Statistical Mechanics of Persistent Currents in Quantum Fluids*

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The Landau condition, $\vec{\nabla}\times\vec{v}_s=0$, for He II and the analogous Meissner effect in superconductors are shown to follow from the existence of persistent currents by the use of a new definition of the quasiequilibrium state. The novel aspect of this definition is the requirement that the effective Hamiltonian defining the density matrix is invariant under infinitesimal quasi-Galilei transformations. In applications to He II and superconductors, the existence of a condensate implies that the density matrix does not have the full symmetry of the effective Hamiltonian. By slightly relaxing the quasiequilibrium conditions we obtain the usual Landau equations for $\vec{v}_s$ (for the case that $\vec{v}_n=0$).

I. INTRODUCTION

Landau's phenomenological two-fluid theory certainly provides the most successful and widely used description of He II, but there are aspects of superfluid flow which cannot be treated within this framework. Two closely related phenomena of particular interest are the quantization of circulation and the Josephson effect, both of which must be imposed *ad hoc* on the two-fluid theory. From the microscopic point of view, the usual argument for the quantization of circulation involves the assumption that the superfluid velocity is given by the gradient of the phase of some sort of condensate wave function.[1] In a recent publication[2] we gave an alternative formulation in which the superfluid velocity $\vec{v}_s(x)$ was introduced as the thermodynamic parameter conjugate to the experimentally observed persistent mass currents. We were then able to carry out a statistical-mechanical derivation of the quantization condition; however, we still had to impose the following conditions on the thermodynamic parameters:

$$\vec{\nabla}\times\vec{v}_s=0,\quad \vec{\nabla}(\mu+\tfrac{1}{2}v_s^2)=0,\quad \vec{\nabla}T=0\ , \tag{1}$$

where $T(x)$ and $\mu(x)$ are, respectively, the local values of the temperature and chemical potential (per unit mass). The first of these equations is the well-known Landau condition, which is a basic postulate of the two-fluid theory, and the remaining equations are consequences of the theory for the steady-state case. In the present paper, we will eliminate this last remaining reference to the two-fluid theory by exhibiting a statistical-mechanical derivation of Eq. (1). Thus, we now have a purely statistical-mechanical description of steady superfluid flows, including the quantization of circulation, which is based solely on the experimental observation of persistent currents.

The Landau condition, which is supposed to hold for general time-dependent flows has been the subject of much discussion.[3] In this connection, Putterman and Uhlenbeck[4] recently considered a version of the two-fluid theory in which the Landau condition is not assumed. They were able to obtain Eq. (1) by imposing thermodynamic stability requirements on the steady-state solutions. Their success in deriving the Landau condition on thermodynamic grounds encouraged us to seek a statistical-mechanical derivation independent of the two-fluid model.

Although this paper refers explicitly to the problem of superfluid He, the techniques are general and can also be applied to superconductors. A formal correspondence between the two problems is provided by the substitution $m\vec{v}_s\to(e/c)\vec{A}$; consequently, we find $\vec{\nabla}\times\vec{A}=0$. In other words, an argument analogous to that yielding the Landau condition can be used to derive the Meissner effect.

Section II contains our definition of a quasiequilibrium state and shows that Eq. (1) constitutes the necessary consistency conditions imposed by the invariance properties of the state. In Sec. III we apply an analogous argument to superconductors and derive the Meissner effect. Section IV discusses a possible generalization to slightly non-quasiequilibrium states which leads to the postulated Landau equations for $\vec{v}_s$ in the time-dependent case. In Sec. V we summarize the assumptions and results.

