$$Y_R\doteq\lambda_2^{-(R-1)}(\tfrac{1}{2}\beta)^{-1}[(1-\lambda_1^{-1}\lambda_2^{-1})(1-\lambda_1^{-1}\lambda_2)(1-\lambda_2^{-2})]^{-1}$$
$$\times\exp\{-4\pi^{-1}e^{-\beta|1-a|}\beta^{-1/2}\textstyle\sum_m\beta^{-m}A_m\Gamma(m+\tfrac{3}{2})\}\ , \qquad (5.11a)$$

(ii) $1-\gamma^2<a<(1-\gamma^2)^{1/2}$:

$$Y_R\doteq(\tfrac{1}{2}\beta)^{-1}2\,\mathrm{Re}\{\lambda_2^{-(R-1)}[(1-\lambda_1^{-1}\lambda_2^{-1})(1-\lambda_1^{-1}\lambda_2)(1-\lambda_2^{-2})]^{-1}$$
$$\times\exp[-4\pi^{-1}e^{-\beta|1-a|}\beta^{-1/2}\textstyle\sum_m\beta^{-m}A_m\Gamma(m+\tfrac{3}{2})]\}\ , \qquad (5.11b)$$

(iii) $0\le a<1-\gamma^2$:

$$Y_R\doteq(\tfrac{1}{2}\beta)^{-1}2\,\mathrm{Re}\{\lambda_2^{-(R-1)}[(1-\lambda_1^{-1}\lambda_2^{-1})(1-\lambda_1^{-1}\lambda_2)(1-\lambda_2^{-2})]^{-1}$$
$$\times\exp[-4\pi^{-1}\beta^{-1/2}e^{-\beta\theta}\textstyle\sum_m\beta^{-n}A_m\Gamma(m+\tfrac{3}{2})]\}\ , \qquad (5.11c)$$

where A_m is defined by Eq. (3.62b). Substitution of (5.11) into (5.7) yields the desired results. Note in particular that $(-1)^R\rho_{yy}(R)$ so obtained is an oscillatory function of R if $0\le a^2<1-\gamma^2$.

C. Ground State of ρ_{yy} ($T=0$)

By substitution of $\beta=\infty$ in (5.10), we obtain

$$Y_R(T=0)\doteq(1/2\pi i)\oint\xi^{R-1}[(1-\lambda_1^{-1}\xi)(1-\lambda_1^{-1}\xi^{-1})$$
$$\times(1-\lambda_2\xi)(1-\lambda_2\xi^{-1})]^{1/2}\,d\xi\ . \qquad (5.12)$$

This is precisely the integral of (Wu 2.29) with α_1 replaced by λ_1^{-1} and α_2 replaced by λ_2. Therefore (i) $1-\alpha^2<a^2<1$:

(i) $1-\gamma^2<a^2<1$

$$Y_R\doteq(\pi R)^{-1/2}\lambda_2^{-R}[(1-\lambda_2^{-2})(1-\lambda_1^{-1}\lambda_2)(1-\lambda_1^{-1}\lambda_2^{-1})]^{-1/2}$$

$$\times[1+\tfrac{1}{4}R^{-1}A_{1>}+\tfrac{3}{16}R^{-2}(A_{2>}-\tfrac{5}{6})+\cdots] , \quad (5.13a)$$

where $A_{1>}$ and $A_{2>}$ are given by (Wu 2.39) as

$$A_{1>}=-\tfrac{1}{2}(x_1-x_2+x_3) , \quad (5.13b)$$

$$A_{2>}=\tfrac{3}{8}(x_1^2+x_2^2+x_3^2)-\tfrac{1}{4}(x_2x_3-x_3x_1+x_1x_2) , \quad (5.13c)$$

where x_1, x_2, x_3 are given by (Wu 2.31) - (Wu 2.33) as

$$x_1=(1+\lambda_1^{-1}\lambda_2^{-1})(1-\lambda_1^{-1}\lambda_2^{-1})^{-1}, \quad (5.13d)$$

$$x_2=(1+\lambda_1^{-1}\lambda_2)(1-\lambda_1^{-1}\lambda_2)^{-1}, \quad (5.13e)$$

$$x_3=(1+\lambda_2^{-2})(1-\lambda_2^{-2})^{-1}. \quad (5.13f)$$

Straightforward algebra then yields

$$\begin{vmatrix} Y_R & Y'_{R+1} \\ Y_{R-1} & Y_R \end{vmatrix} \doteq \pi^{-1}K^2R^{-3}\lambda_2^{-2R}[\tfrac{1}{2}+\tfrac{3}{4}A_{1>}R^{-1}+O(R^{-2})], \quad (5.14)$$

where K is given as

$$K=[(1-\lambda_2^{-2})(1-\lambda_1^{-1}\lambda_2)(1-\lambda_1^{-1}\lambda_2^{-1})]^{-1/2}. \quad (5.15)$$

