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Nonperturbative Analysis of the Resonant Interaction between a Linear and a Nonlinear Oscillator

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The consideration of the resonant interaction between a linear harmonic oscillator (LO) and a (nonlinear) angular momentum oscillator (NO) is motivated by its relation to the interaction between a field mode and a number of atomic systems, its relation to the interaction between three field modes, and the appearance of qualitatively different results quantum mechanically and classically even in cases that can be regarded as macroscopic. The analysis is carried out both classically and quantum mechanically in the rotating-wave approximation, and the time dependence of the energies of the two oscillators is examined, mainly by means of a nonlinear differential equation for the NO energy. In the classical analysis, the problem turns out to be identical to that of a spherical pendulum, where the NO energy L_3 corresponds to the pendulum potential energy, the LO energy n corresponds to the pendulum kinetic energy, and the coupling energy between the two oscillators $\frac{1}{2}\gamma K/\omega$ is proportional to the angular momentum K of the pendulum about the vertical axis, all quantities being expressed in suitable units. A solution for $L_3(t)$ is obtained, and is shown to be described uniquely, apart from a shift in the time origin, by the constants of motion $E(\equiv L_3+n)$ and K. L_3 oscillates periodically, except for the "unstable-equilibrium" solution and the "conical-pendulum" solution. The quantum-mechanical problem is solved exactly for the cases in which the NO is a two-level system and a three-level system, and certain aspects of the exact solution are derived in the case of a four-level system. The constants E and K become a complete set of quantum numbers. It is found that unstable equilibrium, in the sense that $\langle L_3\rangle$ is constant, does not exist, because of spontaneous emission; it is also found that $\langle L_3\rangle$ oscillates sinusoidally for the two-level system, oscillates periodically but not sinusoidally for the three-level system, and oscillates aperiodically for a four- or higher-level system, conclusions about the higher-level systems coming from general considerations and numerical solutions found in the literature. Approximate solutions for NO's with arbitrarily large numbers of levels are obtained. One method of approximation displays spontaneous emission but not aperiodicity. Another approximate method, which contains the statistical spread of the initial phase difference between the oscillations of the two oscillators inherent in the uncertainty principle, displays both spontaneous emission and aperiodicity. It is shown that the aperiodicity – a property which does not exist classically for precise initial conditions – does not disappear even in the macroscopic limit and is due to the averaging performed, in taking the quantum-mechanical expectation value, over an ensemble of periodic oscillations with a spread in frequencies.

I. INTRODUCTION

The oscillation of several types of fields – the electromagnetic or acoustic field, for example – corresponds to that of linear, or harmonic, oscillators. The oscillation of many other systems – atoms or molecules, for example – corresponds to that of nonlinear oscillators. In fact, much of radiation theory and many interesting resonance phenomena may be regarded as the interaction between linear and nonlinear oscillators. It is the purpose of the present paper to analyze in a nonperturbative manner the interaction between two coupled oscillators, one linear and the other nonlinear. Such a coupled system may be regarded as an idealized model for a resonant field mode inside a cavity (or a resonant lumped-constant circuit) coupled to a number of atomic systems. Although one might think, at first, that a number of atomic systems would require the same number of nonlinear oscillators in the model, there are certain conditions, to be mentioned later, under which such systems require only a single nonlinear oscillator. The idealization consists of neglecting dissipation and the coupling between the atomic systems other than through the field. Dissipation has received wide consideration in perturbative treatments, and frequently, dissipation makes it unnecessary to consider anything but a perturbative treatment.[1] Neglect of dissipation will allow emphasis on an interesting long-time aspect of the solution, the absence of a steady-state situation, which has been observed experimentally when the losses are sufficiently low.[2] Our main interest will, in fact, lie in the time dependence of the energy of each of the oscillators. Although the transfer of energy back and forth between two coupled oscillators – the presence of "beats" – is a familiar phenomenon when both oscillators are linear, the nonlinearity of one of the oscillators changes the phenomenon in a basic manner.