II. QUASIEQUILIBRIUM STATES

Since the notion of quasiequilibrium to be used in this paper is modeled on that of absolute equilibrium, we begin by reviewing some basic concepts of equilibrium statistical mechanics. The statistical operator describing an ensemble can be obtained by maximizing the conventional entropy expression (or minimizing the information) subject to the set of constraints on observables appropriate to the ensemble in question. The family of observables usually consists of the total Hamiltonian H together with a set of constants of the motion $\{F_i\}$. The Lagrange-multiplier method yields a statistical operator $\mathfrak{D}$ of the form

$$\mathfrak{D} = \exp[-\Omega - \beta(H - \sum_i \lambda_i F_i)] \ ,$$

where $\beta = 1/kT$ and $\beta\lambda_i$ is the Lagrange multiplier conjugate to F_i. We will call an operator constructed in this way the *ensemble operator* corresponding to the given set of constraints. It follows immediately from the definition that the ensemble operator has the same symmetries as the effective Hamiltonian

$$H_{\mathrm{eff}} = H - \sum_i \lambda_i F_i \ .$$

This behavior should be contrasted with that of the density matrix describing the actual state of the system which often does not have the full symmetry of the effective Hamiltonian. It is this possibility which necessitates the special terminology introduced above. The extra information responsible for this symmetry-breaking behavior is not usually imposed as a further constraint; instead, it is incorporated into the definition of ensemble averages by restricting the sum over states to a suitable subspace of many-body wave functions. This can be done formally by writing the density matrix as a restricted ensemble operator

$$\mathfrak{D}_r \equiv \mathcal{P}\mathfrak{D} \ ,$$

where $\mathcal{P}$ is the appropriate projection operator. To ensure consistency between the density matrix and the original ensemble, $\mathcal{P}$ must be a constant of the motion and also commute with $\mathfrak{D}$. A simple example of a density matrix having broken symmetry is provided by a crystal lattice. The effective Hamiltonian is invariant under continuous translations and rotations whereas the correct density matrix is only invariant under a discrete subgroup (the space group of the lattice).

The various persistent-current phenomena cannot be conveniently described by means of the absolute equilibrium theory outlined above. The usual approach has been to treat these phenomena within the framework of local thermodynamic equilibrium theory. The basic idea of that theory is to partition the system into macroscopically small regions, each described by an equilibrium ensemble operator. The space-time variation of the local thermodynamic parameters is fixed by imposing the usual continuity equations. In this approach, no global ensemble operator describing the entire system is defined. This is in contrast to the work of Mori,[5] who explicitly defines a global ensemble operator for the purpose of deriving the hydrodynamic equations for normal systems.

For our purposes, the definition of a global ensemble operator is essential; the derivation of the conditions in Eq. (1) depends crucially on this idea. Moreover, the quantization of circulation is a consequence of global boundary conditions imposed on the many-body wave functions.[2] We will proceed by generalizing the absolute equilibrium theory just outlined to yield a definition of the quasiequilibrium state which is especially suitable for the description of persistent flows.

By analogy with the absolute equilibrium theory we compute the entropy from the quasiequilibrium ensemble operator $\mathfrak{D}$ in the canonical way,

$$S = -k\,\mathrm{Tr}\,\mathfrak{D}\ln\mathfrak{D} \quad \text{with } \mathrm{Tr}\,\mathfrak{D} = 1 \ . \tag{2}$$

The form of $\mathfrak{D}$ is determined by maximizing S subject to the appropriate set of constraints.[6] We impose the constraints

$$\langle \mathcal{H}(x)\rangle = \mathcal{E}(x), \quad \langle \rho(x)\rangle = \eta(x), \quad \langle \vec{j}(x)\rangle = \vec{J}(x) \ , \tag{3}$$

where $\langle F\rangle \equiv \mathrm{Tr}\,\mathfrak{D}F$; the operators $\mathcal{H}$, ρ, and $\vec{j}$ are, respectively, the densities of energy, mass, and momentum; and the right-hand sides in Eq. (3) are given c-number functions of position. The fact that $\vec{J} \neq 0$ is our definition of the superfluid state. It should be remarked that these functions must vary slowly on the microscopic scale. In the usual way, we introduce a Lagrange multiplier for each constraint [let the multipliers be denoted by $\beta(x)$, $\mu(x)$, $\vec{v}_s(x)$] and find by a standard calculation