The correlation $\rho_{yy}(R)$ becomes

$$\rho_{yy}(R)\doteq-\tfrac{1}{4}(-1)^R\,\pi^{-1}K^2R^{-3}\lambda_2^{-2R}[\tfrac{1}{2}+\tfrac{3}{4}A_{1>}R^{-1}+O(R^{-2})]$$
$$\times[2/(1+\gamma)][\gamma^2(1-a^2)]^{1/4}. \quad (5.16)$$

(ii) $a^2=1-\gamma^2$: For this case $\rho_{yy}(R)$ may be evaluated exactly for all R. Indeed we have

$$b_n=\begin{cases} 0, & n\geq 0 \\ -\alpha, & n=1 \\ \alpha^{n-1}(1-\alpha^2), & n>1. \end{cases} \quad (5.17)$$

Therefore (5.3) becomes an upper triangular determinant, hence for $R\neq 0$

$$\rho_{yy}(R)=0. \quad (5.18)$$

(iii) $a^2<1-\gamma^2$: Wu's analysis can be carried out word for word, but the contribution to the asymptotic series comes from both branch points λ_1^{-1} and λ_2^{-1}, with one contribution being the complex conjugate of the other. Therefore we have

$$Y_R\doteq(\pi R)^{-1/2}\{K\lambda_2^{-R}+K^*\lambda_1^{-R}+\tfrac{1}{4}R^{-1}$$
$$\times(A_{1>}K\lambda_2^{-R}+A_{1>}^*K^*\lambda_1^{-R})$$
$$+\tfrac{3}{16}R^{-2}[(A_{2>}^*-\tfrac{5}{6})K^*\lambda_1^{-R}+(A_{2>}-\tfrac{5}{6})K\lambda_2^{-R}]+\cdots\}, \quad (5.19)$$

where K is given by (5.15).

Define

$$A_{1>}\equiv A_1e^{is}, \quad (5.20a)$$

$$K=ce^{i\epsilon}, \quad (5.20b)$$

$$\lambda_2^{-1}=\alpha e^{i\psi}=\lambda_1^{-1*}. \quad (5.20c)$$

So [using also definitions (4.15)] we have

$$\begin{vmatrix} Y_R & Y_{R+1} \\ Y_{R-1} & Y_R \end{vmatrix} \doteq 4c^2(\pi R)^{-1}\alpha^{2R}$$
$$\times\sin^2\psi[1+\tfrac{1}{2}A_1R^{-1}\cos s+O(R^{-2})]. \quad (5.21)$$

The first two terms do not oscillate with R, but the term of $O(R^{-2})$ does oscillate. This expansion clearly breaks down for $a^2=1-\gamma^2$. Combining (5.7) and (5.21) with (4.14) we obtain

$$\rho_{yy}(R)\doteq(-1)^R[2/(1+\gamma)][\gamma^2(1-a^2)]^{1/4}c^2(\pi R)^{-1}\alpha^{2R}$$
$$\times\sin^2\psi\,[1+\tfrac{1}{2}A_1R^{-1}\cos s+O(R^{-2})]. \quad (5.22)$$

(iv) $a=1$: The generating function specializes to

$$C(\xi)=e^{3\pi i/2}\xi^{-3/2}\left(\frac{1-\lambda_1^{-1}\xi}{1-\lambda_1^{-1}\xi^{-1}}\right)^{1/2}. \quad (5.23)$$

If we consider the case $\gamma=1$

$$C^0(\xi)=e^{3\pi i/2}\xi^{-3/2} , \quad (5.24)$$

then Pfeuty has shown that

$$\rho_{yy}^{(0)}(R)=-(4R^2-1)^{-1}\rho_{xx}^{(0)}(R) . \quad (5.25)$$

For $\gamma\neq 1$ but $\gamma\neq 0$ we may easily combine (5.25) with (5.31) of Wu to find asymptotically

$$\rho_{yy}(R)\doteq-\tfrac{1}{4}(-1)^R\gamma(1+\gamma)\tfrac{1}{8}(\gamma R)^{-9/4}$$
$$\times e^{1/4}2^{1/12}A^{-3}[1+O(R^{-2})] . \quad (5.26)$$

