in region I the scattering of charged particles is due to ordinary plasmons, while the scattering of photons and neutrons is due to ionic sound.

As x increases into region II, the scattering of charged particles and of neutrons is still due to plasmons and ionic sound, respectively. However, the scattering of photons switches from being due to ionic sound to being due to plasmons.

In region III, as mentioned earlier, the electrons and the nuclei become independent. In this region the electrons behave as if they were an ideal Fermi gas and hence the scattering of photons behave as if they were being scattered by an ideal Fermi gas. On the other hand, when x is in region III but $x < \frac{3}{4}\gamma/\sqrt{\delta}$, the scattering of neutrons is due to plasma oscillations of the nuclei. If $x > \frac{3}{4}\gamma/\sqrt{\delta}$, then the nuclei behave as an ideal gas and consequently, the scattering of neutrons is as from an ideal gas of nuclei.

It is interesting to note that it is not until $\frac{3}{4}\gamma/\sqrt{\delta} \ll x$ that the statistics of the nuclei become important. In fact, even though our calculations assume that all species are in the ground state, it is actually only necessary that the electrons satisfy this condition provided $x \ll \frac{3}{4}\gamma/\sqrt{\delta}$.

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Low-Energy Scattering by Long-Range Potentials*

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The phase shift for low-energy scattering by a potential consisting of a long-range and a short-range part, to first order in the long-range potential, is derived using the two-potential formula of scattering theory. For the r^{-4} and for the r^{-6} potentials and for their superposition, $\tan\eta_0$ is calculated up to and including the $k^5 \ln k$ terms, where η_0 is the zero-angular-momentum ($L=0$) phase shift. For the r^{-4} potential and $L>0$, $\tan\eta_L$ is calculated up to and including the $k^{2L+3}\ln k$ term. Finally, the long-range correction to the matrix element for photodissociation of negative ions is calculated.

I. INTRODUCTION

Effective-range theory (ERT) originated in the analysis of low-energy neutron-proton and proton-proton scattering and was an attempt to interpret the data while bypassing the requirement for any detailed knowledge of the short-range nuclear potential. In both cases, the form of the wave function was known explicitly outside the range of the nucleus. It was then, perhaps, natural that the first attempt to analyze low-energy electron-atom scattering, where the dominant long-range interaction is $-\frac{1}{2}\alpha e^2/r^4$ (α is the static electric-dipole polarizability of the atom), should proceed along similar lines; the fact that the exact solution of the radial Schrödinger equation with the r^{-4} potential could be expressed in terms of modified Mathieu functions was used in the analysis. Ultimately, since the modified Mathieu functions are rather complicated, a (modified) ERT was obtained in which the effect of the polarizability was treated in first order; but the formalism, as developed, placed strong emphasis on the detailed properties of the modified Mathieu functions.[1,2]

The significance of the present paper, in this regard, is that these functions need never be introduced into the analysis if one is interested in first-order effects only.

Even in first order, the threshold energy dependence for angular momenta L greater than zero differs from that for short-range potentials. The physics of the situation – namely, why it is that, for sufficiently large L, the coefficients and the form of the energy dependence of the first terms in ERT for any long-range potential are given exactly in the Born approximation – was understood but not deduced mathematically except for the r^{-4} case.[2]

A number of subsequent derivations have been given which do not require the exact solution, including an asymptotic-expansion approach[3] and an iteration of the integral equation of scattering,[4] but these derivations still contain some unnecessary elements of complexity. In this paper, we will present a much simpler derivation based on the well-known two-potential formula of scattering theory. The derivation contains one weakness: the requirement that the short-range potential vanish identically beyond some specified distance d. We assume here that we are concerned with an equivalent one-body potential. The connection with the original many-body problem has been studied,[5,6] though not completely.

Subsequent to the original derivation of an ERT, to first order in α, a formal analysis, good to all orders in α, was developed[6] in terms of a long-range phase shift ρ and functions $\tilde{C}$ and $\tilde{h}$ which are the analogs of C and h which appear in ERT for short-range plus Coulomb interactions. Numerical tables of ρ, $\tilde{C}$, and $\tilde{h}$ have been published,[4] and analytic forms for these functions have also been obtained.[7] The question then arises as to what purpose is served by a rederivation of the more primitive result – the form of ERT valid only to first order in α. For one, the first-order theory often suffices, and it is useful to have a simplified derivation which makes transparent the lack of analyticity in the wave number k of the forms which arise in ERT, and which simplifies the application of ERT to problems involving indirectly the elastic scattering of electrons by atoms. An example is the energy shift of an atom containing a highly excited electron in the presence of a second atom. A second purpose is that it greatly simplifies the determination of further terms in the ERT expansions for the r^{-4} potential and for any long-range potential; we will, in fact, give the ERT expansion for the r^{-4} potential to all powers in k^2. Finally, and most significantly, the considerable simplification of the derivation should point the way to the extension of ERT to the more complicated, but more interesting, case of multichannel scattering involving long-range forces.

