Unstable-Particle Gases: Negative-Lifetime Case*

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The total excluded volume for a gas of stable particles with repulsive interactions is calculated. From this, it is possible to define an effective negative-interaction number density which is formally identical to the previously derived expression for the equilibrium number density of unstable particles. Pointing out that this allows the unstable-particle formalism to be extended to the negative-lifetime case, it is argued that unstable particles with negative lifetimes behave effectively like a gas of unstable holes.

I. INTRODUCTION

In a previous communication,[1] it was shown that to a first approximation, any real gas could be divided into two components, one being an ideal gas of stable particles, and the other being an ideal gas of unstable particles. The collision cross section and the reciprocal of the scattering delay time were used to determine rates of production and destruction in a gas of interacting stable particles. By equating the two rates, a general expression for the equilibrium number density of unstable particles was obtained.

The success of the approach was based on the interpretation of the elastic scattering of two particles as the formation and decay of an unstable particle. The unstable particle was said to be formed the instant the scattering pair "started to interact" and had decayed the instant the scattering pair "stopped interacting." The energy derivative of the phase shift provided the needed expression for the mean life of the state

$$\tau(E) = \hbar \frac{d\eta}{dE} \quad . \tag{1}$$

Clearly, the interpretation breaks down whenever this energy derivative becomes negative (negative lifetimes). Moreover, the expression for the unstable-particle number density also becomes negative.

Negative values for the energy derivative of the scattering phase shift are usually associated with repulsive potentials. In Secs. II and III it will be shown that the excluded or removed free-particle volumes due to these repulsive potentials may be used to define a negative-interaction number density which is formally identical to the expression for the unstable-particle number density [Eq. (2)]. This allows the entire unstable-particle formalism[1] to be extended to the negative-lifetime case. Arguing in Sec. IV that negative number densities may be interpreted in terms of *removed* particle pairs, it is pointed out that introduction of the idea of unstable *pseudoholes* provides the conceptual counterpart for unstable-particle gases when the lifetimes become negative. Before proceeding, we shall first briefly review some of the pertinent formulas for the positive-lifetime case.

In a unispecies gas of stable particles, the total number density of unstable particles is

$$N_u = n_s^2 f \ , \tag{2}$$

where n_s is the number density of *free* stable particles, and f is a sum over orbital angular-momentum states,

$$f \equiv \left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2} \frac{2^{3/2}}{\pi} \sum_l (2l+1) I_l \ , \tag{3}$$

with the integral term I_l given by

$$I_l \equiv \int_0^\infty 2 \frac{d\eta_l}{dE} \sin^2\eta_l \, e^{-E/kT} dE \ . \tag{4}$$

η_l is the lth partial-wave phase shift, and it will be recognized that f is proportional to the cube of the thermal wavelength $\lambda = (2\pi\hbar^2/mkT)^{1/2}$, a small quantity in our low-density high-temperature limit. Equally valid for identical bosons or fermions, Eqs. (2)–(4) were derived for zero-spin particles.

The total number density of all stable particles, free or interacting, in the gas is

$$n = n_s + 2N_u \ . \tag{5}$$

Here, N_u is multiplied by 2 because it takes two stable particles to make up one unstable particle. Substituting Eq. (2) for N_u, we solve for n_s in terms of n:

$$\begin{aligned} n_s &= [-1 + (1+8fn)^{1/2}]/4f \\ &= +n - 2fn^2 + 8f^2n^3 - 40f^3n^4 + \cdots \ . \end{aligned} \tag{6}$$

To third order in the thermal wavelength, this reduces to

$$n_s = n - 2fn^2 \ . \tag{7}$$

II. EXCLUDED VOLUMES AND INTERACTION NUMBER DENSITY

Consider a gas of N stable particles enclosed in a container of volume V. Let us characterize the interactions of these particles with one another as hard sphere. It is evident that because of these hard-sphere collisions, the stable particles will not be free to move in the entire volume V of the container. A certain total excluded volume ΔV will not be accessible to them. The total number density of stable particles, free or interacting, is

$$n \equiv N/V \quad , \tag{8}$$

while the effective free-particle number density will be

$$n_f \equiv \frac{N}{V - \Delta V} = \frac{n}{(1 - \Delta V/V)} \quad . \tag{9}$$

Rearranging terms, we have

$$n = n_f + 2\left[-\tfrac{1}{2}n_f\left(\frac{\Delta V}{V}\right)\right] \quad ,$$

and defining an interaction number density

$$N_I \equiv -\tfrac{1}{2}n_f\left(\frac{\Delta V}{V}\right) \quad , \tag{10}$$

yields

$$n = n_f + 2N_I \quad , \tag{11}$$

which is identical in form to Eq. (5), if N_I assumes the role of an unstable-particle number density. It will be noted from Eq. (10) that positive excluded volumes ΔV imply negative-interaction number densities N_I.