The present problem has motivation beyond the fact that it treats an idealized model of atomic systems interacting with a field mode. Both the linear oscillator (LO) and the type of nonlinear oscillator (NO) to be considered are among the most fundamental and widely used systems in physics. Their interaction will be studied both quantum mechanically and classically, and, in contrast to the case of two linear oscillators, will turn out to have *qualitative* features that are different in the two analyses. This difference, which may appear puzzling at first, since both oscillators could, in principle, be macroscopic systems, contributes toward the insight into the relationship between classical mechanics and quantum mechanics.

II. MODEL

From a mathematical viewpoint, there is only one kind of LO; it may be described by the dimensionless coordinate and momentum q and p, respectively, the Hamiltonian being

$$H_L = \tfrac{1}{2}\hbar\omega(q^2 + p^2) \ , \tag{1}$$

where

$$[q, p]_P = i \ , \tag{2}$$

the bracket $[\ ,\ ]_P$ standing for the commutator in the quantum-mechanical description, and i times the Poisson bracket in the classical description. From the same viewpoint, there are many NO's, with different types of nonlinearities. The one to be considered presently may be called an "angular momentum oscillator." It is described by the dimensionless angular momentum variables l_1, l_2, and l_3, the Hamiltonian being

$$H_N = \hbar\omega l_3 \ , \tag{3}$$

where

$$[l_r, l_s]_P = i l_t \ , \tag{4}$$

with r, s, and t standing for the cyclic permutation of 1, 2, and 3. As is well known, and can be deduced from the equations of motion below, some distinguishing characteristics of the angular momentum oscillator are the facts that its free oscillations [which manifest themselves in $l_1(t)$ and $l_2(t)$] have the single (angular) frequency ω, and – unlike the LO – both the energy and amplitude of oscillation cannot exceed certain maxima, which are determined by the total angular momentum l. This oscillator is a good model for many systems, both microscopic (small l) and macroscopic (large l), among which may be mentioned atomic systems for which only a pair of energy levels participates in a given resonant interaction ("two-level" systems), as well as certain combinations of a number such systems. The details involved in the description of the resonant (single-frequency) behavior of single atoms or molecules by that of angular momentum oscillators have been given elsewhere.[3] The description of combinations of angular momentum oscillators in terms of total angular momentum is well known through the angular momentum addition theorem, and those combinations that have a well-defined total angular momentum correspond to a single angular momentum oscillator that may have a large l.[4] A special case of particular interest to which the present discussion is applicable is that of a large number of two-level systems, all excited to their upper level before being coupled to a resonant-cavity mode.

We need, next, an expression for the coupling between the two oscillators. It can be motivated by considering a specific application. If the LO corresponds to a mode of the electromagnetic field inside a cavity, and if the NO corresponds to a spin system, then the coupling between the magnetic field and magnetic dipole moment can be expressed by[1]

$$H_{LN} = \hbar q\,(\gamma_1 l_1 + \gamma_2 l_2 + \gamma_3 l_3) \ , \tag{5}$$

where the γ's are coupling constants. We consider the case of weak coupling, for which the relationship

$$(\gamma_i/\omega) \ll 1, \quad i = 1, 2, 3 \tag{6}$$

holds. This assumption of weak coupling is not made in order to utilize perturbation-theory approximations, but rather in order to consider a system that displays strong resonance features.

It is convenient to use the familiar non-Hermitian variables (classically, Hermitian conjugation should be read as complex conjugation)

$$\begin{aligned} a &= 2^{-1/2}(q + ip) \ , \quad a^\dagger = 2^{-1/2}(q - ip) \ , \\ l_+ &= 2^{-1/2}(l_1 + il_2) \ , \quad l_- = 2^{-1/2}(l_1 - il_2) \ , \end{aligned} \tag{7}$$

and introduce the associated reduced variables[1] $A, A^\dagger, L_+, L_-, L_3$, specified by

$$\begin{aligned} a &= Ae^{-i\omega t}, \quad a^\dagger = A^\dagger e^{i\omega t} \ , \\ l_+ &= L_+ e^{i\omega t}, \quad l_- = L_- e^{-i\omega t} \ , \quad l_3 = L_3 \ . \end{aligned} \tag{8}$$