(v) $a>1$: In this case Ind $P_y=-1$ and ρ_{yy} can be studied by use of (2.6) and (2.27) of Wu just as ρ_{xx} was studied for $a>1$. We find that

$$\rho_{yy}(R)\doteq-(-1)^R\tfrac{1}{4}\pi^{-1}\lambda_2^R R^{-3/2}$$
$$\times[(1-\lambda_2^2)(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2)^{-2}]^{1/4}$$
$$\times\sum_{m=0}^{\infty}C_m\frac{\Gamma(R)\Gamma(\tfrac{3}{2}+m)}{\Gamma(R+\tfrac{3}{2}+m)} , \quad (5.27)$$

where

$$\left[\frac{1-\lambda_2^2+\lambda_2^2y}{(1-\lambda_1^{-1}\lambda_2+\lambda_1^{-1}\lambda_2y)(1-\lambda_1^{-1}\lambda_2^{-1}-y)}\right]^{1/2}=\sum_{m=0}^{\infty}C_my^m . \quad (5.28)$$

In particular we have

$$C_0 = \{[2\gamma/(1+\gamma)](1-\lambda_1^{-1}\lambda_2)\}^{-1/2}(1-\lambda_2^2)^{1/2}, \tag{5.29a}$$

$$C_1 = \tfrac{1}{2}A_0^3\{[2\gamma/(1+\gamma)](1-\lambda_1^{-1}\lambda_2)+(1-\lambda_2^2)(1-\lambda_1^{-1}\lambda_2) - (1-\lambda_2^2)\lambda_1^{-1}\lambda_2[2\gamma/(1+\gamma)]\}, \tag{5.29b}$$

$$\rho_{yy}(R) \doteq -(-1)^R \tfrac{1}{8}\pi^{-1/2}\lambda_2^R R^{-3/2} \times[(1-\lambda_2^2)^3(1-\lambda_1^{-2})(1-\lambda_1^{-1}\lambda_2^{-1})^{-2}]^{1/4} \times(1-\lambda_1^{-1}\lambda_2)^{-1}[1+\tfrac{3}{2}R^{-1}(C_1/C_0-\tfrac{1}{4})+O(R^{-2})]. \tag{5.30}$$

VI. EVALUATION OF ρ_{zz}

We finally turn our attention to ρ_{zz}, which is given by (2.8c). The magnetization is clearly R independent and will be the first term in the asymptotic expansion of ρ_{zz} for large R.

The next-order term is obtained where $T>0$ by using the partial sum decomposition of $(\tanh\frac{1}{2}\beta\Lambda)/(\frac{1}{2}\beta\Lambda)$ to find

$$G_{-R} = \frac{1}{2\pi i}\oint_{|\xi|=1} d\xi\, \xi^{-(R+1)}(1-\lambda_1^{-1}\xi)(1-\lambda_2^{-1}\xi)(1-\gamma)\frac{\beta}{2}\sum_{k=1}(2k-1)^2\pi^2 S_{2k-1}^{-1}$$

$$= \frac{1}{2\pi i}\oint_{|\xi|=1} d\xi\,\xi^{-(R+1)}(1-\lambda_1^{-1}\xi)(1-\lambda_2^{-1})\frac{1}{1+\gamma}\left(\frac{\beta}{2}\right)^{-1} \times\sum_{k=1}^{\infty} f_{2k-1}g_{2k-1}[(1-\xi^{-1}f_{2k-1})(1-\xi f_{2k-1})(1-\xi g_{2k-1})(1-\xi^{-1}g_{2k-1})]. \tag{6.1}$$