Anticipating our end results, let us assume that the relative excluded volume $\Delta V/V$ may be expressed as

$$\Delta V/V = 2ng \quad , \tag{12}$$

where g is some function, so that

$$N_I = -nn_f g \quad . \tag{13}$$

This resembles the form of Eq. (2), $N_u = n_s^2 f$, and substitution into Eq. (11) will allow us to express n_f in terms of n:

$$n_f = n/(1 - 2ng) = n + 2gn^2 + 4g^2n^3 + 8g^3n^4 + \cdots \quad . \tag{14}$$

For small g the first-order approximation is

$$n_f \simeq n + 2gn^2 \quad . \tag{15}$$

Consequently, the first-order expression for the free-particle number density is the same as in the positive-lifetime case Eq. (7), if $g = -f$.

III. NEGATIVE LIFETIMES AND EXCLUDED VOLUMES

The expression for the mean life, Eq. (1), is actually a relation for the scattering delay time, and when it becomes negative the proper question to ask is "What is the significance of a negative-scattering delay time?" This is best answered by switching off the repulsive potential causing the "negative delay" and recalculating the delay. By definition it will be 0 (the scattering particles pass right through each other) and we must conclude that the negative-time interval predicted in the presence of the interaction is the time the scattering particles *would have* spent in the interaction region in the absence of the interaction.

Care has been taken to explicitly state what might be considered obvious, because knowing this "would-have" time interval is tantamount to knowing the "would-have" interaction region. That is, a knowledge of the amount of time the scattering particles would have spent in the interaction region will enable us to calculate the volume they were excluded from.

If two colliding particles with relative velocity $\vec{v}_{ij} = |\vec{v}_i - \vec{v}_j|$ interact via repulsive forces, the relative distance that would have been traversed in the absence of the interaction is

$$D_{ij} = \vec{v}_{ij}\,\tau(E_{ij}) = \left|\vec{v}_i - \vec{v}_j\right| \hbar\frac{d\eta}{dE_{ij}} \quad . \tag{16}$$

The product of this length with the scattering cross section σ_{ij} will yield the volume that each particle has been excluded from:

$$\Gamma_{ij} = -\sigma_{ij}D_{ij} = -\hbar\left|\vec{v}_i - \vec{v}_j\right| \frac{d\eta}{dE_{ij}}\,\sigma_{ij}(E_{ij}) \,. \tag{17}$$

A minus sign has been introduced here because volume is to be treated as a positive definite quantity whereas Eq. (17) was derived under the assumption that $d\eta/dE_{ij} < 0$.

For Maxwellian velocity distribution functions

$$f_i(\vec{v}_i) = n_i\,(m_i/2\pi kT)^{3/2}\,e^{-m_i\vec{v}_i^2/2kT} \quad , \tag{18}$$

the average excluded volume between any pair of particles is

$$\langle\Gamma_{ij}\rangle = \int\int f_i(\vec{v}_i)f_j(\vec{v}_j)\Gamma_{ij}d^3\vec{v}_i d^3\vec{v}_j \quad . \tag{19}$$

Transforming to c.m. and relative coordinates, the c.m. motion may be integrated out, leaving an integral which depends only on the relative kinetic energy between the colliding particles:

$$\langle\Gamma_{ij}\rangle = -\hbar\left(\frac{8}{\pi\mu_{ij}}\right)^{1/2}(kT)^{-3/2}\int_0^\infty E\frac{dN}{dE}\,\sigma(E)\,e^{-E/kT}dE. \tag{20}$$

Here $\mu_{ij} = m_i m_j/(m_i + m_j)$ is the reduced mass.

