Hydrodynamic Equations for the Condensate and the Depletion of Helium II

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The equations of motion for the reduced density matrices of an interacting Bose system are treated using the concept of off-diagonal long-range order. The equations are decoupled on the one-particle level. With this approximation, the hydrodynamic equations for the densities and the velocities of the condensate and the depletion are derived. The interaction between the two fluid components depends on density variations and on the relative velocity. The latter interaction terms appear as kinetic pressure terms and a Magnus force term. Via the equation for the total momentum, a connection between the velocities of the condensate and the depletion, and the velocities in Landau's two-fluid model is derived. The excitation spectrum of the system is investigated in the phonon region. A comparison of the resulting velocities for the first and second sound with the experimentally determined ones shows that the functional dependence of the condensate density on temperature is similar to that of the superfluid component.

I. INTRODUCTION

In 1956, Penrose and Onsager[1] showed that, for superfluid systems, the reduced density matrix has a long-range off-diagonal order. More precisely, a part of the reduced single-particle density matrix $R_1(\vec{x}, \vec{y})$ factors into the product of two macroscopic wave functions. Generalizing the idea of off-diagonal long-range order (ODLRO), Yang[2] showed that, for superconductors, the reduced two-particle density matrix $R_2(\vec{x}, \vec{y}; \vec{x}', \vec{y}')$ factors. Owing to the interaction between the helium atoms, there are, even at $T = 0\,^{\circ}\mathrm{K}$, only about 10%[3-5] of all the particles in the ground state of the reduced-density-matrix operator. This fraction is called the condensate. The rest of the particles form the depletion. In 1967 Fröhlich[6] stressed that the formalism of the reduced density matrix is useful for the microscopic derivation of hydrodynamic equations. In particular, he showed that the symmetry properties of R_2 alone are sufficient in order to assure that the hydrodynamic equation of a normal fluid has the form of the Navier-Stokes equation. Combining this technique with the factorization property of R_1 for helium II, Fröhlich[7] succeeded in deriving an equation for the macroscopic wave function f of a superfluid. From this equation follow the equations of motion for the density of the condensate ρ_c and its velocity $\vec{v}_c$.[8] An important result was an expression for the transition rate between the condensate and the depletion, which is proportional to $\mathrm{div}\rho_c\rho_d \times(\vec{v}_c - \vec{v}_d)$, where ρ_d and $\vec{v}_d$ are the density and velocity of the depletion, respectively.

However, an equation for $\vec{v}_d$ was still missing. This gap will be filled in this paper, again using the formalism of the reduced density matrix. The hierarchy of equations of motion for the reduced density matrices will be decoupled on the one-particle level. In spite of this serious approximation, one gets the Navier-Stokes equation in the absence of a condensate, if one makes an expansion around equilibrium. Thus, the approximation does not alter the structure of the equation but influences the values of the coefficients only. Specifically, the form of the interaction terms between the condensate and the depletion should not be influenced by this approximation. For constant densities the resulting interaction force is proportional to

$$[\tfrac{1}{2}\nabla(\vec{v}_d - \vec{v}_c)^2 - (\vec{v}_d - \vec{v}_c)\times\mathrm{curl}\vec{v}_d],$$

where the second term describes a Magnus force. An equation for the total momentum is derived, where all the velocity-dependent interaction terms of the four hydrodynamic equations combine to a single term. By comparing the velocity-dependent expressions of the total momentum equation with those of Landau's theory, a connection between $\vec{v}_c$, $\vec{v}_d$ and $\vec{v}_s$, $\vec{v}_n$ is established. The four equations for ρ_c, $\vec{v}_c$, ρ_d, $\vec{v}_d$ do not yet, however, contain the temperature. At present, the temperature dependences of the densities ρ_c and ρ_d are still unknown. Linearizing the four equations around equilibrium, the spectrum of the system is obtained by Fourier analysis. From the phonon part, the velocities of first and second sound are obtained. They depend on the density of the condensate. From a comparison with the experimental sound velocities it is found that ρ_c has approximately the same temperature dependence as the density of the superfluid component ρ_s, i.e.,

$$\rho_c = \rho_c^0[(T_\lambda - T)/T_\lambda]^{\mu},$$

where $\mu < 1$.

II. REDUCED-DENSITY-MATRIX FORMALISM

In this section a brief review will be given of the one- and two-particle density matrices R_1 and R_2 for a system of interacting Bose particles. The

forms of R_1 and R_2 which will be used in this paper will be discussed. The Hamiltonian of the system is

$$H = (\hbar^2/2m)\int d^3x\, \nabla\psi^\dagger \cdot \nabla\psi + \tfrac{1}{2}\int\int d^3x\, d^3y\, V(|\vec{x}-\vec{y}|)\psi^\dagger(\vec{y})\psi^\dagger(\vec{x})\psi(\vec{x})\psi(\vec{y}) , \tag{2.1}$$

where the field operators ψ, $\psi^\dagger$ obey the Bose commutation relations and m is the mass of a He atom. For He, the pair potential $V(r)$ has the form

$$V(r) = \epsilon\,[(\sigma/r)^{12} - (\sigma/r)^6], \tag{2.2}$$

where the values of ϵ, σ are known from experiment,[9] $\epsilon = k_B\, 10.22\,^\circ\mathrm{K}$, $\sigma = 2.56$ Å. The reduced one-particle density matrix R_1 is defined as

$$R_1(\vec{x},\vec{x}') = \mathrm{Tr}\hat{\rho}\,\psi^\dagger(\vec{x}')\psi(\vec{x}) = R_1^*(\vec{x}',\vec{x}) , \tag{2.3}$$

where $\hat{\rho}$ is the density-matrix operator. Penrose and Onsager[1] showed that, for a superfluid, R_1 can be written as

$$R_1(\vec{x},\vec{x}') = f^*(\vec{x}')f(\vec{x}) + \tilde{R}_1(\vec{x},\vec{x}'), \tag{2.4}$$

where $f = \rho_c^{1/2}e^{i\theta}$ is the condensate wave function. For equilibrium and for zero temperature, various numerical calculations of R_1 have been carried out.[1,3-5] Under these conditions R_1 reduces to (see Appendix A)

$$R_1(\vec{x},\vec{x}') = N\int \prod_{i=2}^{N} d^3x_i\, \varphi_0(\vec{x},\vec{x}_2,\ldots,\vec{x}_N) \times \varphi_0(\vec{x}',\vec{x}_2,\ldots,\vec{x}_N), \tag{2.5}$$

where N is the total number of He atoms and φ_0 the ground-state wave function of the system. The calculations show that the condensate density ρ_c is, at $T = 0\,^\circ\mathrm{K}$, about 10% of the total density ρ. The incoherent part $\tilde{R}_1$, which describes the depletion, decays within a coherence length $\lambda_0 \sim 4$ Å (see Fig. 1).