Therefore if $\gamma\neq 0$, we have

$$G_{-R} = (1+\gamma)^{-1}(\tfrac{1}{2}\beta)^{-1}\sum_{k=1}^{\infty}\{f_{2k-1}^R g_{2k-1}[(1-f_{2k-1}^2)(1-f_{2k-1}g_{2k-1})(1-f_{2k-1}^{-1}g_{2k-1})]^{-1}\{(1-\lambda_1^{-1}f_{2k-1})(1-\lambda_2^{-1}f_{2k-1}) + g_{2k-1}^R f_{2k-1}[(1-g_{2k-1}^2)(1-f_{2k-1}g_{2k-1})(1-g_{2k-1}^{-1}f_{2k-1})]^{-1}(1-\lambda_1^{-1}g_{2k-1})(1-\lambda_2^{-1}g_{2k-1})\}. \tag{6.2}$$

As in (M Sec. 2A), when $T\neq 0$ and R is sufficiently large, only the first term contributes to an asymptotic expansion, and the second-order term in ρ_{zz} can be easily written down using the relations $G_R(\gamma) = G_{-R}(-\gamma)$.

Therefore for any a and T fixed and positive

$$\rho_{zz} \sim m_z^2 - \beta^{-2}(1-\gamma^2)^{-1}(1-f_1g_1)^{-2}(g_1^{-1}-f_1^{-1})^{-2} \times(A_1f_1^R+\bar{A}_2g_1^R)(\bar{A}_1f_1^R+\bar{A}_2g_1^R), \tag{6.3}$$

where

$$A_1 = (1-f_1^2)^{-1}(1-\lambda_1^{-1}f_1)(1-\lambda_2^{-1}f_1), \tag{6.4a}$$

$$\bar{A}_1 = (1-f_1^2)^{-1}(1-\lambda_2f_1)(1-\lambda_1f_1), \tag{6.4b}$$

$$A_2 = (1-g_1^2)^{-1}(1-\lambda_1^{-1}g_1)(1-\lambda_2^{-1}g_1), \tag{6.4c}$$

$$\bar{A}_2 = (1-g_1^2)^{-1}(1-\lambda_2g_1)(1-\lambda_1g_1). \tag{6.4d}$$

As in the previous sections of this paper, there are two cases to consider according to whether f_k (and g_k) is real or complex. Consider first the case

$$\gamma^2[a^2-(1-\gamma^2)] < (1-\gamma^2)(\pi/\beta)^2. \tag{6.5}$$

Then f_k is complex, and (6.3) specializes to

$$\rho_{zz} \sim m_z^2 - \beta^{-2}(1-\gamma^2)^{-1}(1-|f_1|^2)^{-2}(g_1^{-1}-f_1^{-1})^{-2} \times\mathrm{Re}[A_1g_1^R]\mathrm{Re}[\bar{A}_1g_1^R]. \tag{6.6}$$

The case $\gamma=0$ may be studied by letting $\gamma\to 0$ in (6.6), and we find that for all a and $T>0$,

$$\rho_{zz} \sim m_z^2 - \beta^{-2}(1-\gamma^2)^{-1}(1-|f_1|^2)^{-2} \times(g_1^{-1}-f_1^{-1})^{-2}(\mathrm{Re}A_1g_1^R)^2. \tag{6.7}$$

It is clear from (6.6) and (6.7) that the approach of ρ_{zz} to m_z^2 is not monotonic. Indeed, when $a=0$ the oscillations in ρ_{zz} are quite pronounced and $\rho_{zz}(R) = 0$ when R is even, while if R is odd it decays as

$$\rho_{zz} \sim -(1-\gamma^2)^{-1}\beta^{-2}|f_1|^2(1-|f_1|^4)^{-2}|g_1|^{2R}A_1\bar{A}_1. \tag{6.8}$$

However, as a^2 increases, the wavelengths of these oscillations become longer and longer until at

$$\gamma^2[a^2-(1-\gamma^2)]=(1-\gamma^2)(\pi/\beta)^2 \qquad (6.9)$$

g_1 and f_1 become real and ρ_{zz} becomes monotonic.

When $T>0$, the other case to consider is

$$\gamma^2[a^2-(1-\gamma^2)]>(1-\gamma^2)(\pi/\beta)^2 . \qquad (6.10)$$

Then f_1 and g_1 are real, $f_1<g_1<1$, and (6.3) reduces to

$$\rho_{zz}\sim m_z^2-\beta^{-2}(1-\gamma^2)^{-1}(1-f_1g_1)^{-2}(g_1^{-1}-f_1^{-1})^{-2}A_1\bar{A}_1g_1^{2R}. \qquad (6.11)$$

It should be noted that (6.11) holds for all values of a satisfying (6.10) and that there is no distinction $a<1$ or $a>1$. However, as seen from (3.12), g_1 is not a monotonic function of a but rather is a function which reaches a maximum value as a increases towards 1 and then decreases to zero as $a\to\infty$.