The two-potential derivation is presented in Sec. II. It is used in Secs. III–V to calculate the phase shift for $V+U$, where V is the short-range potential and U is the r^{-4} potential, the r^{-6} potential, and a superposition of the two. In Sec. VI, we calculate the long-range correction to the matrix element for the photodetachment of an electron from a negative ion.

II. TWO-POTENTIAL DERIVATION

We use the following notation. L is the orbital-angular-momentum quantum number. U and V are the long- and short-range potentials, respectively, and $\bar{U}=(2m/\hbar^2)U$. Here, V vanishes identically for $r>d$ and need not be known, and U vanishes identically for $r<d$ and is presumed known. η_{Ls}, A_{Ls}, r_{Ls} and η_L, A_L, r_L are the phase shift, scattering length, and effective range associated with V and $U+V$, respectively. $I(L)$, $J(L)$, and $K(L)$ are integrals defined by Eqs. (2.5)–(2.7). k is the wave number of the scattered particle. $r^{-1}u_L$ is the eigenfunction of the radial Schrödinger equation of energy $E=\hbar^2k^2/2m$ with both U and V present. $r^{-1}u_{Ls}$ satisfies the same equation with only V acting. u_L and u_{Ls} satisfy the boundary conditions

$$u_L(0)=u_{Ls}(0)=0\ ,$$

$$u_L\to\sin(kr-\tfrac{1}{2}L\pi)+\tan\eta_L\cos(kr-\tfrac{1}{2}L\pi),\quad\text{as}\quad r\to\infty$$

$$u_{Ls}\to\sin(kr-\tfrac{1}{2}L\pi)+\tan\eta_{Ls}\cos(kr-\tfrac{1}{2}L\pi),\quad\text{as}\quad r\to\infty\ .$$

From conventional ERT, we have

$$k^{-2L-1}\tan\eta_{Ls}=-A_{Ls}-\tfrac{1}{2}r_{Ls}A_{Ls}^2k^2+O(k^4)\ . \qquad (2.1)$$

By multiplying the radial Schrödinger equation containing V alone by u_L and that containing $U+V$ by u_{Ls}, subtracting and integrating, we obtain the standard two-potential formula,[8]

$$\tan\eta_L=\tan\eta_{Ls}-k^{-1}\int_d^\infty u_{Ls}\bar{U}u_L\,dr\ . \qquad (2.2)$$

The range of integration is limited from d to ∞ because U vanishes for $r<d$. Since we are concerned with first-order effects due to U, and since u_L differs from u_{Ls} by a term of order U, we arrive at

$$\tan\eta_L=\tan\eta_{Ls}-k^{-1}\int_d^\infty u_{Ls}^2(r)\bar{U}(r)\,dr+O(U^2)\ . \qquad (2.3)$$

For $r>d$, u_{Ls} represents a free wave with phase shift η_{Ls} due to the short-range potential V. Hence, we find

$$u_{Ls}=krj_L(kr)-\tan\eta_{Ls}krn_L(kr),\quad r>d\ .$$

j_L and n_L are the usual spherical Bessel and Neumann functions, respectively. Substituting into (2.3) and adding and subtracting $[\tan\eta_{Ls}krj_L(kr)]^2$ in the integrand, we find

$$\tan\eta_L=\tan\eta_{Ls}-k^{-1}(1-\tan^2\eta_{Ls})I(L)+2k^{-1}\tan\eta_{Ls}J(L)$$
$$-k^{-1}\tan^2\eta_{Ls}K(L)+O(U^2)\ , \qquad (2.4)$$

where

$$I(L)\equiv\int_d^\infty(kr)^2j_L^2(kr)\bar{U}(r)\,dr\ , \qquad (2.5)$$

$$J(L)\equiv\int_d^\infty(kr)^2j_L(kr)n_L(kr)\bar{U}(r)\,dr\ , \qquad (2.6)$$

$$K(L) \equiv \int_d^\infty (kr)^2 [j_L^2(kr) + n_L^2(kr)] \bar{U}(r)\, dr \; . \qquad (2.7)$$

The determination of an ERT is thereby reduced to the relatively trivial problem of determining $I(L)$, $J(L)$, and $K(L)$ as expansions in k and $\ln k$, and of regrouping terms to eliminate the parameters due to the purely short-range potential while introducing the minimum number of parameters to characterize the actual short-range plus long-range potential. [The appearance of $\ln|k|$ terms exhibits the lack of analyticity of $\tan\eta_L$ as a function of k. This nonanalyticity associated with long-range forces is well known from dispersion theory. In the present context, the nonanalyticity of $\tan\eta_L$ is evident from the fact that the integrals $I(L)$ and $J(L)$ are infinite for $\mathrm{Im}k \neq 0$; this is particularly easy to see for $L=0$. We would like to thank Dr. E. Gerjuoy for this observation.]