R_1 satisfies the following equation of motion:

$$i\hbar\dot{R}_1(\vec{x},\vec{x}') = (\hbar^2/2m)(\nabla'^2 - \nabla^2)R_1(\vec{x},\vec{x}') + \int d^3y\,[V(|\vec{x}-\vec{y}|) - V(|\vec{x}'-\vec{y}|)]R_2(\vec{x},\vec{y};\vec{x}',\vec{y}) . \tag{2.6}$$

From the definition of

$$R_2(\vec{x},\vec{y};\vec{x}',\vec{y}') = \mathrm{Tr}\hat{\rho}\psi^\dagger(\vec{y}')\psi^\dagger(\vec{x}')\psi(\vec{x})\psi(\vec{y}) \tag{2.7}$$

follows

$$R_2(\vec{x},\vec{y};\vec{x}',\vec{y}') = R_2(\vec{y},\vec{x},\vec{x}',\vec{y}') = R_2(\vec{y},\vec{x};\vec{y}',\vec{x}') = R_2^*(\vec{x}',\vec{y}';\vec{x},\vec{y}),\ \int d^3y\, R_2(\vec{x},\vec{y};\vec{x}'\vec{y}) = (N-1)R_1(\vec{x},\vec{x}') . \tag{2.8}$$

These conditions suggest the following splitting[7]:

$$R_2(\vec{x},\vec{y};\vec{x}',\vec{y}) = [R_1(\vec{x},\vec{x}')R_1(\vec{y},\vec{y}) + R_1(\vec{x},\vec{y})R_1(\vec{y},\vec{x}') - f^*(\vec{x}')f(\vec{x})|f(\vec{y})|^2 + \tilde{R}_2(\vec{x},\vec{y};\vec{x}',\vec{y})] \times S(|\vec{x}-\vec{y}|)S(|\vec{x}'-\vec{y}|), \tag{2.9}$$

where the screening functions S have been introduced to eliminate the divergent hard-core part of the potential. The first term in (2.9) fulfills up to order $O(1)$ the last condition of (2.8), while the contribution of the other three terms is only of the order $O(1)$. For equilibrium, the diagonal part of R_2 is given by (see Appendix A)

$$R_2(\vec{x},\vec{y};\vec{x},\vec{y}) = \rho^2 g(|\vec{x}-\vec{y}|) . \tag{2.10}$$

Here $g(r)$ is the pair correlation function which can be measured by neutron diffraction experiments[10] and which has been calculated numerically.[3,5] If one combines these results with (2.9), one determines the spatial variation of $\tilde{R}_2$ (see Fig. 1). $\tilde{R}_2$ describes only properties of the depletion. If one neglects $\tilde{R}_2$ one still gets, in the absence of a condensate, the Navier-Stokes equation (see Sec. IV). On the other hand, Fröhlich[6] has shown that the Navier-Stokes equation follows quite generally from the symmetry properties of R_2. Neglecting $\tilde{R}_2$, thus, does not alter the structure of the resulting hydrodynamic equation for the depletion alone, but influences only the values of the coefficients (viscosities). Furthermore, the explicit interaction between the condensate and the depletion is fully contained in the first three terms of (2.9). In the following, $\tilde{R}_2$ is therefore neglected in spite of its relatively large contribution for real He (see Fig.

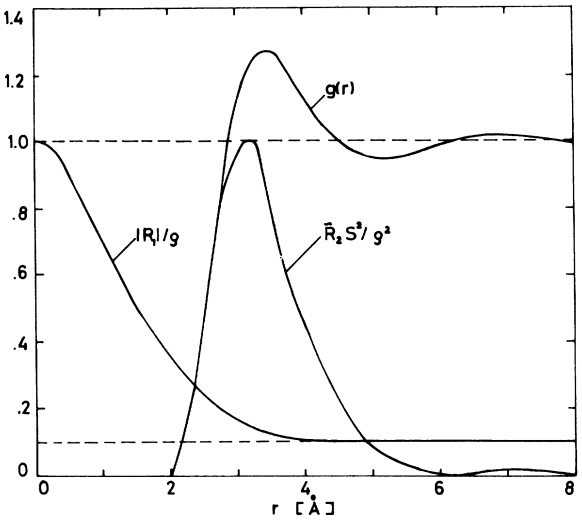

FIG. 1. Pair correlation function g and the absolute value $|R_1|/\rho$ of the reduced one-particle density matrix at $T = 0\,^\circ\mathrm{K}$ according to Ref. 3. $\tilde{R}_2$ is the nonfactorable part of the two-particle reduced density matrix R_2. The screening function S was taken to be the pair-function squared of Ref. 3.

1). With this approximation, one avoids treating the full hierarchy of higher-order reduced-density-matrix equations. In the framework of this theory, the coefficients which are influenced by the above approximation are not determined anyhow. They have to be taken from other calculations or from experiment.

III. HYDRODYNAMIC EQUATIONS

The equation of motion (2.6) of R_1 will now be treated using the forms (2.4) for R_1 and (2.9) with $\tilde{R}_2 = 0$ for R_2. To obtain the equation of motion for the condensate wave function f, the nonlocal limit $|\vec{x} - \vec{x}'| > \lambda_0$ has to be considered. In this limit, the depletion part $\tilde{R}_1(\vec{x}, \vec{x}')$ is already zero. One gets the following equation:

$$i\hbar \dot{f}(\vec{x}) + (\hbar^2/2m)\nabla^2 f(\vec{x}) = \int d^3y\, \bar{V}(|\vec{x} - \vec{y}|)[f(\vec{y})\tilde{R}_1(\vec{x}, \vec{y}) + f(\vec{x})R_1(\vec{y}, \vec{y})] , \tag{3.1}$$

where $\bar{V}(r) = V(r)S(r)$ is the screened potential. This equation has already been treated by Fröhlich.[7] The resulting equation and a physical interpretation of it are given in Appendix B.