The reduced variables can be regarded as unitary transformations of the original variables, given by

$$A = e^{-(i/\hbar)H_0 t} a e^{(i/\hbar)H_0 t}, \ \text{etc.}, \tag{9a}$$

where

$$H_0 = H_L + H_N \ . \tag{9b}$$

(This unitary transformation may be used to obtain the reduction of variables other than those explicitly shown above.) One easily sees that when the oscillators are uncoupled the reduced variables are constant, and for weak coupling, which we are considering, they vary slowly compared to the natural oscillations. The interaction Hamiltonian can now be written

$$H_{LN}=\tfrac{1}{2}\hbar\,(\gamma A^{\dagger}L_{-}+\gamma A\,L_{-}e^{-2i\omega t}+\sqrt{2}\,\gamma_3 L_3 A\,e^{-i\omega t})$$
$$+\mathrm{H.\,c.}\ , \tag{10}$$

where H. c. is the Hermitian conjugate and

$$\gamma=\gamma_1+i\gamma_2\ .$$

In the analysis of a resonant interaction, it is customary to approximate by ignoring the rapidly oscillating terms in the interaction Hamiltonian of Eq. (10).[5] The significance of this kind of approximation has been examined[1] and it can be shown that, qualitatively speaking, it amounts to ignoring slight shifts in the oscillating frequencies, that is, reactive effects. In addition to this approximation [the validity of which depends on the inequality (6)], we adopt a simplification, for notational reasons, which has little physical significance; we assume γ to be real, which is equivalent to the assumption $\gamma_2=0$. In resonance phenomena it matters little whether the coupling takes place through one or through both oscillating coordinates, and the entire analysis can be carried out in the same manner with a complex γ. The resulting simplified interaction Hamiltonian is

$$H_{LN}=\tfrac{1}{2}\hbar\gamma\,(AL_{+}+A^{\dagger}L_{-})\ . \tag{11}$$

The approximation contained in this interaction Hamiltonian is frequently referred to as the rotating-wave approximation.[6]

The present problem is not a new one. Either the identical problem or similar problems have been studied previously,[4,5,7-10] mostly in connection with a number of two-level systems coupled to a cavity mode. The methods employed previously include perturbation theory,[4,7] numerical computation,[8-10] and other techniques. There will be occasion to compare some of these analyses to the present results, not all of which are new, but are needed for a self-consistent and coherent discussion.

It may be of interest to note that the present problem is formally identical to that of a mode (or linear oscillator) of frequency ω, interacting with two other modes, of frequencies ω_1 and ω_2, respectively, where

$$\omega=\omega_2-\omega_1\ , \tag{12}$$

through an interaction Hamiltonian

$$H_I=\tfrac{1}{4}\sqrt{2}\,\hbar\gamma\,(AA_1A_2^{\dagger}+A^{\dagger}A_1^{\dagger}A_2)\ , \tag{13}$$

the operators being labeled according to the frequencies of the modes to which they belong. The formal similarity of the two problems is due to the fact that the variables $A_1A_2^{\dagger}$ and $A_1^{\dagger}A_2$ obey the same commutation rules as $\sqrt{2}\,L_{+}$ and $\sqrt{2}\,L_{-}$, with $\frac{1}{2}(A_2^{\dagger}A_2-A_1^{\dagger}A_1)$ corresponding to L_3.[11] Since, as will be seen, the equations of motion for the reduced variables are obtained from the interaction Hamiltonian only, the solution of the present problem leads to a solution of the problem specified by Eqs. (12) and (13). The latter problem is that of an idealized parametric amplifier.[10]

III. EQUATIONS OF MOTION

The equations of motion in terms of the reduced variables are given by the relationships[1]

$$i\hbar\dot{R}=[R,\,H_{LN}]_P\ , \tag{14}$$

where R stands for any reduced variable. Introducing the dimensionless time

$$\tau=\tfrac{1}{2}\gamma t\ , \tag{15}$$

and noting that the reduced variables satisfy the rules

$$\begin{aligned}&[A,\,A^{\dagger}]_P=1, &&[L_{+},\,L_{-}]_P=L_3\ ,\\ &[L_{+},\,L_3]_P=-L_{+}, &&[L_{-},\,L_3]_P=L_{-},\end{aligned} \tag{16}$$

with all other brackets vanishing, we have

$$\begin{aligned}&A'=-iL_{-}, &&A^{\dagger\prime}=iL_{+},\\ &L_{+}'=-iA^{\dagger}L_3, &&L_{-}'=iA\,L_3,\\ &L_3'=-i\,(A\,L_{+}-A^{\dagger}L_{-}),\end{aligned} \tag{17}$$

where the prime indicates differentiation with respect to τ. These equations of motion are valid both classically and quantum mechanically.