One method of performing the integrations is given in Ref. 4. The integrals are evaluated to all orders in k in the Appendices.

Since we are here concerned only with terms linear in U, the effects of the different long-range tails contained in U can be studied separately and then effectively added. We begin with the r^{-4} component.

III. r^{-4} POTENTIAL

The interaction of a charged particle with a neutral polarizable system in a spherically symmetric ground state has a long-range interaction $U(r) = -\frac{1}{2}\alpha e^2/r^4$. We then have

$$\bar{U}(r) = \frac{2m}{\hbar^2} U(r) = -\frac{\alpha/a_0}{r^4} \equiv -\frac{\beta^2}{r^4} \; , \quad r > d$$

with β having the dimensions of a length. For electrons of a few eV incident on an atom, a partial-wave decomposition is useful. The $L=0$ and $L>0$ cases must be treated separately. We begin with $L=0$.

Using (2.1) and the results of Appendix A in (2.4), we find

$$\tan\eta_0 = -A_0 k - \tfrac{1}{3}\pi(\beta k)^2 - \tfrac{4}{3}A_{0s}\beta^2 k^3 \ln|2kd| - \tfrac{1}{2} r_0 A_0^2 k^3 + \tfrac{1}{3}\pi\beta^2 A_{0s}^2 k^4 - \tfrac{2}{3}\beta^2 r_{0s} A_{0s}^2 k^5 \ln|2kd| + O(\beta^4, k^5) \; , \qquad (3.1)$$

where

$$A_0 = A_{0s} + \beta^2 \Delta \; , \qquad (3.2a)$$

$$\Delta = -(d^2 - A_{0s} d + \tfrac{1}{3}A_{0s}^2)/d^3 \; , \qquad (3.2b)$$

$$r_0 A_0^2 = r_{0s} A_{0s}^2 + \beta^2 \delta \; , \qquad (3.3a)$$

$$\delta = -[\tfrac{2}{3}d^4 - 2A_{0s}^2 d^2 + \tfrac{2}{3} r_{0s} A_{0s}^3 - r_{0s} A_{0s}^2 d - \tfrac{4}{3}(2C - \tfrac{11}{3})A_{0s} d^3]/d^3 \; . \qquad (3.3b)$$

Note that Δ and δ are independent of β. C is Euler's constant.

Since A_0 and r_0 differ from A_{0s} and r_{0s}, respectively, by terms of order β^2, we can rewrite (3.1) as

$$\tan\eta_0 = -(kA_0 + \tfrac{1}{2} r_0 A_0^2 k^3)(1 + \tfrac{4}{3}\beta^2 k^2 \ln|2kd|) - \tfrac{1}{3}\pi\beta^2 k^2 (1 - k^2 A_0^2) + O(\beta^4, k^5) \; . \qquad (3.4)$$

The short-range parameters A_{0s} and r_{0s}, which will not normally be of any particular interest, no longer appear. The terms in k, k^2, and $k^3 \ln k$ are identical with those obtained previously.[1,2,4] [$\tilde{A}$ and A^* of Eq. (A11) of Ref. 4 differ from A by terms of order β^2.] The terms in k^3, k^4, and $k^5 \ln k$ are new. We note that (3.4) gives six terms in the energy expansion of $\tan\eta_0$ with only two experimental parameters. It would be easy to introduce additional parameters, starting with the shape-dependent parameter for the short-range potential, and retain correspondingly more terms in the energy expansion of $\tan\eta_0$, since we have evaluated the integrals I, J, and K to all orders in k.

We now turn to $L>0$. On using (2.1) and the results of Appendix B in (2.4), we find

$$\tan\eta_L = \tfrac{1}{8}\pi g(L)(\beta k)^2 - A_L k^{2L+1}[1 - \tfrac{1}{2} g(L)(\beta k)^2 \ln|2kd|] + O(\beta^4, k^{2L+3}) \; , \qquad (3.5)$$

where

$$g(L) \equiv [(L + \tfrac{3}{2})(L + \tfrac{1}{2})(L - \tfrac{1}{2})]^{-1} \; ,$$

$$A_L = A_{Ls} + \beta^2 \left(\frac{\pi d^{2L-1}}{2^{2L+2}(2L-1)[\Gamma(L+\frac{3}{2})]^2} + \frac{A_{Ls}}{(2L+1)d^2} + \frac{[(2L)!]^2 A_{Ls}^2}{2^{2L}(2L+3)(L!)^2 d^{2L+3}} \right) \; .$$

In the derivation of (3.5), A_{Ls} rather than A_L appears in the $\ln k$ term but, as before, A_{Ls} can be replaced by A_L since the error introduced is of order β^4.