Subtracting (3.1) from the original equation (2.6) yields an equation for the depletion:

$$\begin{aligned} i\hbar \dot{\tilde{R}}_1(\vec{x}, \vec{x}') + (\hbar^2/2m)(\nabla^2 - \nabla'^2)\tilde{R}_1(\vec{x}, \vec{x}') &= \int d^3y [\bar{V}(|\vec{x} - \vec{y}|) - \bar{V}(|\vec{x}' - \vec{y}|)] \\ &\times [\tilde{R}_1(\vec{x}, \vec{x}')R_1(\vec{y}, \vec{y}) + \tilde{R}_1(\vec{x}, \vec{y})\tilde{R}_1(\vec{y}, \vec{x}')] \\ &+ \int d^3y\, \bar{V}(|\vec{x} - \vec{y}|) f(\vec{x}) f^*(\vec{y}) \tilde{R}_1(\vec{y}, \vec{x}') \\ &- \int d^3y\, \bar{V}(|\vec{x}' - \vec{y}'|) f^*(\vec{x}') f(\vec{y}) \tilde{R}_1^*(\vec{y}, \vec{x}) . \end{aligned} \tag{3.2}$$

If one wants to derive the hydrodynamic equations, one does not need the full information contained in (3.2). It is sufficient to treat the equation in the local limit[6] $|\vec{x} - \vec{x}'| < \lambda_0$:

$$\tilde{R}_1(\vec{x}, \vec{x}') = \tilde{R}_1(\vec{x}, \vec{x}) + (\vec{x}' - \vec{x}) \lim_{\vec{x}' \to \vec{x}} \nabla' \tilde{R}_1(\vec{x}, \vec{x}') . \tag{3.3}$$

The connection with the macroscopic density ρ_d and the current density $\vec{j}_d$ is obtained via the following formulas:

$$\begin{aligned} \rho_d(\vec{x}) &= \tilde{R}_1(\vec{x}, \vec{x}), \\ \vec{j}_d(\vec{x}) &= \rho_d \vec{v}_d = (\hbar/2mi) \lim_{\vec{x}' \to \vec{x}} (\nabla - \nabla')\tilde{R}_1(\vec{x}, \vec{x}'). \end{aligned} \tag{3.4}$$

Thus, the first term in the expansion gives the continuity equation for ρ_d, and the second term the equation of motion for the velocity field $\vec{v}_d$ of the depletion. To treat the integrals on the right-hand side of (3.1) and (3.2), R_1 is expanded in these expressions around the origin of the potential up to the second order.

$\tilde{R}_1$ is now expressed as

$$\tilde{R}_1(\vec{x}, \vec{x}') = \sigma(\vec{x}, \vec{x}')\, e^{i\beta(\vec{x}, \vec{x}')} , \tag{3.5}$$

where

$$\sigma(\vec{x}, \vec{x}') = \sigma(\vec{x}', \vec{x}) , \quad \beta(\vec{x}, \vec{x}') = -\beta(\vec{x}', \vec{x}) , \tag{3.6}$$

$$f(\vec{x}) = \rho_c^{1/2}(\vec{x})\, e^{i\theta(\vec{x})} . \tag{3.7}$$

The velocity of the condensate is given by

$$\vec{v}_c(\vec{x}) = (\hbar/m)\nabla\theta(\vec{x}) , \tag{3.8}$$

with

$$\mathrm{curl}\vec{v}_c(\vec{x}) = 0 . \tag{3.9}$$

The local limits of σ, β, and their derivatives up to third order have now to be expressed in terms of ρ_d and $\vec{v}_d$. From (3.4) one sees immediately that

$$\rho_d(\vec{x}) = \sigma(\vec{x}, \vec{x}) , \quad \vec{v}_d(\vec{x}) = (\hbar/m) \lim_{\vec{x}' \to \vec{x}} \nabla\beta(\vec{x}, \vec{x}') . \tag{3.10}$$

The rest of the formulas are derived and listed in Appendix C. From these formulas it can be seen that the local limits of the second- and third-order derivatives of σ cannot be expressed only in derivatives of ρ_d alone. A symmetric kinetic-energy tensor

$$T_{ij}(\vec{x}) = -\frac{\hbar^2}{2m} \lim_{\vec{x}' \to \vec{x}} \frac{\partial^2 \sigma(\vec{x}, \vec{x}')}{\partial(\vec{x} - \vec{x}')_i \, \partial(\vec{x} - \vec{x}')_j} \tag{3.11}$$

has to be introduced. In the case of the derivatives of β the antisymmetric tensor $\frac{1}{2}(\partial v_i/\partial x_j - \partial v_j/\partial x_i)$ appears. This tensor can be expressed in terms of components of $\mathrm{curl}\vec{v}_d$. The appearance of these two tensors is characteristic for the depletion. The condensate is fully described in terms of ρ_c and the longitudinal $\vec{v}_c$, alone.

The results of the lengthy but straightforward calculation will now be listed together with the hydrodynamic equations for the condensate which follow from Eq. (3.1)[7,8]:

$$\dot{\rho}_c + \partial_i \rho_c v_{ci} = -(\kappa/\rho)\, \partial_i [\rho_c \rho_d (\vec{v}_d - \vec{v}_c)_i] = -\Gamma , \tag{3.12}$$

$$\dot{\rho}_d + \partial_i \rho_d v_{di} = \Gamma , \tag{3.13}$$

$$m\rho_c \frac{D_c v_{ci}}{Dt} = -w_0 \rho_c \partial_i (\rho_t + 2\rho_d) + \kappa_c \partial_i T_{ii} + A_i[\rho] + \kappa_c \partial_i [\tfrac{1}{2} m \rho_d (\vec{v}_d - \vec{v}_c)^2] , \tag{3.14}$$

$$m\rho_d \frac{D_d v_{di}}{Dt} = -2w_0 \rho_d \partial_i \rho + \sum_{(iil)} \partial_l [(\kappa - 1) T_{il}] - (\kappa_c - 1)\partial_i T_{ii} + B_i[\rho] + \kappa_d \{\partial_i [\tfrac{1}{2} m \rho_c (\vec{v}_d - \vec{v}_c)^2] - m\rho_c [(\vec{v}_d - \vec{v}_c) \times \mathrm{curl}\vec{v}_d]_i\} . \tag{3.15}$$