We now turn to the case $T=0$, where we have already seen in ρ_{xx} and ρ_{yy} that the cases $a=1$ or $\gamma=0$ are to be distinguished. When $T=0$, we may set $\beta=\infty$ in (2.17) to find

$$G_R=-\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi e^{-i\phi(R+1)} \times\left(\frac{(1-\lambda_1^{-1}e^{i\phi})(1-\lambda_2^{-1}e^{i\phi})}{(1-\lambda_1^{-1}e^{-i\phi})(1-\lambda_2^{-1}e^{-i\phi})}\right)^{1/2}. \qquad (6.12)$$

This is simply expanded as $R\to\infty$, and using (2.7) we find the following results:

(a) $\gamma=0$:

$$\rho_{zz}(R)=\begin{cases} m_z^2-\left(\dfrac{\sin(R\cos^{-1}a)}{\pi R}\right)^2, & a<1,\ R\neq0\\ \frac{1}{4}, & a\geq1\end{cases} \qquad (6.13)$$

which is an exact and not an asymptotic result;

(b) $a=1,\ \gamma\neq0$:

$$\rho_{zz}(R)\sim m_z^2-\tfrac{1}{4}(\pi R)^{-2}[1+\tfrac{1}{4}(\gamma R)^{-2}+O(R^{-3})]; \qquad (6.14)$$

(c) $a^2=1-\gamma^2,\ \gamma\neq0$:

$$\rho_{zz}(R)=m_z^2,\qquad R\neq0 \qquad (6.15)$$

which also is an exact result;

(d) $\gamma\neq0,\ 0\leq a^2<1-\gamma^2$:

$$\rho_{zz}(R)\sim m_z^2-\alpha^{2R}R^{-2}\pi^{-1}\,\mathrm{Re}\Big\{e^{i\phi(R+1)}[(1-e^{2i\phi})/(1-\alpha^2e^{-2i\phi})]^{1/2} \times\left[1-\tfrac{3}{4}R^{-1}\left(\frac{1}{2}+\frac{1}{1-e^{2i\phi}}+\frac{\alpha^2}{1-\alpha^2}+\frac{\alpha^2e^{-2i\phi}}{1-\alpha^2e^{-2i\phi}}\right)\right]\Big\}$$
$$\times\mathrm{Re}\Big\{e^{i\phi(R-1)}\left(\frac{(1-\alpha^2e^{-2i\phi})}{(1-e^{2i\phi})}\right)^{1/2}\left[1+\tfrac{1}{2}R^{-1}\left(\frac{1}{4}+\frac{1}{1-e^{2i\phi}}+\frac{\alpha^2}{1-\alpha^2}+\frac{\alpha^2e^{-2i\phi}}{1-\alpha^2e^{-2i\phi}}\right)\right]\Big\}\ ; \qquad (6.16)$$

(e) $1-\gamma^2<a^2<1$:

$$\rho_{zz}(R)\sim m_z^2-\tfrac{1}{8}\lambda_2^{-2R-2}\pi^{-1}[1-R^{-1}(x_1+x_2+x_3)], \qquad (6.17)$$

where the x_i are given by (5.13);

(f) $1<a$:

$$\rho_{zz}\sim\tfrac{1}{4}-\tfrac{1}{8}\lambda_2^{2R}R^{-2}\pi^{-1}[1-R^{-1}(x_1+x_2-x_3)]. \qquad (6.18)$$

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APPENDIX

For easy reference, we give in this Appendix a dictionary of $f(t)$ for the various ranges of t and the parameters involved. Note that $g(t)=f^*(-t)$.