IV. r^{-6} POTENTIAL

We assume now that

$$\bar{U}(r) = -\gamma^4 r^{-6}, \quad r > d$$

where γ, defined by this equation, has the dimensions of a length. In electron-atom scattering, this form of potential arises as a correction to the static dipole polarizability β^2/r^4 potential and has its origins in both the static quadrupole polarizability of the atom and the dynamical response of the atom to the incident electron.[9] The r^{-6} potential also appears, of course, as the dominant long-range term in atom-atom interactions.

Restricting ourselves to $L=0$ and using Eq. (2.1) and the results of Appendix C in Eq. (2.4), we find

$$\tan\eta_0 = -A_0 k - \tfrac{1}{2} r_0 A_0^2 k^3 + \tfrac{1}{15}\pi\gamma^4 k^4 + \tfrac{4}{15} A_0 \gamma^4 k^5 \ln|2kd| + O(k^5, \gamma^8) \; , \qquad (4.1)$$

where

$$A_0 = A_{0s} + \gamma^4 \Delta' \quad , \tag{4.2a}$$

$$\Delta' = -d^{-5}(\tfrac{1}{3}d^2 - A_{0s}d + \tfrac{1}{5}A_{0s}^2) \quad , \tag{4.2b}$$

$$r_0 A_0^2 = r_{0s}A_{0s}^2 + \gamma^4 \delta' \quad , \tag{4.3a}$$

$$\delta' = d^{-5}[\tfrac{2}{3}d^2(d^2 + A_{0s}^2) - A_{0s}d(\tfrac{4}{3}d^2 - r_{0s}A_{0s}) - \tfrac{2}{5}r_{0s}A_{0s}^3] \quad . \tag{4.3b}$$

Δ' and δ' are independent of γ.

In the derivation of (4.1), the coefficient of the $\ln k$ term contains A_{0s} rather than A_0, but the replacement by A_0 is legitimate since the error introduced is of order γ^8. Equation (4.1) does not appear to have been obtained previously.

V. SUPERPOSITION OF r^{-4} AND r^{-6} POTENTIALS

We now assume that

$$\bar{U}(r) = -\beta^2 r^{-4} - \gamma^4 r^{-6}, \quad r > d \quad .$$

(We could generalize this somewhat by choosing different d's for the two components, but we will not.) Restricting ourselves to the $L = 0$ case, the integrals required for the use of (2.4) are, of course, those used in the previous sections. With A_0 and r_0 defined by

$$A_0 = A_{0s} + \beta^2\Delta + \gamma^4\Delta' \quad , \tag{5.1}$$

$$r_0 A_0^2 = r_{0s}A_{0s}^2 + \beta^2\delta + \gamma^4\delta' \quad , \tag{5.2}$$

where Δ, δ, Δ', and δ' are defined by Eqs. (3.2b), (3.3b), (4.2b), and (4.3b), we find

$$\begin{aligned}\tan\eta_0 = &-A_0 k - \tfrac{1}{3}\pi(\beta k)^2 - \tfrac{4}{3}A_0\beta^2k^3\ln|2kd| - \tfrac{1}{2}r_0A_0^2k^3 \\ &+ \tfrac{1}{3}\pi(\beta^2A_0^2 + \tfrac{1}{5}\gamma^4)k^4 \\ &- \tfrac{2}{3}(\beta^2 r_0 A_0 - \tfrac{2}{5}\gamma^4)A_0k^5\ln|2kd| + O(k^5, \beta^2\gamma^4, \beta^4, \gamma^8) \quad .\end{aligned} \tag{5.3}$$

VI. PHOTODETACHMENT

The radial dipole-matrix element required for the evaluation of the photodetachment cross section σ_{photo} is[10]

$$M_{L'L} = \int_0^\infty v_{L'}(r) r u_L(r)\, dr \quad , \tag{6.1}$$

where $v_{L'}$ and u_L are r times the radial wave function of the active electron in the bound and continuum states, respectively, and $L' = L \pm 1$. u_L satisfies the radial Schrödinger equation

$$\left(\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} - \frac{2m}{\hbar^2}V + k^2\right)u_L = \bar{U}u_L \quad , \tag{6.2}$$

where $\hbar^2k^2/2m$ is the energy of the emergent electron, that is, the difference between the energy of the incident photon and the binding energy of the active electron. The boundary conditions on u_L are

$$u_L(0) = 0$$

and

$$u_L(r) \to \sin(kr - \tfrac{1}{2}L\pi + \eta_L), \quad r \to \infty \quad . \tag{6.3}$$