$$\mathfrak{D} = \exp\{-\Omega - \int d^3x\ \beta(x)[\mathcal{H}(x) - \vec{v}_s(x)\cdot\vec{j}(x) - \mu(x)\rho(x)]\} \ , \tag{4}$$

where Ω is the normalization constant, $\beta(x)$ is the reciprocal temperature, and $\mu(x)$ is the local chemical potential (per unit mass). Since we are interested in steady-state phenomena, the thermodynamic parameters will be assumed to be time independent. For persistent superfluid flows this approximation is essentially exact. The parameter $\vec{v}_s$ may be interpreted as the superfluid velocity since, in the steady state, the normal fluid is necessarily at rest with respect to the container walls.

We now have an expression for $\mathfrak{D}$ which satisfies the necessary condition of time independence. This expression by itself, however, does not constitute a complete definition of quasiequilibrium. To see what is lacking, recall that the absolute equilibrium ensemble operator, in addition to being independent of time, also has simple Galilean transformation properties. In particular, it is invariant under rotations and spatial translations. It is the analog of the Galilean transformation properties which is missing in our definition. As a guide to the correct replacement of the usual Galilei group, we will express the theory in terms of quantities which are invariant under Galilean transformations. Note that this is done implicitly in the usual absolute equilibrium theory by evaluating the various observables (e.g., the total Hamiltonian, the total momentum, etc.) in the center-of-mass frame for the whole system. More generally, the correct procedure is to express the theory in terms of observables from which the coherent motion has been eliminated. In our case, the relevant coherent motion is not that of the local center of mass but instead is described by $\vec{v}_s$, since that is the parameter conjugate to the measured persistent current. Thus, we should formulate the theory in terms of local observables referred to the local frame in which the coherent motion vanishes; i.e., the frame for which $\vec{v}_s(x)=0$. First, we introduce the Hamiltonian density $\mathcal{H}_0$ referred to the local frame; it is expressed in terms of observables referred to the laboratory frame by

$$\mathcal{H}_0=\mathcal{H}-\vec{v}_s\cdot\vec{j}+\tfrac{1}{2}\rho v_s^2 \quad . \tag{5}$$

Note that $\mathcal{H}_0$ is obtained from $\mathcal{H}$ by the replacement

$$(\hbar/i)\vec{\nabla}\to(\hbar/i)\vec{\nabla}-m\vec{v}_s \quad ,$$

where m is the mass of the He atom. The same replacement leads to the appropriate local momentum density

$$\vec{j}_0=\vec{j}-\rho\vec{v}_s \quad . \tag{6}$$

Thus, we can express $\mathfrak{D}$ in terms of explicitly invariant quantities by

$$\mathfrak{D}=\exp\{-\Omega-\int d^3x\ \beta[\mathcal{H}_0-(\mu+\tfrac{1}{2}v_s^2)\rho]\} \quad . \tag{7}$$

If we consider the subgroup of spatial translations, with generators given by

$$\vec{P}=\int d^3x\ \vec{j} \quad , \tag{8}$$

then the form of the local momentum density given in Eq. (6) suggests that the natural replacement for $\vec{P}$ is[7]

$$\vec{\Pi}=\int d^3x\ \vec{j}_0=\vec{P}-\int d^3x\ \rho\vec{v}_s \quad . \tag{9}$$