It is useful to introduce the notation

$$n\equiv A^{\dagger}A,\quad E\equiv n+L_3,\quad K\equiv A\,L_{+}+A^{\dagger}L_{-}, \tag{18}$$

where, in units of $\hbar\omega$, n is the energy of the LO (without the zero-point energy), E is the sum of the energy of both oscillators, and $\frac{1}{2}(\gamma/\omega)K$ is the coupling energy. From the equations of motion it follows immediately that E and K are constants of motion. Another constant, of course, is the square of the total angular momentum of the NO,

$$L^2=L_1^2+L_2^2+L_3^3=L_{+}L_{-}+L_{-}L_{+}+L_3^2\ . \tag{19}$$

Equations (17) yield

$$\begin{aligned}L_3''&=-i\,(A'\,L_{+}+A\,L_{+}')+\mathrm{H.\,c.}\\ &=-2\,L_{-}L_{+}-2AA^{\dagger}L_3.\end{aligned} \tag{20}$$

Noting that

$$2\,L_{-}L_{+}=L^2-L_3^2-[L_{+},\,L_{-}] \tag{21}$$

and that

$$AA^{\dagger}=E-L_3+[A,\,A^{\dagger}], \tag{22}$$

we obtain, by substitution in Eq. (20),

$$L_3''=3L_3^2-2(E+[A,\,A^{\dagger}])\,L_3+[L_{+},\,L_{-}]-L^2\ . \tag{23}$$

It is convenient to use the notation

$$[A, A^\dagger] = \lambda, \quad [L_+, L_-] = \lambda L_3, \quad L^2 = L_0(L_0 + \lambda), \tag{24}$$

where

$$\lambda = 0, \quad \text{classically}, \tag{25a}$$

$$\lambda = 1, \quad \text{quantum mechanically}. \tag{25b}$$

We can then write

$$L_3'' = 3L_3^2 - (2E + \lambda) L_3 - L_0(L_0 + \lambda), \tag{26}$$

where L_0 is the total angular momentum, classically, and the corresponding quantum number, quantum mechanically. Equation (26) is an ordinary nonlinear second-order differential equation for the dynamical variable L_3 in both the classical and the quantum-mechanical formalism. We will refer to it as the second-order energy equation.

For certain purposes, it will be convenient to work with the *normalized quantities*

$$\mathcal{L} \equiv L_3/L_0, \quad \mathcal{N} \equiv n/L_0, \quad \mathcal{E} \equiv E/L_0 = \mathcal{N} + \mathcal{L}, \quad \mathcal{K} \equiv K/L_0^{3/2}, \quad \epsilon \equiv \lambda/L_0, \tag{27}$$

and the "normalized" time

$$s \equiv L_0^{1/2}, \qquad \tau = \tfrac{1}{2}\gamma L_0^{1/2} t \,. \tag{28}$$

The second-order energy equation, in terms of the normalized quantities, becomes

$$\mathcal{L}_{ss} = 3\mathcal{L}^2 - (2\mathcal{E} + \epsilon)\mathcal{L} - (1 + \epsilon), \tag{29}$$

where the subscript s indicates differentiation with respect to s.