We recast (6.2) as an integral equation:

$$u_L = \cos\rho_L u_L' + \int_0^\infty G_L(r, r')\bar{U}(r')u_L(r')\, dr' \quad , \tag{6.4}$$

where

$$\rho_L = \eta_L - \eta_{Ls} = O(U)$$

and the Green's function is given by

$$G_L(r, r') = -u_L'(r_<)u_L''(r_>) \quad ,$$

with u_L' and u_L'' the regular and irregular solutions to (6.2) with $\bar{U} = 0$ and asymptotic behavior

$$u_L' \to \sin(kr - \tfrac{1}{2}L\pi + \eta_{Ls}), \quad r \to \infty$$

$$u_L'' \to k^{-1}\cos(kr - \tfrac{1}{2}L\pi + \eta_{Ls}), \quad r \to \infty \quad .$$

We can rewrite (6.4) as

$$u_L = u_L' + \int_0^\infty G_L(r, r')\bar{U}(r')u_L'(r')\, dr' + O(U^2) \quad . \tag{6.5}$$

Substituting (6.5) in (6.1), we get

$$\begin{aligned}M_{L'L} = &\int_0^\infty dr\, v_{L'}(r) r [u_L'(r) \\ &- \int_0^\infty u_L'(r_<)u_L''(r_>)\bar{U}u_L'(r')\, dr'] + O(U^2) \quad .\end{aligned} \tag{6.6}$$

Since $v_{L'}$ represents a bound state, we can choose the cutoff distance d so that the contribution to $M_{L'L}$ of the integral for $r > d$ is negligible and since $U = 0$ for $r < d$, (6.6) becomes

$$M_{L'L} = M_{L'L}'(1 - B_L) + O(U^2) \quad , \tag{6.7}$$

where

$$B_L = \int_d^\infty u_L''(r)\bar{U}(r)u_L'(r)\, dr \quad , \tag{6.8}$$

$$M_{L'L}' \equiv \int_0^d v_{L'}(r) r u_L'(r)\, dr \quad , \tag{6.9}$$

since we can now set $r_> = r'$ and $r_< = r$. Furthermore, $V = 0$ for $r > d$ and hence, for $r > d$,

$$u_L'(r) = krj_L(kr)\cos\eta_{Ls} - krn_L(kr)\sin\eta_{Ls} \, , \tag{6.10a}$$

$$u_L''(r) = -k[krn_L(kr)\cos\eta_{Ls} + krj_L(kr)\sin\eta_{Ls}] \quad . \tag{6.10b}$$

Substituting the last two equations in (6.8) we obtain

$$\begin{aligned}B_L = &-k^{-1}\sin 2\eta_{Ls} I(L) - k^{-1}\cos 2\eta_{Ls} J(L) \\ &+ \tfrac{1}{2}k^{-1}\sin 2\eta_{Ls} K(L) \quad ,\end{aligned}$$

where $I(L)$, $J(L)$, and $K(L)$ are defined by (2.5), (2.6), and (2.7). By using the effective-range expansion (2.1) and the results of Appendices A and B, B_L can be evaluated explicitly.

For the photodissociation of a negative ion the dominant long-range interaction is r^{-4}. For this case, we find

$$B_0=\left(\frac{\beta}{d}\right)^2\left(\frac{A_{0s}}{3d}-\frac{1}{2}\right)-\frac{2}{3}\,\beta^2k^2\ln|2kd|-\Bigg[\frac{2}{3}\left(C-\frac{11}{6}\right)$$

$$+\frac{A_{0s}^2}{3d^3}\left(A_{0s}-\tfrac{1}{2}r_{0s}\right)$$

$$-\frac{2}{d}A_{0s}\Bigg]\beta^2k^2+\frac{2}{3}\,\pi A_{0s}\beta^2k^3+O(k^4)\ , \qquad (6.11)$$

$$B_0\equiv B_{00}+B_{01}\beta^2k^2\ln|2kd|+B_{02}\beta^2k^2+B_{03}\beta^2k^3+O(k^4)\ , \qquad (6.12)$$

and for $L>0$, we have

$$B_L=\left(\frac{\beta}{d}\right)^2\left(\frac{b_{L0}A_{Ls}}{2(\omega+1)d^{2L+1}}-\frac{1}{4\omega}\right)+\tfrac{1}{4}g(L)\beta^2k^2\ln|2kd|$$

$$+\Bigg[\frac{A_{Ls}}{d^{2L+1}}\left(\frac{b_{L1}}{2\omega}+\frac{r_{Ls}A_{Ls}b_{L0}}{4(\omega+1)d^2}\right)+\tfrac{1}{4}g(L)\big[C+\psi(\tfrac{1}{2})$$

$$-\psi(\omega)+\tfrac{1}{2}\omega(\omega^2+1)g(L)-(1/2\omega)\big]\Bigg]\beta^2k^2+O(k^4),$$

$$B_L\equiv B_{L0}+B_{L1}\beta^2k^2\ln|2kd|+B_{L2}\beta^2k^2+O(k^4)\ . \qquad (6.13)$$

The ψ function is the logarithmic derivative of the Γ function and $\omega=L+\frac{1}{2}$.