In order to gain some insight into the significance of the operator $\vec{\Pi}$, we continue the program of expressing all quantities in Galilean invariant form by replacing the usual field operator $\psi(x)$ with the field operator $\phi(x)$ referred to the local frame. Clearly, the two operators are related by a Galilean transformation, but the fact that $\vec{v}_s$ varies from point to point prevents us from using the usual formula. However, the required generalization is easily seen to be

$$\phi(x)=e^{iF(x)}\psi(x) \quad , \tag{10}$$

where

$$F(x,t)=-(m/\hbar)\int_\gamma(d\vec{y}\cdot\vec{v}_s-\tfrac{1}{2}\,d\tau\ v_s^2) \quad , \tag{11}$$

and γ is an integration path in space-time ending at the point $(\vec{x},t)$. This definition of ϕ is analogous to that employed by Dirac and Mandelstam in their path-dependent formulations of quantum electrodynamics.[8] The significance of $\vec{\Pi}$ is made clear by the following calculation:

$$\begin{aligned}[\phi(x),\vec{\Pi}]&=e^{iF(x)}[\psi(x),\vec{\Pi}]\\&=e^{iF(x)}[\psi(x),\vec{P}-\int d^3y\ \rho(y)\vec{v}_s(y)]\\&=e^{iF(x)}[(\hbar/i)\vec{\nabla}\psi(x)-m\vec{v}_s(x)\psi(x)]\\&=(\hbar/i)\vec{\nabla}(e^{iF}\psi)=(\hbar/i)\vec{\nabla}\phi \quad .\end{aligned}$$

In other words, $\vec{\Pi}$ is the generator of spatial translations for the field $\phi(x)$. In a similar way, we can construct appropriate modifications of each of the remaining generators of infinitesimal Galilei transformations; but we omit the explicit expressions, since they are not needed in what follows. The infinitesimal transformations induced by the modified generators will be called quasi-Galilei transformations.

We are thus led to complete the definition of quasiequilibrium by requiring the ensemble operator to have the same invariance under quasi-Galilei transformations as the equilibrium ensemble operator has under ordinary Galilei transformations. In particular, the ensemble operator must be invariant under quasitranslations. The necessary and sufficient condition for this invariance is that the $\mathfrak{D}$ commutes with the generators $\vec{\Pi}$; this is equivalent to the condition

$$\vec{C}\equiv\int d^3x\ \beta\{[\mathcal{H}_0,\vec{\Pi}]-(\mu+\tfrac{1}{2}v_s^2)[\rho,\vec{\Pi}]\}=0 \quad . \tag{12}$$

Using Eq. (9), we easily find

$$[\rho,\vec{\Pi}]=(\hbar/i)\vec{\nabla}\rho$$

and

$$[\mathcal{H}_0(x),\vec{\Pi}]=[\mathcal{H}_0(x),\vec{P}]-\int d^3y\ [\mathcal{H}_0(x),\rho(y)]\vec{v}_s(y) \quad .$$

According to Eq. (5), we have

$$[\mathcal{H}_0(x),\rho(y)]=[\mathcal{H}(x),\rho(y)]-\vec{v}_s\cdot[\vec{j}(x),\rho(y)] \quad ,$$

where the operators are given by

$$\begin{aligned}\rho&=m\psi^\dagger\psi \quad ,\\ \vec{j}&=(\hbar/2i)(\psi^\dagger\vec{\nabla}\psi-\vec{\nabla}\psi^\dagger\psi) \quad ,\\ \mathcal{H}(x)&=(\hbar^2/2m)\vec{\nabla}\psi^\dagger(x)\cdot\vec{\nabla}\psi(x)+\tfrac{1}{2}\int d^3y\\ &\quad\times\psi^\dagger(x)\psi^\dagger(y)u(y-x)\psi(y)\psi(x) \quad ,\end{aligned}$$

and $u(y-x)$ is the interaction potential. A straightforward calculation yields

$$\begin{aligned}[\vec{j}(x),\rho(y)]&=-i\hbar\rho(x)\vec{\nabla}_x\delta(x-y) \quad ,\\ [\mathcal{H}_0(x),\rho(y)]&=-i\hbar\vec{j}_0(x)\cdot\vec{\nabla}_x\delta(x-y) \quad ,\\ [\mathcal{H}_0(x),P_l]&=-i\hbar[\nabla_l\mathcal{H}-(v_s)_k\nabla_l j_k+\tfrac{1}{2}v_s^2\nabla_l\rho] \quad .\end{aligned} \tag{13}$$