I. $a^2 > 1-\gamma^2$; $a > 1-\gamma^2$

$$f(t)=\frac{a+\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)t^2\}^{1/2}}{1-\gamma^2}-\left[\left(\frac{a+\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)t^2\}^{1/2}}{1-\gamma^2}\right)^2-1\right]^{1/2},\quad t>0 \tag{A1}$$

$$f(t)=\frac{a-\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)t^2\}^{1/2}}{1-\gamma^2}-\left[\left(\frac{a-\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)t^2\}^{1/2}}{1-\gamma^2}\right)^2-1\right]^{1/2},\quad -(1-a)<t<0 \tag{A2}$$

$$f(t)=\frac{a-\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)t^2\}^{1/2}}{1-\gamma^2}-i\left[-\left(\frac{a-\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)t^2\}^{1/2}}{1-\gamma^2}\right)^2+1\right]^{1/2},\quad -(1+a)<t<-(1-a) \tag{A3}$$

$$f(t)=\frac{a-\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)t^2\}^{1/2}}{1-\gamma^2}+\left[\left(\frac{a-\{\gamma^2[a^2-(1-\gamma^2)]+(1-\gamma^2)t^2\}^{1/2}}{1-\gamma^2}\right)^2-1\right]^{1/2},\quad t<-(1+a). \tag{A4}$$

II. $a^2 < 1-\gamma^2$; $a > 1-\gamma^2$

$$f(t)=\frac{a+[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}-\left[\left(\frac{a+[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}\right)^2-1\right]^{1/2},\quad t>0 \tag{A5}$$

$$f(t)=\frac{a+i[-(1-\gamma^2)t^2+\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}-\left[\left(\frac{a+i[-(1-\gamma^2)t^2+\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}\right)^2-1\right]^{1/2},\quad |t|<\theta \tag{A6}$$

$$f(t)=\frac{a-[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}-\left[\left(\frac{a-[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}\right)^2-1\right]^{1/2},\quad -(1-a)<t<-\theta \tag{A7}$$

$$f(t)=\frac{a-[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}-i\left[-\left(\frac{a-[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}\right)^2+1\right]^{1/2},\quad -(1+a)<t<-(1-a) \tag{A 8}$$

$$f(t)=\frac{a[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}+\left[\left(\frac{a-[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}\right)^2-1\right]^{1/2},\quad t<-(1+a). \tag{A9}$$

III. $a < 1-\gamma^2$

$$f(t)=\frac{a+[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}-\left[\left(\frac{a+[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}\right)^2+1\right]^{1/2},\quad t>1-a \tag{A10}$$

$$f(t)=\frac{a+[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}-i\left[-\left(\frac{a+[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}\right)^2+1\right]^{1/2}, \qquad \theta<t<1-a \tag{A11}$$

$$f(t)=\frac{a+i[-(1-\gamma^2)t^2+\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}-i\left[-\left(\frac{a+i[-(i-\gamma^2)t^2+\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}\right)^2+1\right]^{1/2}, \quad |t|<\theta \tag{A12}$$

$$f(t)=\frac{a-[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}-i\left[\left(\frac{a-[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}\right)^2-1\right]^{1/2}, \quad -(1+a)<t<-\theta \tag{A13}$$

$$f(t)=\frac{a-[(1-\gamma^2)t^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}+\left[\left(\frac{a-[(1-\gamma^2)t^2-\gamma^2-\gamma^2(1-\gamma^2-a^2)]^{1/2}}{1-\gamma^2}\right)^2-1\right]^{1/2}, \qquad t<-(1+a). \tag{A14}$$

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*Present address: Department of Mathematics, MIT, Cambridge, Mass. 02139.

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Generalization of Boltzmann's Kinetic Theory. The Lorentz Gas.

J. Coste* and J. Peyraud*

Laboratoire de Physique des Plasmas,† *Batiment* 211, *Faculté des Sciences*, 91, *Orsay*, *France*

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We present in this paper a new formulation of the kinetics of the Lorentz gas, based on an analysis of statistically independent collisional events. We give a method in which the general collision operator is expanded in terms of functions of the density, and we carefully treat the μ^s approximation (μ is the density parameter and s is the dimensionality of the gas). The method is applied to the calculation of the self-diffusion coefficient.

INTRODUCTION

From the point of view of Boltzmann's early work in kinetic theory, one considers a particle of a gas at time t with velocity $\vec{v}(t)$ and evaluates the transition probabilities associated with the various possible binary collision processes. Molecular chaos (MC) is then assumed, wherein each collision is a random event having no correlation with the past history of the particle. One can then derive the increment $d\Pi(t)$ of the particle distribution function $f(\vec{v}, t)$ between times t and $t+dt$ in terms of these transition probabilities and $f(\vec{v}, t)$. The kinetic equation thus obtained is local in time, and in the case of the Lorentz gas, the successive deflections of the velocity vector constitute a Markov chain.