The transition rate Γ is proportional to the inter-

action constant

$$\kappa = w_2 m \rho/\hbar^2 , \tag{3.16}$$

where w_2 is one-third of the second moment of the screened potential

$$w_2 = \tfrac{1}{3}\int \bar{V}(r) r^2 d^3 r$$

and ρ is the total density, $\rho = \rho_c + \rho_d$. In κ_c and κ_d, ρ is replaced by ρ_c and ρ_d, respectively. Correspondingly, w_0 is the zeroth momentum of the screened potential, $w_0 = \int \bar{V}(r) d^3 r$. The values of w_0 and w_2 are strongly dependent on the screening function S and are not yet known. D_c/Dt and D_d/Dt are convective derivatives $\partial/\partial t + v_i \partial_i$, with velocities $\vec{v}_c$ and $\vec{v}_d$, respectively, where $\partial_i = \partial/\partial x_i$ and the summation convention is used. Furthermore, the notation

$$\sum_{\{ikl\}} a_{ikl} = a_{ikl} + a_{kli} + a_{lik}$$

is used. The force densities $\vec{A}[\rho]$ and $\vec{B}[\rho]$ contain only third-order derivatives of the densities,

$$A_l[\rho] = \frac{\hbar^2}{2m}\rho_c \partial_l \left[(1-\kappa_d)\frac{\Delta \rho_c^{1/2}}{\rho_c^{1/2}} - \kappa_d\left(\frac{\Delta\rho_c}{\rho_d} + \frac{5}{4}\frac{\Delta\rho_d}{\rho_d} + \frac{1}{2}\frac{\nabla\rho_d\cdot\nabla\rho_c}{\rho_d\rho_c}\right)\right], \tag{3.17}$$

$$B_l[\rho] = -(\hbar^2/2m)(\kappa/\rho)[\rho_d \partial_l \Delta\rho + \tfrac{1}{8}\partial_l \Delta\rho_d^2 + \rho_d \partial_l(\rho_c^{1/2}\Delta\rho_c^{1/2}) + \tfrac{1}{4}(\Delta\rho_d)\partial_l\rho_c + \tfrac{1}{2}(\partial_i\rho_d)\partial_l\partial_i\rho_c] . \tag{3.18}$$

Equations (3.12)–(3.15) would form a complete set of differential equations for the densities and velocities of the condensate and the depletion if an additional equation for the kinetic-energy tensor (3.11) could be given which couples this quantity back to the densities and velocities. One could try to find this missing information by going further in the Taylor expansion (3.3). It will, however, be shown in the following discussion that, for $\rho_c = 0$, one obtains from (3.15) the Navier-Stokes equation by writing down for T_{ij} and ρ_d the simplest expansions around equilibrium which are allowed by the symmetry properties of these quantities. In Sec. IV also the velocity-dependent interaction terms in (3.12)–(3.15) will be considered in detail and the equation for the total momentum will be given. Using the equation for the total momentum, comments will be made on the connection of this theory with Landau's two-fluid description. An investigation of the excitation spectrum contained in these equations will follow. From the phonon part the velocities of first and second sound are calculated. A comparison with experiments indicates a rough form of the temperature dependence of ρ_c. Here again the connection with Landau's model will be discussed.

IV. DISCUSSION OF TWO-FLUID EQUATIONS

In order to derive the Navier-Stokes equation for a normal fluid, i.e., $\rho_c = 0$ and $\rho_d = \rho$, the density ρ and the kinetic-energy tensor T_{ij} are expanded around their equilibrium values ρ_0 and $T_0\delta_{ij}$. Because in an ordinary fluid all material constants are scalars, no vectorial or tensorial coefficients can appear in the expansions. The simplest linear expansions are then

$$\rho = \rho_0 + a(p - p_0) + b\partial_i v_i , \tag{4.1}$$

$$T_{ij} = T_0\delta_{ij} + c(p-p_0)\delta_{ij} + d(\partial_j v_i + \partial_i v_j - \tfrac{2}{3}\partial_k v_k \delta_{ij}) + e\partial_k v_k \delta_{ij} , \tag{4.2}$$

where $a^{-1} = \partial p/\partial\rho = mu^2$, with u being the sound velocity. p_0 is the equilibrium value of the pressure p. Inserting (4.1) and (4.2) into (3.15) one gets the Navier-Stokes equation

$$m\rho \frac{D\vec{v}}{Dt} = -\nabla p + (\xi + \tfrac{1}{3}\eta)\nabla\cdot(\nabla\cdot\vec{v}) + \eta\Delta\vec{v} , \tag{4.3}$$

where

$$\eta = 2(\kappa - 1)d , \quad \xi = Cb + De ,$$

with

$$C = 5(\kappa/\rho)T_0 - 2w_0\rho_0 , \quad D = 5\kappa - 2 , \tag{4.4}$$

and the subsidiary condition

$$Ca + Dc = -1 .$$

Here ξ and η are the viscosity coefficients. The expansion coefficients of (4.1) and (4.2) can in principle be determined via the Wigner distribution function

$$f(\vec{p},\vec{x}) = (2\pi)^{-3}\int R_1(\vec{x}',\vec{x}'') e^{-i\vec{p}(\vec{x}'-\vec{x}'')} d^3(\vec{x}'-\vec{x}'') , \tag{4.5}$$

where $\vec{x} = \frac{1}{2}(\vec{x}' + \vec{x}'')$. The density, the current density, and the kinetic-energy tensor are then given by

$$\rho(\vec{x}) = \int d^3p f(\vec{p},\vec{x}) , \quad \vec{j}(\vec{x}) = \int d^3p\, \vec{p} f(\vec{p},\vec{x}) ,$$
$$T_{ij}(\vec{x}) = (1/2m)\int d^3p\, p_i p_j f(\vec{p},\vec{x}) - \tfrac{1}{2}\rho m v_i v_j . \tag{4.6}$$

From the equation of motion for R_1 (2.6) one gets an equation for the Wigner distribution. Approximating this equation by a Boltzmann equation, one could then apply the Chapman-Enskog procedure[11] to calculate the coefficients.