We next substitute these results into Eq. (12) for $\vec{C}$. In those terms involving gradients of operators, we integrate by parts. After collecting terms and using some vector identities we finally obtain

$$(i\hbar)^{-1}\vec{C} = \int d^3x \, \{\vec{\nabla}\beta[\mathcal{H}_0 - (\mu + \tfrac{1}{2}v_s^2)\rho] - \beta\rho\vec{\nabla}(\mu + \tfrac{1}{2}v_s^2) - \beta\vec{j}_0 \times (\vec{\nabla}\times\vec{v}_s)\} \quad . \tag{14}$$

It is shown in Appendix A that the operators $\mathcal{H}_0$, ρ, and $\vec{j}_0$ may be regarded as mutually independent; consequently, the invariance condition $\vec{C} = 0$ can only be satisfied if their coefficients vanish. Therefore, we have derived the Landau condition $\vec{\nabla}\times\vec{v}_s = 0$, as well as the other conditions in Eq. (1). Thus, from our point of view, Eq. (1) represents the necessary consistency conditions imposed by the invariance of the ensemble operator under quasitranslations. Since all of the desired conditions have been obtained from quasitranslation invariance, we should show that the required invariance properties for the remainder of the quasi-Galilei transformations do not impose any new conditions. The necessary calculations are similar to the one given above; consequently, we omit them and simply give the result. In each case, the correct transformation property is guaranteed by the conditions in Eq. (1).

In the discussion of the absolute equilibrium theory at the beginning of this section, we stressed the possibility that the density matrix describing the actual state of a system could have less symmetry than the ensemble operator because of the inclusion of additional information. This possibility also exists for the quasiequilibrium theory developed here; that is, the quasiequilibrium density matrix may fail to have the full quasi-Galilean invariance of the ensemble operator. In applying this theory to superfluid flows in He II, the quasi-Galilean invariance of the ensemble operator is indeed broken by the existence of a condensate. In fact, it is shown in Appendix B that the quasitranslation invariance of the ensemble operator $\mathfrak{D}$ yields

$$\langle \vec{j}(x)\rangle = \langle\rho\rangle\,\vec{v}_s(x) \quad ;$$

that is, it describes a system which is *entirely* composed of superfluid at all temperatures. Consequently, the density matrix describing persistent currents in He II corresponds to broken quasitranslation invariance.

So far, we have not considered the problems arising from possible singularities in the thermodynamic parameters; however, we must allow for singularities in $\vec{v}_s$ in order to describe vortices. Allowing $\vec{v}_s$ to become singular on a vortex line is a mathematical idealization of the structure of the vortex core and, for singular $\vec{v}_s$, the many-body wave functions are required to vanish if any argument lies on a vortex line. More explicitly, we expect a δ-function behavior in $\vec{\nabla}\times\vec{v}_s$ concentrated on the vortex line, and it is necessary to show that these singularities do not influence the calculation of $\vec{C}$. In Appendix A, the singular term $\vec{\nabla}\times\vec{v}_s(x)$ appears multiplied by the mass density operator $\rho(x)$, but the boundary conditions on the wave functions ensure that all matrix elements of $\rho(x)$ vanish if $\vec{x}$ lies on a vortex line; therefore, the δ-function singularities make no contribution and the argument remains valid.

III. MEISSNER EFFECT

In this section, we will show that an argument analogous to that just given for superfluid He can be applied to the problem of superconductivity to yield the Meissner effect. In the presence of a static external magnetic field with vector potential $\vec{A}_e$, the Hamiltonian density is

$$\mathcal{H}_e = \mathcal{H} - (1/c)\,\vec{j}\cdot\vec{A}_e + (e/2mc^2)\,\rho A_e^2 \quad , \tag{15}$$

where $\mathcal{H}$ is the Hamiltonian density in the absence of any field; $\vec{j}$ and ρ are given by