We can also obtain a first-order energy equation involving the constants E and K. From the expressions for L_3' and K, we have

$$L_3'^2 + K^2 = 2(AA^\dagger L_+ L_- + A^\dagger A L_- L_+), \tag{30}$$

which yields, in a manner similar to that used in the derivation of Eq. (26),

$$L_3'^2 = 2L_3^3 - (2E + \lambda)L_3^2 - [2L_0(L_0 + \lambda) - \lambda]L_3 + L_0(L_0 + \lambda)(2E + \lambda) - K^2. \tag{31}$$

In normalized quantities one can write this equation, with some rearrangement, as

$$\mathcal{L}_s^2 = 2[\mathcal{L} - (\mathcal{E} + \tfrac{1}{2}\epsilon)][\mathcal{L}^2 - (1 + \epsilon - \tfrac{1}{2}\epsilon^2)] + \epsilon^2(\mathcal{E} + \tfrac{1}{2}\epsilon) - \mathcal{K}^2 \,. \tag{32}$$

Once we solve for the NO energy L_3 we can immediately obtain the LO energy n, since their sum is the constant E (which is determined by the initial conditions). In the following discussion the NO energy will be our chief concern. It is important to note, in the quantum-mechanical considerations involving the energy equation, that E may be treated as a c number since it commutes with L_3, L_3', and L_3''. (K, however, cannot be so treated.)

IV. LIMITING SOLUTIONS OF ENERGY EQUATION

A detailed consideration of the energy equation, for arbitrary E, will require separate treatment of the classical and quantum-mechanical cases. In the limiting situations of high and low energy, however, such specialization is not required. Let us consider, first, the situation

$$\mathcal{E} \gg 1. \tag{33}$$

Since the largest numbers associated with $\mathcal{L}$ are of the order of unity, Eq. (29) becomes, approximately,

$$\mathcal{L}_{ss} = -2\mathcal{E}\mathcal{L}, \tag{34}$$

the same equation as that for the coordinate of a harmonic oscillator, and leads to the solution

$$\mathcal{L}(s) = \mathcal{L}(0)\cos\Omega^{(s)}s + [\mathcal{L}_s(0)/\Omega^{(s)}]\sin\Omega^{(s)}s\,, \tag{35}$$

where $\Omega^{(s)} \equiv (2\mathcal{E})^{1/2}$, the frequency with respect to s. [With respect to τ, the frequency is $(2E)^{1/2}$, and with respect to t, the frequency is $\frac{1}{2}\gamma(2E)^{1/2}$.] Equation (35) gives essentially the same result as that obtained for an angular momentum oscillator subject to a prescribed classical driving force. This is to be expected from the correspondence principle, since the assumption $\mathcal{E} \gg 1$ is equivalent to the assumption that the linear oscillator is in a high quantum-number state. Formally, Eq. (35) is the same in both the classical and quantum-mechanical descriptions. The difference between the two descriptions lies in the fact that $\mathcal{L}(0)$ and $\mathcal{L}_s(0)$ are noncommuting variables (or operators) in the quantum-mechanical description, and cannot both be specified precisely, in principle, in contrast to the possibility of precise specifications in the classical description.

For the opposite limiting case of small energy, we set (noting that the ground state corresponds to $\mathcal{E} = -1$)

$$\mathcal{E} = -1 + \xi, \quad \xi \ll 1 \tag{36}$$

combined with $L_0 \gg 1$ or $\epsilon \ll 1$. We can expect under these conditions that $\mathcal{L}$ will differ only slightly from -1, so that we may also set

$$\mathcal{L} = -1 + \tfrac{1}{2}\xi + \zeta\,, \tag{37}$$

and treat ξ, ζ, and ϵ as small quantities of first order. To lowest order, Eq. (29) then becomes

$$\zeta_{ss} = -4\zeta\,, \tag{38}$$

which is, again, the same equation as that for the coordinate of a harmonic oscillator, with $\mathcal{L}$ oscillating about $-1 + \frac{1}{2}\xi$, but now with the frequency independent of the energy. The frequency with respect to s is 2; with respect to τ it is $2L_0^{1/2}$; and with respect to t it is $\gamma L_0^{1/2}$. This result is consistent with the fact that an angular momentum oscillator which is only slightly excited behaves as a linear oscillator. The oscillation of energy, back and forth, between two linear coupled oscillators – the phenomenon of beats – is well known to have a period independent of the energy.