By factoring out the term independent of k, and by using the fact that B_L is proportional to β^2, we can write $|M_{L'L}|^2$ for $L=0$ and for $L>0$ as

$$|M_{L'L}|^2=(1-2B_{L0})\ |M'_{L'L}|^2(1-2B_{L1}\beta^2k^2\ln|2kd|$$

$$-2B_{L2}\beta^2k^2\ -2B_{03}\beta^2k^3\delta_{L0})+O(\beta^4,k^4)\ , \qquad (6.14)$$

where δ_{L0} is the Kronecker δ function.

The ln$k$ correction term in (6.14) had previously been obtained[11] by a straightforward analysis of the Mathieu functions that appear in the wave functions, but the sign of the ln$k$ term was given incorrectly.

The result contained in (6.14) can be used to extrapolate σ_{photo} data down to threshold in an attempt to determine the binding energy of the electron. However, since the $|M_{L'L}|^2$ factor is itself energy dependent, being of the form $A+Bk^2+\cdots$, the correction terms in β^2k^2, and in β^2k^3 for $L=0$, are not particularly useful. Furthermore, without a knowledge of $M'_{L'L}$, (6.14) gives only the *form* of σ_{photo}. However, if there is available reasonable data for the elastic scattering of an electron by the atom that remains upon photodissociation of the ion – the (approximate) binding energy of the electron in the ion would also be useful – one should be able to obtain reasonably accurate estimates of the effective one-body potentials U and V, and thereby to obtain reasonable approximations to $v_{L'}$ and u'_L and ultimately to $M'_{L'L}$. It should therefore be possible, given the elastic scattering data, to obtain an explicit and reasonably accurate expression for σ_{photo}.

APPENDIX A: EVALUATION OF $I(0)$, $J(0)$, AND $K(0)$ FOR AN r^{-4} POTENTIAL

$I(0)$

With

$$U(r)=-\beta^2r^{-4}\ , \qquad (A1)$$

$I(0)$, defined by (2.5), becomes, setting $kr=z$,

$$I(0)=-\beta^2k^3\int_{kd}^{\infty}z^{-4}\sin^2z\,dz\ . \qquad (A2)$$

Integrating by parts three times, we get

$$I(0)=-\tfrac{2}{3}(\beta k)^2[2a^{-3}+a^{-1}(1-2a^{-2})\cos a$$

$$+a^{-2}\sin a-\tfrac{1}{2}\pi+\mathrm{Si}(a)]\ , \qquad (A3)$$

where

$$a=2kd\ ,$$

$$\mathrm{Si}(a)=\int_0^a x^{-1}\sin x\,dx\ .$$

Using the power series for the sin, cos, and Si functions, we can write (A2) as

$$I(0)=\pi\beta^2k^3\left(\frac{1}{3}+\frac{1}{2}\sum_{m=0}^{\infty}{}'(2m-1)^{-1}a_{0m}(kd)^{2m-1}\right)\ , \qquad (A4)$$

where

$$a_{0m}=\frac{(-)^m(2m+1)!}{2^{2m+1}m!\,(m+1)!\,[\Gamma(m+\frac{3}{2})]^2}\ .$$

$J(0)$

Using (A1), $J(0)$, defined by (2.6), becomes

$$J(0)=\tfrac{1}{2}\beta^2\int_d^{\infty}r^{-4}\sin 2kr\,dr$$

$$=\tfrac{1}{2}(\beta^2/d^3)\int_1^{\infty}z^{-4}\sin az\,dz\ . \qquad (A5)$$

The last integral is formula 3.761-3 of Ref. 12:

$$\int_1^{\infty}z^{-4}\sin az\,dz=\tfrac{1}{6}a^3[\mathrm{Ci}(a)+a^{-1}(2a^{-2}-1)\sin a$$

$$+a^{-2}\cos a]\ , \qquad (A6)$$

where

$$\mathrm{Ci}(a)=-\int_a^{\infty}x^{-1}\cos x\,dx$$

$$=C+\ln|a|+\sum_{m=1}^{\infty}{}'\frac{(-)^ma^{2m}}{2m(2m)!}\ , \qquad (A7)$$

$$C=\text{Euler's constant}=0.57722\ldots\ .$$

Note that considered as a principal value, $\mathrm{Ci}(-a)=\mathrm{Ci}(a)$. (The expansion for the cosine integral is misprinted in Ref. 12, Eq. 8.232-2, where the ln term appears with the wrong sign.)