$$\vec{j} = (e\hbar/2mi)(\psi^\dagger\vec{\nabla}\psi - \vec{\nabla}\psi^\dagger\circ\psi) \quad , \rho = e\psi^\dagger\psi \quad , \tag{16}$$

and ψ is the electron field operator. The correct gauge-invariant expression for the electron current in the presence of $\vec{A}_e$ is

$$\vec{j}_e = \vec{j} - (e/mc)\rho\vec{A}_e \quad . \tag{17}$$

For normal metals, one has $\langle\vec{j}_e(x)\rangle = 0$ almost exactly; but for a superconductor, which can support persistent currents, this equation need not hold.[9] In the latter case, we can describe the system by a quasiequilibrium density matrix as in Sec. II. Thus, we maximize the canonical entropy expression, Eq. (2), subject to the constraints

$$\langle\mathcal{H}_e(x)\rangle = \mathcal{E}(x), \quad \langle\rho(x)\rangle = \eta(x), \quad \langle\vec{j}_e(x)\rangle = \vec{J}(x) \tag{18}$$

and obtain the ensemble operator

$$\mathfrak{D} = \exp\{-\Omega - \int d^3x \,\beta(x)\,[\mathcal{H}_e(x) - (1/c)\,\vec{A}_i(x)\cdot\vec{j}_e(x) - \Phi(x)\rho(x)]\} \quad , \tag{19}$$

where $\beta(x)$, $\Phi(x)$, and $(1/c)\vec{A}_i(x)$ are the Lagrange multipliers for the constraints in Eq. (18). If we introduce the explicit expressions for $\mathcal{H}_e$ and $\vec{j}_e$ into Eq. (19), we can rewrite it in the form

$$\mathfrak{D} = \exp\{-\Omega - \int d^3x \,\beta[\mathcal{H}_0 - (\Phi + eA_i^2/2mc^2)\rho]\} \quad , \tag{20}$$

where

$$\mathcal{H}_0 \equiv \mathcal{H} - (1/c)\,\vec{j}\cdot\vec{A} + (eA^2/2mc^2)\rho \quad \text{and} \quad \vec{A} \equiv \vec{A}_e + \vec{A}_i \quad . \tag{21}$$

The Lagrange multiplier $\vec{A}_i$ has the obvious physical interpretation of the induced vector potential, while Φ is an average electrostatic potential.

To apply the second part of our definition for quasiequilibrium, we must identify the quasitranslations. In the present case, the key lies in gauge rather than Galilean invariance, but the formal procedure is identical. That is, we make the substitution

$$(\hbar/i)\vec{\nabla} \to (\hbar/i)\vec{\nabla} - (e/c)\vec{A}$$

to obtain the modified translation generator

$$\vec{\Pi} = (m/e) \int d^3x\, \vec{j}_0(x) \quad , \vec{j}_0 = \vec{j} - (e/mc)\rho\vec{A} \quad . \tag{22}$$

The arguments in favor of this definition are identical to those given in Sec. II with appropriate changes in terminology. Just as in Sec. II, the requirement that $\mathfrak{D}$ commutes with $\vec{\Pi}$ leads to the consistency conditions

$$\vec{\nabla}\times\vec{A} = 0, \quad \vec{\nabla}\beta = 0, \quad \vec{\nabla}[\Phi + eA_i^2/2mc^2)] = 0 \quad . \tag{23}$$

Note that the first of these equations, which represents the Meissner effect, involves the total $\vec{A}$, whereas the last equation only involves the induced field $\vec{A}_i$. The latter condition represents a kind of electrostatic Bernoulli's equation.[10]