The velocity-dependent interaction terms in Eqs. (3.12)–(3.15) will now be discussed. One sees

that all these terms are proportional to κ, i.e., they are only present for an extended potential. A δ-potential approximation which is often used in the He theories would fail to produce these terms. It is interesting to note that the transition rate Γ vanishes not only for a homogeneous flow or a flow where $\vec{v}_d = \vec{v}_c$, but also for a vortex-type solution, where only an angular-velocity component exists which is, as are the densities, a function of r alone. If, however, in the presence of such a vortex, $\vec{v}_d$ has a laminar part flowing perpendicular to the vortex axis, a finite transition rate results.

The velocity-dependent force densities in (3.14) and (3.15) are derivatives of quadratic forms in the relative velocity $\vec{v}_d - \vec{v}_c$ and represent momentum transfers due to the transitions between the two fluids. The last term in the depletion equation is a Magnus force which couples the rotational motion in the depletion to the relative velocity. The remaining terms represent kinetic pressures. The flow will adjust itself to make these forces vanish. For constant densities the last two terms of (3.15) combine to yield

$$m\kappa_d\rho_c[(\vec{v}_d - \vec{v}_c)\cdot\nabla](\vec{v}_d - \vec{v}_c) . \tag{4.7}$$

Combining the full set of the two-fluid Eqs. (3.12)–(3.15), one obtains the equation for the total momentum density:

$$\begin{aligned}
&\vec{P} = m(\rho_c\vec{v}_c + \rho_d\vec{v}_d) ,\\
&\dot{P}_k = -w_0\partial_k(\rho^2 - \tfrac{1}{2}\rho_c^2) + \partial_k T_{ii} + \sum_{\{iik\}} \partial_i[(\kappa-1)T_{ik}]\\
&\qquad + A_k[\rho] + B_k[\rho] - \partial_i T^0_{ik;cd}\\
&\qquad + \tfrac{1}{2}m(\kappa/\rho)\sum_{\{iik\}} \partial_i[\rho_c\rho_d(v_{di} - v_{ci})(v_{dk} - v_{ck})] ,
\end{aligned} \tag{4.8}$$

where

$$T^0_{ik;cd} = m(\rho_c v_{ci} v_{ck} + \rho_d v_{di} v_{dk}) \tag{4.9}$$

is the usual stress tensor. One sees that all the velocity-dependent interaction terms in (3.12)–(3.15) combine to a single term in (4.8). From this equation, a connection with Landau's two-fluid model can be established. There, one has $\vec{P} = m(\rho_s\vec{v}_s + \rho_n\vec{v}_n)$, where the subscripts s and n refer to the super and normal component, respectively. Under the assumption that both theories are equivalent in the regime where the densities do not yet depend on the velocities, one gets from a comparison of the total momentum equations

$$\begin{aligned}
\partial_i T^0_{ik;sn} = \partial_i T_{ik;cd} - \tfrac{1}{2}m(\kappa/\rho)\\
\times \sum_{\{iik\}} \partial_i[\rho_c\rho_d(v_{di} - v_{ci})(v_{dk} - v_{ck})] .
\end{aligned} \tag{4.10}$$

Because of the form of the interaction term, $\vec{v}_c$, $\vec{v}_d$ and $\vec{v}_s$, $\vec{v}_n$ are not merely linked by an algebraic equation but by three quadratic first-order partial differential equations. This is to be expected because both $\vec{v}_c$ and $\vec{v}_s$ are longitudinal fields with vanishing curl. From (4.10) and the expressions for ρ and $\vec{P}$ it follows that, whenever $\vec{v}_c = \vec{v}_d$, all four velocities are equal: $\vec{v}_s = \vec{v}_n = \vec{v}_c = \vec{v}_d$.

Finally, the excitation spectrum of the four hydrodynamic equations (3.12)–(3.15) will be investigated in the long-wavelength limit, i.e., only the phonon part will be considered at present. For this purpose, the four equations are linearized around the constant-equilibrium values ρ_c^0, ρ_d^0, $\vec{v}_c = \vec{v}_d = \vec{0}$. The kinetic-energy tensor is expanded in the same way as for the derivation of the Navier-Stokes equation [see (4.2)]. The terms in (4.2) proportional to derivatives of the velocities are neglected, i.e., no dissipative processes are considered:

$$T_{ij} = [T_0 + c(p - p_0)]\delta_{ij} . \tag{4.11}$$

The pressure p is also expanded:

$$p - p_0 = \frac{\partial\rho}{\partial\rho_d}(\rho_d - \rho_d^0) + \frac{\partial\rho}{\partial\rho_c}(\rho_c - \rho_c^0) , \tag{4.12}$$

and used to eliminate the pressure in (4.11). The deviations of the densities and velocities from equilibrium are taken to be longitudinal plane waves,

$$\rho_{c,d} = \rho^0_{c,d} + \rho'_{c,d}e^{i(\Omega t + kx)} , \quad v_{x;c,d} = v'_{x;c,d}e^{i(\Omega t + kx)} . \tag{4.13}$$

For the phonon part of the spectrum, only terms linear in k have to be considered. The terms $\vec{A}[\rho]$ and $\vec{B}[\rho]$ are of third order in k and will be important for the roton part of the spectrum, which is however not investigated here. The secular equation of the four resulting linear equations determines the dispersion law

$$\Omega = ku , \tag{4.14}$$

where the sound velocity u is given by

$$u^2_{\mathrm{I,II}} = \tfrac{1}{2}F[1 \pm (1 - 4G/F^2)^{1/2}] , \tag{4.15}$$

where

$$F = \frac{1}{m}\frac{\partial p}{\partial\rho} + \frac{1}{m}\frac{\rho_c}{\rho}\varphi(\rho_c) , \quad G = \frac{1-\kappa}{m^2}\frac{\rho_c}{\rho}\gamma(\rho_c) . \tag{4.16}$$