V. CLASSICAL DESCRIPTION

In the classical description, the second-order energy equation [Eq. (29)] reads

$$\mathcal{L}_{ss}=3\,\mathcal{L}^2-2\,\mathcal{E}\mathcal{L}-1\,, \tag{39}$$

and the first-order equation [Eq. (32)] reads

$$\begin{aligned}\mathcal{L}_s^2&=2[(\mathcal{L}-\mathcal{E})(\mathcal{L}^2-1)-\tfrac{1}{2}\mathcal{K}^2]\\ &\equiv F_c(\mathcal{L})\equiv 2\,\varphi(\mathcal{L},\mathcal{K})\,,\end{aligned} \tag{40}$$

where the notation $F_c(\mathcal{L})$ and $\varphi(\mathcal{L},\mathcal{K})$ is introduced for later use. Equation (39) follows from Eq. (40) by differentiation (for $\mathcal{L}_s\neq 0$).

It is useful to consider the phases of oscillation of the two oscillators θ_1 and θ_2 defined by

$$A=n^{1/2}e^{-i\theta_1},\qquad A^\dagger=n^{1/2}e^{i\theta_1}\quad, \tag{41a}$$

$$L_\pm=2^{-1/2}(L_0^2-L_3^2)^{1/2}e^{\pm i\theta_2}\quad. \tag{41b}$$

[Note that classically, $AA^\dagger=A^\dagger A=n$ and $L_+L_-=L_-L_+=\frac{1}{2}(L_0^2-L_3^2)$.] Substituting these expressions into those for L_3' and K [Eqs. (17) and (18)], we obtain

$$\mathcal{L}_s=[2(\mathcal{E}-\mathcal{L})(1-\mathcal{L}^2)]^{1/2}\sin\theta \tag{42}$$

and

$$\mathcal{K}=[2(\mathcal{E}-\mathcal{L})(1-\mathcal{L}^2)]^{1/2}\cos\theta\,, \tag{43}$$

where θ is the phase difference between the oscillations of the two oscillators given by

$$\theta=\theta_2-\theta_1\,. \tag{44}$$

These expressions are consistent with Eq. (40). They also allow a simple derivation of Eq. (40) from Eq. (39). Integration of the latter equation yields

$$\begin{aligned}\mathcal{L}_s^2&=2\{(\mathcal{L}-\mathcal{E})(\mathcal{L}^2-1)-[\mathcal{L}(0)-\mathcal{E}][\mathcal{L}^2(0)-1]\}\\ &\quad+\mathcal{L}_s^2(0)\,.\end{aligned} \tag{45}$$

Obtaining $\mathcal{L}_s^2(0)$ from Eq. (42), and recalling that $\mathcal{K}=\mathcal{K}(0)$, one gets immediately Eq. (40). Any solution of Eq. (40) is also a solution of Eq. (39), except the solution

$$\mathcal{L}(s)=\mathcal{L}(0),\quad \cos\theta=1,$$

where $\mathcal{L}(0)$ is arbitrary. The arbitrariness is removed by Eq. (39).

Equation (40) has the form of a well-known equation, that for the vertical position of a spherical pendulum (a point mass constrained to move on a sphere in a gravitational field). If we consider a spherical pendulum of unit mass, unit radius, total energy $\mathcal{E}$, and angular momentum h about the vertical axis through the center, then its vertical coordinate z (referred to the center) in a unit gravitational field is given by the differential equation[12]

$$\dot{z}^2=2[(z-\mathcal{E})(z^2-1)-\tfrac{1}{2}h^2]\,. \tag{46}$$

(Note that h is a constant of motion since the force of gravity exerts no torque about the z axis.) It is easy to show that the angular momentum is given by

$$h=[2(\mathcal{E}-z)(1-z^2)]^{1/2}\cos\beta\,, \tag{47}$$

where β is the angle of the velocity with the circle of latitude (that is, with the horizontal direction) on the unit sphere at z.

Comparison of Eqs. (40) and (43) on the one hand, with Eqs. (46) and (47) on the other, shows that the (normalized) energy of the NO behaves exactly like the potential energy of the (normalized) spherical pendulum, with the phase difference between the two oscillators θ corresponding to the angle of the deviation from horizontal motion β. It follows immediately that the energy of the LO corresponds to the kinetic energy of the spherical pendulum.