IV. SMALL DEVIATIONS FROM QUASIEQUILIBRIUM

The conclusions reached in Secs. I–III are all rigorous consequences of the definition of quasiequilibrium. In this section, we propose a generalization to slightly nonquasiequilibrium states. The ensemble operator is assumed to have the form given in Eq. (4), but the thermodynamic parameters are allowed to depend on time as well as position. We choose to work in the Schrödinger picture so that the operators have no time dependence, and we use the effective Hamiltonian, defining $\mathfrak{D}$ as the generator of time translations

$$H' = \int d^3x\, \theta(\mathcal{H} - \vec{v}_s\cdot\vec{j} - \mu\rho) \quad , \tag{24}$$

where $\theta(x) \equiv T_0/T(x)$, and T_0 is the spatial average of $T(x)$; i.e.,

$$T_0 = V^{-1}\int_V d^3x\, T(x) \quad ,$$

where V is the volume of the system. This choice for H' is analogous to the replacement of the usual translation generator $\vec{P}$ by the quasitranslation generator $\vec{\Pi}$. The time dependence of the parameter $\vec{v}_s$ is determined by imposing the Heisenberg equation of motion

$$i\hbar\frac{\partial\vec{\Pi}}{\partial t} = [\vec{\Pi}, H'] \quad . \tag{25}$$

On the other hand, the time dependence of $\vec{\Pi}$ is explicitly given by Eq. (9) as

$$\frac{\partial\vec{\Pi}}{\partial t} = -\int d^3x\, \rho\frac{\partial\vec{v}_s}{\partial t} \tag{26}$$

since we are using the Schrödinger picture. Thus, we have

$$\int d^3x\, \rho\frac{\partial\vec{v}_s}{\partial t} = \frac{1}{i\hbar}[H', \vec{\Pi}] \quad , \tag{27}$$

and, by using Eq. (14), we find

$$\vec{\nabla}\theta = 0, \quad \vec{\nabla}\times\vec{v}_s = 0 \quad , \tag{28a}$$

$$\frac{\partial\vec{v}_s}{\partial t} + \vec{\nabla}(\mu + \tfrac{1}{2}v_s^2) = 0 \quad . \tag{28b}$$

The definition of $\theta(x)$, together with the first of Eqs. (28), yields $\theta \equiv 1$; this fact was used in writing the third equation. Note that this equation is useful in deriving the Josephson effect[1] in He II. We have thus obtained a specialization of the Landau two-fluid equations for the case of no temperature gradients and no normal fluid motion. The restricted form of the equations is due to the restricted form of the nonquasiequilibrium ensemble operator.

The corresponding calculation for a superconductor leads to

$$-\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = \vec{\nabla}\left(\Phi + \frac{e}{2mc^2}A_i^2\right)$$

or (29)

$$\vec{E} = (e/2mc^2)\,\vec{\nabla}A_i^2 \quad ,$$

where $\vec{E}$ is the electric field. Similar relations are given by London.[10]

V. SUMMARY

The principal results of this paper are the derivations of the Landau condition $\vec{\nabla}\times\vec{v}_s = 0$, and the Meissner effect $\vec{\nabla}\times\vec{A} = 0$. These conditions, in turn, imply the quantization of circulation[2] and flux,[11] respectively; consequently, these effects can be understood on a purely statistical-mechanical basis with no reference to phenomenological theories.

We start from the experimental observation of persistent currents and proceed to construct a suitable statistical-mechanical description of such states. This description is provided by our definition of the quasiequilibrium state. A density matrix $\mathfrak{D}_r$ describes a quasiequilibrium state if it is a restriction of a quasiequilibrium ensemble operator $\mathfrak{D}$: i.e., (a) $\mathfrak{D}_r = \mathcal{P}\mathfrak{D}$ for some projection operator $\mathcal{P}$; (b) $\mathfrak{D}$ is time independent and maximizes the entropy, subject to the constraints appropriate to the system in question; and (c) $\mathfrak{D}$ is invariant under quasi-Galilei transformations. The generators of the infinitesimal quasi-Galilei transformations are constructed from a modified momentum density operator whose form depends on the physical nature of the persistent current. For He II, the modified momentum density is simply the momentum density referred to the local superfluid rest frame. For superconductors, the usual momentum density is replaced by the standard gauge-invariant form in Eq. (22).