Using (A5), (A6), and the power series for the sine and cosine, we have

$$J(0)=\tfrac{1}{2}\beta^2k/d^2+\tfrac{2}{3}\beta^2k^3\ln|2kd|+\tfrac{1}{3}(2C-\tfrac{11}{3})\beta^2k^3+O(k^5)\ . \qquad (A8)$$

$K(0)$

With (A1), $K(0)$, defined by (2.7), becomes

$$K(0)=\int_d^\infty (kr)^2[j_0^2(kr)+n_0^2(kr)]\bar{U}(r)\,dr$$
$$=-\tfrac{1}{3}\beta^2/d^3\ . \tag{A9}$$

APPENDIX B: EVALUATION OF $I(L)$, $J(L)$, AND $K(L)$ FOR AN r^{-4} POTENTIAL, $L>0$

$I(L)$

With (A1), $I(L)$, defined by (2.5), becomes

$$I(L)=-\tfrac{1}{2}\pi\beta^2k^3\int_{kd}^\infty z^{-3}J_{L+1/2}^2(z)\,dz\ . \tag{B1}$$

For $L>0$, the integrand is finite for $z=0$, and we can write the integral as the difference of two integrals with ranges 0 to ∞ and 0 to kd, respectively. The integral with range 0 to ∞ is $\frac{1}{4}g(L)$ (Ref. 12, Eq. 6.574-2). In the other integral use the power series for the square of a Bessel function (Ref. 13, Eq. 5.4-6):

$$J_{L+1/2}^2(z)=\sum_{m=0}^\infty a_{Lm}z^{2L+2m+1}\ ,$$

where

$$a_{Lm}=\frac{(-)^m(2L+2m+1)!}{2^{2L+2m+1}m!\,(2L+m+1)!\,[\Gamma(L+m+\frac{3}{2})]^2}\ .$$

We then have

$$I(L)=-\tfrac{1}{2}\pi\beta^2k^3\left(\tfrac{1}{4}g(L)-\sum_{m=0}^\infty(2L+2m-1)^{-1}\right.$$
$$\left.\times\, a_{Lm}(kd)^{2L+2m-1}\right)\ . \tag{B2}$$

$J(L)$

With (A1), $J(L)$, defined by (2.6), becomes

$$J(L)=(-)^L\tfrac{1}{2}\pi\beta^2k^3\int_{kd}^\infty z^{-3}J_{L+1/2}^2(z)J_{-L-1/2}(z)\,dz\ . \tag{B3}$$

Using the power series (Ref. 13, Eq. 5.4-7), we find

$$J_{L+1/2}(z)J_{-L-1/2}(z)=\sum_{m=0}^\infty b_{Lm}z^{2m}\ ,$$

where

$$b_{Lm}=\frac{(-)^m(2m)!}{2^{2m}(m!)^2\Gamma(L+m+\frac{3}{2})\Gamma(-L+m+\frac{1}{2})}\ ,$$

and dividing the range of integration into two parts, we get

$$J(L)=(-)^L\tfrac{1}{2}\pi\beta^2k^3\left(\frac{b_{L0}}{2(kd)^2}-b_{L1}\ln|2kd|\right.$$
$$\left.-\sum_{m=2}^\infty\frac{b_{Lm}}{2m-2}(kd)^{2m-2}+B\right),$$

where

$$B=\int_1^\infty z^{-3}J_{L+1/2}(z)J_{-L-1/2}(z)\,dz+b_{L_1}\ln 2$$
$$-\tfrac{1}{2}b_{L0}+\sum_{m=2}^\infty\frac{b_{Lm}}{2m-2}\ . \tag{B4}$$

It can be shown[7] that

$$B=\frac{(-)^{L+1}}{2\pi}g(L)\left(C+\psi(\tfrac{1}{2})-\psi(\omega)\right.$$
$$\left.-\frac{1}{2\omega}+\frac{\omega^2+1}{2(\omega^2-1)}\right),$$

where

$$\omega=L+\tfrac{1}{2}\ .$$

We therefore have

$$(4/\beta^2k)J(L)=(\omega d^2)^{-1}-g(L)k^2\ln|2kd|-g(L)k^2[C+\psi(\tfrac{1}{2})$$
$$-\psi(\omega)-(2\omega)^{-1}\quad+\tfrac{1}{2}g(L)\omega(\omega^2+1)]$$
$$-(-)^L\pi d^{-2}\sum_{m=2}^\infty(m-1)^{-1}b_{Lm}(kd)^{2m}\ . \tag{B5}$$

$K(L)$

$K(L)$, defined by (2.7), can be rewritten as

$$K(L)=-\tfrac{1}{2}\pi\beta^2k^3\int_{kd}^\infty z^{-3}[J_{L+1/2}^2(z)+J_{-L-1/2}^2(z)]\,dz\ . \tag{B6}$$

Using the identity (Ref. 13, Eq. 9.62)

$$J_{L+1/2}^2(z)+J_{-L-1/2}^2(z)=\frac{2}{\pi z}\sum_{m=0}^L C_{Lm}z^{2m-2L}\ ,$$

where

$$C_{Lm}=\frac{2^{2m-2L}(2L-m)!\,(2L-2m)!}{m!\,[(L-m)!]^2}\ ,$$

we have

$$K(L)=\frac{\beta^2}{d^3}\sum_{m=0}^L\frac{b_{Lm}(kd)^{2m-2L}}{2m-2L-3}\ . \tag{B7}$$