The formal solution of Eq. (40) or (46) is well known,[12] and is expressed in terms of the roots of the cubic polynomial $F_c(\mathcal{L})$. If we label these roots – which are determined by the initial conditions (energies and phase relationship) – $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$, with

$$\mathcal{L}_1\leq\mathcal{L}_2\leq\mathcal{L}_3\,, \tag{48}$$

the solution is given in terms of the standard form of of the Jacobian elliptic function $\mathrm{sn}(x,k)$ by

$$\mathcal{L}(s)=\mathcal{L}_1+(\mathcal{L}_2-\mathcal{L}_1)\,\mathrm{sn}^2[p(s-s_0),k]\,, \tag{49a}$$

where

$$p=[\tfrac{1}{2}(\mathcal{L}_3-\mathcal{L}_1)]^{1/2}\,, \tag{49b}$$

s_0 is a constant of integration, and the modulus k is given by

$$k=[(\mathcal{L}_2-\mathcal{L}_1)/(\mathcal{L}_3-\mathcal{L}_1)]^{1/2}\,. \tag{49c}$$

Since the sn function oscillates periodically between -1 and $+1$, we see that $\mathcal{L}(s)$ oscillates periodically between $\mathcal{L}_1$ and $\mathcal{L}_2$, the period of oscillation (in s) being given by

$$P=2\int_{\mathcal{L}_1}^{\mathcal{L}_2}d\mathcal{L}\,[F_c(\mathcal{L})]^{-1/2}\,, \tag{50}$$

which can also be written as[12]

$$P=2I/p\,, \tag{51a}$$

where I is the elliptic integral

$$I\equiv\int_0^1 dy\,[(1-y^2)(1-k^2y^2)]^{-1/2}\,. \tag{51b}$$

Thus, in general, energy of the amount

$$\Delta=L_0(\mathcal{L}_2-\mathcal{L}_1) \tag{52}$$

(in units of $\hbar\omega$) oscillates back and forth between the linear and nonlinear oscillators with a period

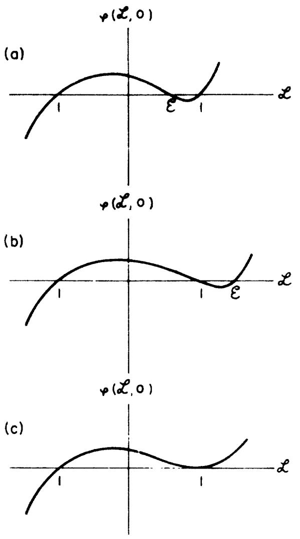

FIG. 1. Graph of $\varphi(\mathcal{L}, 0)$. (a) $\mathcal{E} < 1$; (b) $\mathcal{E} > 1$; (c) $\mathcal{E} = 1$.

(in t) given by

$$T = 2\gamma^{-1} L_0^{-1/2} P. \tag{53}$$

Qualitatively, this oscillation is analogous to the well-known oscillation of energy between two linear coupled oscillators, or the phenomenon of beats. In the case of linear oscillators, however (as was seen earlier for low excitation energy), the period of the oscillation is independent of the initial conditions and depends only on the coupling strength.

A. Classification of Solutions

It is evident that the three roots of $F_c(\mathcal{L})[\equiv 2\varphi(\mathcal{L},\mathcal{K})]$, which determine the solution $\mathcal{L}(s)$ except for a translation along the s (or time) axis, are themselves determined by $\mathcal{E}$ and $\mathcal{K}$. The dependence of the roots on these constants is illuminated by reference to the graph of $\varphi(\mathcal{L}, 0)$, shown in Fig. 1 for the three cases $\mathcal{E} < 1$, $\mathcal{E} > 1$, and $\mathcal{E} = 1$. For $\mathcal{K} = 0$, $\mathcal{L}$ oscillates from -1 to the smaller of the two quantities $\mathcal{E}$ and 1. (In the case $\mathcal{E} = 1$, the period of oscillation is infinite, since $\mathcal{L}_2 = \mathcal{L}_3$, and the modulus k is unity. This special case will be considered later in greater detail.) The corresponding situation for the pendulum is that of zero angular momentum, or "simple-pendulum" motion, where the motion lies in a plane containing the z axis. [Simple-pendulum solutions are discussed in Ref. 5.] Consider now $\mathcal{E}$ fixed, and let $\mathcal{K}^2$ increase from zero. The graph of $\varphi(\mathcal{L}, \mathcal{K})$ is obtained from that of $\varphi(\mathcal{L}, 0)$ by lowering it by an amount $\frac{1}{2}\mathcal{K}^2$. We see that, as $\mathcal{K}^2$ increases, the roots $\mathcal{L}_1$ and $\mathcal{L}_2$ approach each other, while $\mathcal{L}_3$ increases. The amplitude of oscillation of $\mathcal{L}$ decreases, and becomes zero when $\mathcal{L}_1$ and $\mathcal{L}_2$ merge. $\mathcal{K}^2$ then (when $\mathcal{L}_1 = \mathcal{L}_2$) assumes its largest value $\mathcal{K}^2_{\max}$. It follows that, for this value, $\mathcal{L}$ is a constant which, as can be easily ascertained, is given by