The functions φ and γ are given by

$$\begin{aligned}
&\varphi(\rho_c) = \rho w_0(\kappa x - 1) + \frac{2c}{\rho}\frac{\partial p}{\partial x}(\kappa - 1)^2 ,\\
&\gamma(\rho_c) = 2(w_0\rho)^2(1-x) - \frac{\partial p}{\partial\rho}w_0\rho[1 + 3c\kappa(1-x)]\\
&\qquad + \frac{c}{\rho}\frac{\partial p}{\partial x}w_0\rho[2(1+x) - \kappa(8 - x + 3x^2)]\\
&\qquad + 3c\frac{\kappa}{\rho}\frac{\partial p}{\partial\rho}\frac{\partial p}{\partial x}[1 - c(2 - 5\kappa)] ,
\end{aligned} \tag{4.17}$$

where the variables $x=\rho_d/\rho$ and ρ are used instead of ρ_c and ρ_d. The equilibrium value $T_0\delta_{ij}$ of the kinetic-energy tensor has been approximated by its value at T_λ with the aid of the subsidiary condition (4.4):

$$2w_0\rho-5\frac{\kappa}{\rho}T_0=\frac{\partial p}{\partial\rho}[1-c(2-5\kappa)], \qquad (4.18)$$

where $\partial p/\partial\rho=m u_I^2(T=T_\lambda)$. This approximation is expected to be rather good because the depletion density ρ_d varies between T_λ and 0 only by 10%.

In a two-fluid model there are two possible eigenmodes for longitudinal excitations. In the first mode, the velocities of the two components are in phase. The two density modulations are also in phase with each other and therefore this mode corresponds to the normal sound (first sound). In the second-sound mode, the two velocities and also the two density modulations have a relative phase of 180°. This is the second-sound branch. The first-sound branch corresponds to the positive sign in (4.15), while the second-sound branch corresponds to the negative sign, because less energy is stored in this mode.

For $\rho_c\to 0$, i.e., $T/T_\lambda-1\to-0$, Eq. (4.16) shows that $u_I^2\to(1/m)(\partial p/\partial\rho)$ and $u_{II}^2\to 0$, in agreement with experiment. To get an estimate of the behavior of the two branches in the whole region $0\leq\rho_c\leq 0.1\rho$ (i.e., $T_\lambda\geq T\geq 0$) the following set of parameters is used: $\rho=2.2\times10^{22}$ cm^{-3}, $u_I(T=T_\lambda)=2.2\times10^4$ cm sec^{-1}, $\partial p/\partial x=0$ for $T=T_\lambda$ and $\partial p/\partial x=-3\times10^6$ erg cm^{-3} for $T=0$, $w_0=-\alpha\,2.65\times10^{-37}$ erg cm^3, $\kappa=-32\alpha$, $\alpha=0.35$, $c=0.08$. The values of $\partial p/\partial x$ are estimated from experimental data[12] for $\partial p/\partial T$. w_0 and κ are calculated from (2.2) with a cutoff radius $r=\sigma$. To allow for a slightly different screening, the factor α is introduced. The values of α and c are chosen so that $u_I(T=0)=2.4\times10^4$ cm sec^{-1} and $u_{II}(T=0)=(1/\sqrt{3})\,u_I(T=0)$. The variation of the two branches is schematically given in Fig. 2. For this set of parameters, also, the eigenvectors have been calculated and are found to have the correct phase relations for both modes. If one assumes in the vicinity of T_λ, a temperature dependence of the condensate density

$$\rho_c=\rho_c^0[(T_\lambda-T)/T_\lambda]^\mu\,, \quad \mu<1 \qquad (4.19)$$

both branches have near T_λ the same form as the experimentally determined curves (see Fig. 3).

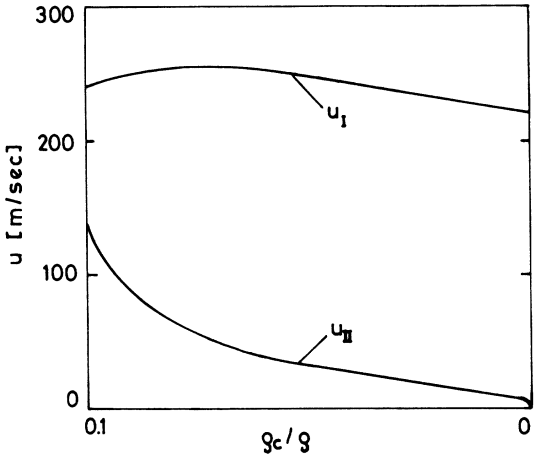

FIG. 2. Schematical diagram of the dependence of the velocities of first and second sound on the condensate density.

Equation (4.19) shows that the condensate density varies with the temperature in a manner similar to that of the supercomponent ρ_s in Landau's theory.

To establish a complete connection between Landau's two-fluid description and the present one, it will be necessary to introduce thermodynamical quantities in the equations for the condensate and the depletion. Though this full thermodynamical theory does not yet exist, in this paper a relation between the velocities in both theories was established and good evidence for a relation between the densities was obtained.

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APPENDIX A

In order to establish the connection between numerical calculations[1,3-5] and experiments[10] the reduced density matrices R_1 and R_2 are given in various representations. The definition (2.3) reads in the x representation

$$R_1(\vec{x},\vec{x}')=\int\prod_{i=1}^{N}d^3x_i\langle\vec{x}_1,\ldots\vec{x}_N|\,\tilde{\rho}\psi^\dagger(\vec{x}')\psi(\vec{x})\,|\vec{x}_1,\ldots,\vec{x}_N\rangle$$

$$=\sum_{j=1}^{N}\int\prod_{i=1}^{N}d^3x_i\,\langle\vec{x}_1,\ldots,\vec{x}_N|\,\hat{\rho}\,\psi^\dagger(\vec{x}')\,|\vec{x}_1,\ldots,\vec{x}_{j-1},\vec{x}_{j+1},\ldots,\vec{x}_N\rangle\,\delta(\vec{x}-\vec{x}_j).$$