In a less rigorous way, we have given a possible generalization to slightly nonquasiequilibrium states which yields the usual Landau equations for $\vec{v}_s$. However, there is no provision for normal fluid motion so that these equations are only valid for situations in which the normal fluid motion has already been dissipated.

APPENDIX A

We wish to prove that the operator condition $\vec{C}=0$, where $\vec{C}$ is given by Eq. (14), leads to Eq. (1). Since $\vec{C}$ is to be the zero operator, it must commute with any operator; therefore, a necessary condition for $\vec{C}=0$ is given by

$$\vec{Q}\equiv[\rho(g),[\rho(g),\vec{C}]]=0 \quad ,$$

where

$$\rho(g)\equiv\int d^3x\ \rho(x)g(x) \quad ,$$

and g is a real-valued test function. First, note that $[\rho(g),[\rho(g),\vec{j}(x)]]\equiv 0$, since the commutator $[\rho,\vec{j}]$ is proportional to ρ. Then, according to Eq. (14), we have

$$\vec{Q}=\int d^3x\ \vec{\nabla}\beta(x)[\rho(g),[\rho(g),\mathcal{H}_0(x)]] \quad .$$

Substituting for the inner commutator from Eq. (13) leads to

$$\vec{Q}=i\hbar\int d^3x\ \vec{\nabla}\beta\,\nabla_l g[\rho(g),j_l(x)]$$
$$=-\hbar^2\int d^3x\ |\vec{\nabla}g|^2\rho\vec{\nabla}\beta \quad .$$

Since $|\nabla g|^2\rho$ is a positive operator, the condition $\vec{Q}=0$ requires $\vec{\nabla}\beta=0$.

The first term in $\vec{C}$ has now been eliminated so we next impose

$$[\rho(g),\ \vec{C}]=0 \quad ,$$

which yields

$$\int d^3x\ \rho\vec{\nabla}g\times(\vec{\nabla}\times\vec{v}_s)=0 \quad .$$

Since g is arbitrary, we must have $\vec{\nabla}\times\vec{v}_s=0$.

The third term in $\vec{C}$ has now been eliminated so that we are left with

$$\int d^3x\ \rho\vec{\nabla}(\mu+\tfrac{1}{2}v_s^2)=0 \quad ,$$

but again ρ is a positive operator so we must have $\vec{\nabla}(\mu+\frac{1}{2}v_s^2)=0$.

APPENDIX B

The ensemble operator $\mathfrak{D}$ satisfies $[\mathfrak{D},\vec{\Pi}]=0$; consequently, we have

$$\langle\nabla_k\rho(x)\rangle=\mathrm{Tr}\,\mathfrak{D}(i/\hbar)[\rho,P_k]$$
$$=(i/\hbar)\mathrm{Tr}\,\mathfrak{D}[\rho,\Pi_k]$$
$$=(i/\hbar)\mathrm{Tr}[\mathfrak{D},\Pi_k]\rho$$
$$=0 \quad ;$$

i.e., $\langle\rho(x)\rangle=\langle\rho\rangle$, a constant. Similarly we find

$$\langle\nabla_k j_l(x)\rangle=(i/\hbar)\mathrm{Tr}\,\mathfrak{D}[j_l(x),\ P_k]$$
$$=(i/\hbar)\mathrm{Tr}\,\mathfrak{D}\{[j_l(x),\Pi_k]+\int d^3y$$
$$\times\ v_k(y)[j_l(x),\rho(y)]\}$$
$$=\langle\rho(x)\rangle\nabla_l v_k(x)$$
$$=\nabla_l(\langle\rho\rangle v_k(x)) \quad ,$$

so that

$$\langle\vec{j}(x)\rangle=\vec{I}+\langle\rho\rangle\vec{v}(x) \quad ,$$

where $\vec{I}$ is constant. Since the current must vanish at large distances from all vortex lines we must have $\vec{I}=0$.

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