APPENDIX C: EVALUATION OF $I(0)$, $J(0)$, AND $K(0)$ FOR AN r^{-6} POTENTIAL TO LOWEST ORDERS IN k

We now consider

$$\bar{U}=-\gamma^4/r^6\ . \tag{C1}$$

$I(0)$, defined by (2.5), becomes

$$I(0)=-\gamma^4k^5\int_{kd}^\infty z^{-6}\sin^2 z\,dz\ . \tag{C2}$$

By integrating by parts twice and then using the double-angle formula, we can write (C2) as

$$I(0)=-\gamma^4k^5\left(\frac{\sin^2 kd}{5(kd)^5}+\frac{\sin 2kd}{20(kd)^4}+\frac{1}{30(kd)^3}\right.$$
$$\left.-\frac{1}{5}\int_{kd}^\infty z^{-4}\sin^2 z\,dz\right)\ . \tag{C3}$$

The integral in (C3) has already been evaluated in obtaining (A3). Using that result, and the power series for the sine function, we have

$$I(0)=-\gamma^4k^5\left(\frac{1}{3(kd)^3}-\frac{1}{3kd}+\frac{\pi}{15}+O(k)\right)\ . \tag{C4}$$

With (C1), we have from (2.6),

$$J(0)=\tfrac{1}{2}\gamma^4d^{-5}\int_1^\infty z^{-6}\sin az\,dz\ , \tag{C5}$$

where

$z = r/d$ and $a = 2kd$.

Using formula 3.761-3 of Ref. 12 and the power series for the sin, cosin, and Ci functions, we have

$$J(0) = \gamma^4 k^5 [\tfrac{1}{4}(kd)^{-4} - \tfrac{1}{3}(kd)^{-2} - \tfrac{2}{15}\ln|2kd| + O(k^0)] \quad . \tag{C6}$$

From (C1) and (2.7), we have

$$K(0) = -\gamma^4 \int_d^\infty r^{-6}\, dr = -\tfrac{1}{5}\gamma^4 d^{-5} \quad . \tag{C7}$$

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Quantum-Mechanical Relation for a Magnetized Electron Gas

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The following theorem is proven: The xx and zz components of the electron energy-momentum tensor $t_{\mu\nu}$ and the magnetization M of an electron gas in a constant magnetic field B satisfy the relation $t_{zz} - t_{xx} = BM$. This relation is valid at any density, temperature, and magnetic field strength, if the system is in thermal equilibrium. Since the electromagnetic energy-momentum tensor $t_{\mu\nu}$ is anisotropic itself, this relation makes the *total* energy-momentum tensor $T_{\mu\nu} = t_{\mu\nu} + \tau_{\mu\nu}$ become a scalar. Because t_{xx} and t_{zz} can simply describe the electron pressure in these directions, the above theorem states that the difference $p_{zz} - p_{xx}$ equals twice the magnetic field energy, since $M = 4\pi B$.

Given an electron gas in a uniform magnetic field B, the total energy-momentum tensor can be written as the sum of the electron parts $t_{\mu\nu}$, plus the field contribution $\tau_{\mu\nu}$, i.e.,

$$T_{\mu\nu} = t_{\mu\nu} + \tau_{\mu\nu} , \tag{1}$$

where[1]

$$t_{\mu\nu} = \tfrac{1}{2}\hbar c(\bar{\psi}\gamma_\mu \partial_\nu \psi - \partial_\nu \bar{\psi}\gamma_\mu \psi) \tag{2}$$

and[2]

$$\tau_{\mu\nu} = (4\pi)^{-1}(-B_\mu B_\nu + \tfrac{1}{2}\delta_{\mu\nu}B^2). \tag{3}$$

The wave function ψ satisfies the Dirac equation for an electron in a constant magnetic field.[1]

The wave function ψ is the product of a plane wave $e^{ip_z z}$ since a constant magnetic field in the z direction does not alter the electron motion along that axis and a Laguerre polynomial of the variable $x^2 + y^2$ representing the harmonic motion in the plane perpendicular to the magnetic field. The exact form of the eigenvalue is given by[1]

$$E/mc^2 \equiv \epsilon(x, n) = (1 + x^2 + 2nB/B_q)^{1/2},$$

$$x \equiv p_z/mc,$$

$$B_q = m^2c^3/e\hbar = 4.\ 10^{14} \text{ G}. \tag{4}$$

The x and y momenta in the combination $p_x^2 + p_y^2$ have been changed into the discrete quantum expression $2nB/B_q$, where $n = 0, 1, \ldots$ is the index of the harmonic-oscillator levels.

Making use of the exact spinors and eigenvalues, the energy-momentum tensor was computed in