Integration over x_j and application of the creation operator $\psi^\dagger$ yields

$$R_1(\vec{x},\vec{x}') = \sum_{j=1}^{N} \int \prod_{i=1,i\neq j}^{N} d^3x_i \langle \vec{x}_1,\ldots,\vec{x}_{j-1},\vec{x},\vec{x}_{j+1},\ldots,\vec{x}_N|\hat{\rho}|\vec{x}_1,\ldots,\vec{x}_{j-1},\vec{x}',\vec{x}_{j+1},\ldots,\vec{x}_N\rangle$$

$$= N\int \prod_{i=2}^{N} d^3x_i \langle \vec{x},\vec{x}_2,\ldots,\vec{x}_N|\hat{\rho}|\vec{x}',\vec{x}_2,\ldots,\vec{x}_N\rangle . \quad \text{(A1)}$$

The density-matrix operator is now written as

$$\hat{\rho} = \sum_n e^{-\beta(E_n-\mu N)}\,|n\rangle\langle n|,$$

where $|n\rangle$ are the eigenvectors of the Hamiltonian. At $T=0\,^\circ$K only the ground state survives. Therefore, one gets

$$R_1(\vec{x},\vec{x}') = N\int \prod_{i=2}^{N} d^3x_i\, \varphi_0(\vec{x},\vec{x}_2,\ldots,\vec{x}_N)\varphi_0(\vec{x}',\vec{x}_2,\ldots,\vec{x}_N), \quad \text{(A2)}$$

where the ground-state wave function is

$$\varphi_0(\vec{x}_1,\ldots,\vec{x}_N) = \langle \vec{x}_1,\ldots,\vec{x}_N|0\rangle .$$

A reduced-density-matrix operator $\hat{R}_1$ may be defined as[1]

$$\hat{R}_1 = N\,\mathrm{Tr}_{2,\ldots,N}\,\hat{\rho}$$

$$= N\int \prod_{i=2}^{N} d^3x_i \langle \vec{x}_2,\ldots,\vec{x}_N|\hat{\rho}|\vec{x}_2,\ldots,\vec{x}_N\rangle .$$

A comparison with (A1) shows that

$$\langle \vec{x}|\hat{R}_1|\vec{x}'\rangle = R_1(\vec{x},\vec{x}') . \quad \text{(A3)}$$

Introducing the number representation for $\hat{R}_1$, one gets

$$\hat{R}_1|m\rangle = r_m|m\rangle, \quad R_1(\vec{x},\vec{x}') = \sum r_m \varphi_m^*(\vec{x}')\varphi_m(\vec{x}) ,$$

with

$$\varphi_m(\vec{x}) = \langle \vec{x}|m\rangle .$$

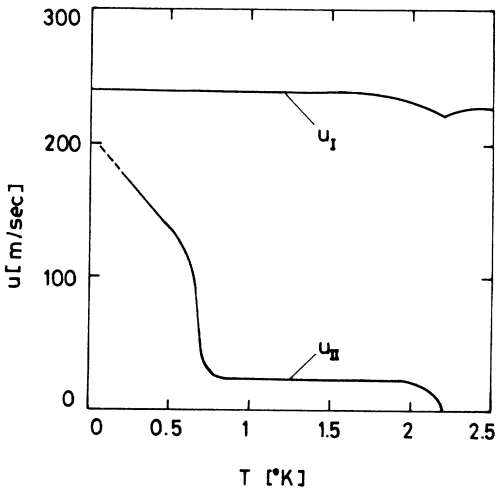

FIG. 3. Experimental temperature dependence of the velocities of first and second sound according to Ref. 12.

The condensate is therefore the ground state of $\hat{R}_1$. For $T=0\,^\circ$K, the pair correlation function g is defined[3] as

$$g(|\vec{x}-\vec{y}|) = \rho^{-2} \sum_{i,j=1,i\neq j}^{N} \int \prod_{l=1}^{N} d^3x_l\, \delta(\vec{x}-\vec{x}_i)\delta(\vec{y}-\vec{x}_j)$$

$$\times \varphi_0^2(\vec{x}_1,\ldots,\vec{x}_N) . \quad \text{(A4)}$$

Using the same techniques as above one can show that

$$R_2(\vec{x},\vec{y};\vec{x},\vec{y}) = \mathrm{Tr}\hat{\rho}\psi^\dagger(\vec{x})\psi^\dagger(\vec{y})\psi(\vec{y})\psi(\vec{x}) = \rho^2 g(|\vec{x}-\vec{y}|) . \quad \text{(A5)}$$

APPENDIX B

The equation of motion (3.1) for the order parameter f will not be discussed further. An interpretation of the resulting differential equation is obtained[13] from a comparison with the phenomenological theory of Pitaevskii.[14]

After a Taylor expansion of f and $\hat{R}_1$ around the origin of the potential, (3.1) takes the form[7]

$$i\hbar\dot{f} + \frac{\hbar^2}{2m}\Delta f - Uf = -\frac{1}{2}\frac{\kappa_d}{\rho_d^2}\left(\frac{i\hbar}{m}\rho_d\nabla + \rho_d v_d\right)^2 mf . \quad \text{(B1)}$$

Here potential U is

$$U = w_0\rho + w_0\rho_d + \frac{\hbar^2}{2m}\frac{\kappa}{\rho}\left(\Delta\rho + \tfrac{1}{4}\Delta\rho_d - \frac{2m}{\hbar^2}T_{ii}\right) . \quad \text{(B2)}$$

The symbols w_0, κ_d, and T_{ij} are defined in Sec. III. The complex differential equation (B1) is nonlinear in f owing to the ρ_c dependence of the potential U. In the linear regime near T_λ where $\rho_c/\rho \ll 1$ and for a slowly varying depletion density ρ_d, Eq. (B1) can be put in the form[13]

$$i\hbar\dot{f} + \frac{\hbar^2}{2m}\Delta f - (\mu+\mu_c)mf$$

$$= -\kappa_d\left\{\left[\frac{1}{2}\left(\frac{i\hbar}{m}\nabla + v_d\right)^2 + \mu_c\right]mf\right\} , \quad \text{(B3)}$$

if one splits, formally, U into

$$U = m[\mu + \mu_c(1-\kappa_d)] . \quad \text{(B4)}$$

The chemical potentials are defined as

$$m\mu = \frac{\partial E_{\rm pot}}{\partial \rho} \quad \text{and} \quad \mu_c = \frac{\partial E_{\rm pot}}{\partial \rho_c} \quad . \tag{B5}$$

In the considered regime one shows that (B4) holds indeed. For this purpose the potential energy

$$E_{\rm pot} = \tfrac{1}{2} \int d^3y \, V(|\vec{x}-\vec{y}|) R_2(\vec{x},\vec{y};\vec{x},\vec{y}) \tag{B6}$$

is evaluated by a Taylor expansion around the origin of the potential V using the form (2.9) for R_2. The result is

$$\mu_c = 0, \quad m\mu = 2w_0\rho - (\kappa/\rho) T_{ii} \quad . \tag{B7}$$

A comparison with (B2) shows that for $\rho_c/\rho \ll 1$ and $\rho_d \approx \rho = \text{const}$, (B4) is fulfilled.