In the case of uncharged spinless particles, the well-known relation

$$\sigma_l = (4\pi/k^2)(2l+1)\sin^2\eta_l \quad , \tag{21a}$$

or for identical bosons or fermions

$$\sigma_l = [1 \pm (-1)^l](4\pi/k^2)(2l+1)\sin^2\eta \quad , \tag{21b}$$

where k is the wave number, may be substituted for the scattering cross section:

$$\langle\Gamma_{ij}\rangle_l = -(1+\delta_{ij})\left(\frac{2\pi\hbar^2}{\mu_{ij}kT}\right)^{3/2}\frac{2}{\pi}(2l+1) \times\int_0^\infty \frac{d\eta_l}{dE}\sin^2\eta_l\, e^{-E/kT}dE\ . \tag{22}$$

This is the average volume for the lth angular-momentum partial wave and the factor $(1+\delta_{ij})$ arises from the quantum-mechanical fact that the scattering cross section is twice as large for identical particles. For all angular-momentum waves the average total volume is just the sum over these states:

$$\langle\Gamma_{ij}\rangle = \sum_l\langle\Gamma_{ij}\rangle_l\ . \tag{23}$$

In a gas containing two types of particles i and j, Eq. (23) gives the average volume that each individual is excluded from due to their mutual interaction. If the gas contains N_i type-i particles, then the total excluded volume of these particles due to their interaction with type-j particles is

$$\Delta V_i(j) = N_i\langle\Gamma_{ij}\rangle = -N_i(1+\delta_{ij})\left(\frac{2\pi\hbar^2}{\mu_{ij}kT}\right)^{3/2} \times\frac{2}{\pi}\sum_l(2l+1)\int_0^\infty\frac{d\eta_l}{dE}\sin^2\eta_l\, e^{-E/kT}dE\ . \tag{24}$$

In a monatomic gas of N identical particles, setting $\mu_{ij}=\frac{1}{2}m$ and $n=N_i/V$ in (24) will result in

$$\frac{\Delta V}{V} = -2n\left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2}\sum_l\frac{2^{3/2}}{\pi}(2l+1) \times\int_0^\infty 2\frac{d\eta_l}{dE}\sin^2\eta_l\, e^{-E/kT}dE\ . \tag{25}$$

Comparison with Eq. (12) reveals that

$$g \equiv -\left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2}\frac{2^{3/2}}{\pi}\sum_l(2l+1)I_l\ , \tag{26}$$

$$I_l \equiv \int_0^\infty 2\frac{d\eta_l}{dE}\sin^2\eta_l\, e^{-E/kT}dE\ . \tag{27}$$

Expect for sign $(g=-f)$ these are identical to Eqs. (3) and (4) of the positive-lifetime case. (3)

From (11) in (13),

$$N_I = -n_f^2 g/(1+2n_f g),$$

which to lowest order in $f\,(f=-g)$ is

$$N_I = n_f^2 f. \tag{28}$$

Therefore, to third order in the thermal wavelength, the interaction number density N_I is equivalent to the unstable-particle number density N_u of Eq. (2).

Using the concept of excluded volumes, we have demonstrated that it is possible to define an interaction number density N_I which is formally identical to the unstable-particle number density N_u. In a monatomic gas where the total number density of particles, free or interacting, is n, we note that for $d\eta/dE \geqslant 0$, $n=n_s+2N_u$ with $n_s\leqslant n$ and $N_u\geqslant 0$ while for $d\eta/dE<0$, $n=n_f+2N_I$ with $n_f>n$ and $N_I<0$.

The previously derived corrections to the equations of chemical and nuclear statistical equilibrium[1] were just the differences between the total number density of a constituent in a gas and its effective *free* number density. This, of course, was equal to the number density of unstable particles $[(\Delta n_i=\sum_j N_{ij}(1+\delta_{ij})$, to be exact$]$ in the gas that the constituent forms with itself and the other constituents. For $d\eta/dE<0$ the corrections are still the differences between the total and the effective free number densities, where, in this case, the interaction number density N_I replaces the unstable-particle number density N_u. The equivalence of N_I and N_u, however, indicates that as far as these corrections are concerned the unstable-particle formalism may be extended to the negative-lifetime case. The question of what the interpretation becomes under these circumstances will now be dealt with.

IV. UNSTABLE PSEUDOHOLES

The elastic scattering of two particles has been interpreted as the formation and decay of an unstable particle whenever $d\eta/dE\geqslant 0$. In order to construct a similar interpretation when $d\eta/dE<0$ we shall again use the technique of switching off the interaction and asking what *would have been*. Without the repulsive forces between them, the scattering particles *would have* entered the interaction region and *would have* spent a mean time there given by $\tau(E)=\hbar|d\eta/dE|$. While there, the centers of mass of all "colliding" pairs *would have* moved along with a Maxwellian velocity distribution. The rate at which the colliding particles *would have* entered their mutual-interaction regions is determined by their scattering cross sections as discussed in Ref. 1.

Without the repulsive interactions then, we *would have* had a gas of noninteracting particle pairs, each pair being localized for a positive finite time interval in some small interaction volume. The velocity distribution function for these particle pairs *would have* been Maxwellian, and *en masse* they *would have* exerted some net pressure on the gas containing walls. With the repulsive interactions, we have a *removed* gas of noninteracting localized particle pairs. Consequently, the *particle* number density of this *removed* gas will be negative and the replacement of "unstable pseudohole gas" for the cumbersome phrase "removed noninteracting localized finite-lifetime particle pair gas" would therefore seem appropriate.

We have argued in the subjunctive sense for the adoption of the term "unstable pseudohole gas" when the lifetimes of unstable particles become negative. Regarded as a gas of particles, pseudoholes have negative number densities given by Eq. (28) and from Ref. 1 tend to exert negative pressures:

$$\frac{P_I}{kT} = n^2\left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2} \frac{2^{3/2}}{\pi} \sum_l (2l+1) \times \int_0^\infty \frac{d\eta_l}{dE} (4\sin^2\eta_l - 1)\, e^{-E/kT} dE \,. \tag{29}$$

But these are effective properties. We may say that a gas of unstable particles with negative lifetimes behaves as if it were a gas of holes. But as individuals, pseudoholes lack the identity of unstable particles because it is impossible even to consider writing down localized wave functions describing them. Unlike holes or bubbles in a degenerate Fermi sea, pseudoholes only exist in the contrary sense of removed-particle pairs. As such, they are only intended as a conceptual convenience with which to extend the unstable-particle formalism to the negative-lifetime case. The formal equivalence of Eqs. (2), (3), and (4) with Eqs. (28), (26), and (27), respectively, was demonstrated without recourse to this interpretation, and should therefore indicate the permissibility of extending the formulas of Ref. 1 when $d\eta/dE < 0$.

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[1]Michael D. Delano, Phys. Rev. A 1, 1175 (1970).

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N-Fold Photoelectric Counting Statistics of Mixed Chaotic and Coherent Light*

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General formulas for the factorial cumulants of the photocounting distribution and the cumulants of the integrated intensity are obtained for chaotic light superposed with coherent light, for experiments involving N different counting intervals. The chaotic light may be partially polarized, need not be time stationary or single mode, may differ in frequency from the coherent light, and may have any spectral distribution.

I. INTRODUCTION

Chaotic light superposed with coherent light is one of a small number of cases for which an exact solution can be found for the probability distribution of photoelectric counts, or for arbitrarily high moments of this distribution. As in other areas of physics, exactly solvable problems in photoelectric counting facilitate physical understanding of problems which are less tractable theoretically, such as the counting statistics of scattered light.

The probability distribution of the number of photoelectrons produced in a counting interval T by chaotic-plus-coherent light incident on a single detector has been studied theoretically by several authors.[1-10] Closed general formulas for the generating function, or the factorial moments, or the probability distribution itself, were derived by Lachs,[1] Glauber,[3] and Magill and Soni,[4] assuming that: (i) The incident light is fully polarized; (ii) the counting time is short compared to the coherence time of the chaotic light; and (iii) the incident light is confined to a single mode of the electromagnetic field (which implies that the mean frequency of the chaotic light ω_T equals the frequency of the coherent light ω_C). Without using (ii) or (iii), Korenman calculated both the generating function and the factorial cumulants of the photocount distribution[5]; the latter have proved to be a particularly simple and useful form for summarizing the complete photocounting statistics. Assumptions (ii) and (iii) were removed in the formalism set up by Lachs, although closed formulas for the generating function and the higher-factorial moments were obtained only in the limit of short counting times.[6] Jakeman and Pike calculated the photocount generating function without (ii) or (iii) for the case of chaotic light with a Lorentzian spectrum[8]; results for an arbitrary spectral distribution were obtained by Peřina and Horâk.[9] Recently, Jaiswal and Mehta derived the factorial cumulants without any of these assumptions.[10]

Photoelectric counting experiments involving more than one counting interval and more than one