Equation (B3) can now be interpreted as follows: Putting the left-hand side equal to zero one obtains the Gross-Pitaevskii[15] equation in its linearized form. This equation contains no dynamic interaction with the depletion. On the other hand, one obtains the Ginzburg-Pitaevskii[16] equation by putting the curly bracket on the right-hand side equal to 0. This equation determines the order parameter in equilibrium. In a nonequilibrium situation, the curly bracket is not equal to zero and acts as a driving force[14] in (B3). The factor κ_d is the kinetic coefficient which is real in contrast to the imaginary coefficient introduced by Pitaevskii[13,14]

APPENDIX C

In (3.5) and (3.6) the amplitude and the phase of $\tilde{R}_1$ are introduced. In order to derive the connections between the local limits of the derivatives of σ and β and the quantities ρ_d, V_d, T_{ij} [see (3.10) and (3.11)], and their derivatives, σ and β are written in their most general form:

$$\sigma(\vec{x},\vec{y}) = \sum_{\{n_i\}\{m_j\}} a_{\vec{n}\vec{m}} \prod_{i,j=1}^{3} x_i^{n_i} y_j^{m_j} \quad , \tag{C1}$$

$$\beta(\vec{x},\vec{y}) = \sum_{\{n_i\}\{m_j\}} b_{\vec{n}\vec{m}} \prod_{i,j=1}^{3} x_i^{n_i} y_j^{m_j} \quad , \tag{C2}$$

with

$$a_{\vec{n}\vec{m}} = a_{\vec{m}\vec{n}}, \quad b_{\vec{n}\vec{m}} = -b_{\vec{m}\vec{n}} \quad . \tag{C3}$$

Equation (B3) follows from the symmetries (3.6). The notation $\partial_i = \partial/\partial x_i$, $\partial_i' = \partial/\partial x_i'$, and $\lim = \lim$ as $\vec{x}' \to \vec{x}$ is used. With the aid of (B1)–(B3) the following formulas are derived:

a. First derivatives:

$$\lim \partial_i \sigma(\vec{x},\vec{x}') = \lim \partial_i' \sigma(\vec{x},\vec{x}') = \tfrac{1}{2}\partial_i \rho_d \, ,$$

$$\lim \partial_i \beta(\vec{x},\vec{x}') = -\lim \partial_i' \beta(\vec{x},\vec{x}') = (m/\hbar) v_{di} \, ;$$

b. Second derivatives:

$$\lim \partial_i \partial_j \sigma(\vec{x},\vec{x}') = \lim \partial_i' \partial_j' \sigma(\vec{x},\vec{x}') = \tfrac{1}{4}\partial_i \partial_j \rho_d - (2m/\hbar^2) T_{ij} \, ,$$

$$\lim \partial_i \partial_j' \sigma(\vec{x},\vec{x}') = \lim \partial_i' \partial_j \sigma(\vec{x},\vec{x}') = \tfrac{1}{4}\partial_i \partial_j \rho_d + (2m/\hbar^2) T_{ij} \, ,$$

$$\lim \partial_i \partial_j \beta(\vec{x},\vec{x}') = -\lim \partial_i' \partial_j' \beta(\vec{x},\vec{x}') = (m/2\hbar)(\partial_i v_{dj} + \partial_j v_{di}) \, ,$$

$$\lim \partial_i \partial_j' \beta(\vec{x},\vec{x}') = -\lim \partial_i' \partial_j \beta(\vec{x},\vec{x}') = (m/2\hbar)(\partial_j v_{di} - \partial_i v_{dj}) \, ;$$

c. Third derivatives:

$$\lim \partial_i \partial_j \partial_k \sigma(\vec{x},\vec{x}') = \lim \partial_i' \partial_j' \partial_k' \sigma(\vec{x},\vec{x}') \, ,$$

$$\lim \partial_i \partial_j' \partial_k' \sigma(\vec{x},\vec{x}') = \lim \partial_i' \partial_j \partial_k \sigma(\vec{x},\vec{x}') \, ,$$

$$\lim (\partial_i \partial_j \partial_k' + \partial_i \partial_j' \partial_k') \sigma(\vec{x},\vec{x}') = \tfrac{1}{4}\partial_i \partial_j \partial_k \rho_d + (2m/\hbar^2)\partial_j T_{ik} \, ,$$

$$\lim (\partial_i \partial_j \partial_k + \partial_i \partial_j \partial_k') \sigma(\vec{x},\vec{x}') = \tfrac{1}{4}\partial_i \partial_j \partial_k \rho_d - (2m/\hbar^2)\partial_k T_{ij} \, ,$$

$$\lim \partial_i \partial_j \partial_k \beta(\vec{x},\vec{x}') = -\lim \partial_i' \partial_j' \partial_k' \beta(\vec{x},\vec{x}') \, ,$$

$$\lim \partial_i \partial_j \partial_k' \beta(\vec{x},\vec{x}') = -\lim \partial_i' \partial_j' \partial_k \beta(\vec{x},\vec{x}') \, ,$$

$$\lim (\partial_i \partial_i \partial_j' - \partial_i' \partial_i' \partial_j) \beta(\vec{x},\vec{x}') = (m/\hbar)\partial_j \partial_i v_{di} \, ,$$

$$\lim (\partial_i \partial_j \partial_k + \partial_i \partial_j' \partial_k + \partial_i' \partial_j \partial_k' + \partial_i \partial_j \partial_k') \beta(\vec{x},\vec{x}') = (m/\hbar)\,\partial_j \partial_k v_{di} \, .$$

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