$$\mathcal{L} = \mathcal{L}_1 = \mathcal{L}_2 = \tfrac{1}{3}[\mathcal{E} - (\mathcal{E}^2 + 3)^{1/2}], \tag{54}$$

and θ is a constant, given by

$$\theta = 0 \text{ or } \pi. \tag{55}$$

The corresponding pendulum motion is that of a conical pendulum. Thus the interchange of energy between the two oscillators (for a given $\mathcal{E}$) varies from a maximum to zero as $\mathcal{K}^2$ varies from 0 to $\mathcal{K}^2_{\max}$, the latter quantity being given, from Eqs. (43), (54), and (55), by

$$\mathcal{K}^2_{\max} = \tfrac{4}{27}[(\mathcal{E}^2 + 3)^{3/2} - \mathcal{E}^3 + 9\mathcal{E}]. \tag{56}$$

The corresponding motion of the spherical pendulum (for a fixed total energy) varies from that of the simple pendulum to that of the conical pendulum.

As noted previously, $\mathcal{E}$ and $\mathcal{K}$ determine the solutions, except for a shift in the time origin. We can regard these two quantities, in analogy with quantum-mechanical usage, as determining the "state" of the system. As may be anticipated, these will, indeed, be a complete set of operators in the quantum-mechanical description. It should be noted that $+K$ and $-K$ correspond to different values of θ, and thus to different phase relationships between the oscillations of the two oscillators, but to the same energy behavior (except, again, for a possible shift in the time origin). In the case of the pendulum, $+K$ and $-K$ obviously correspond to rotation about the z axis in opposite senses.

B. Unstable Equilibrium and Almost Unstable Equilibrium

We will now examine the case $\mathcal{E} = 1$, $\mathcal{K} = 0$ [Fig. 1 (c)] in greater detail. The formal expressions for $\mathcal{L}$ and its period of oscillation, Eqs. (49)–(51), show that $\mathcal{L}$ oscillates between -1 and $+1$, but that, as mentioned previously, the period of oscillation is infinite. In this case, we have

$$F_c(\mathcal{L}) = 2(1 - \mathcal{L}^2)(1 - \mathcal{L}). \tag{57}$$

The roots of this polynomial are

$$\mathcal{L}_1 = -1, \quad \mathcal{L}_2 = \mathcal{L}_3 = 1.$$

The integral expression for the period, Eq. (50), diverges at the upper limit because of the coincidence of the two upper roots. This divergence and its cause are also demonstrated explicitly by the elliptic integral of Eq. (51), which becomes infinite as the modulus k approaches unity, and the fact that $k = 1$ only if $\mathcal{L}_2 = \mathcal{L}_3$. It is obvious that in this instance $\mathcal{L}(s) = 1$ is a solution of the energy equation (of both first and second order), the corresponding pendulum solution being that of unstable equilibrium. We shall refer to the conditions $\mathcal{E} = \mathcal{L} = 1$ (or $\mathcal{L} = 1$, $\mathcal{N} = 0$), in which the NO is excited to its highest energy and the LO is unexcited, as "unstable equilibrium" conditions. There are other solutions for the case $\mathcal{E}=1$, $K=0$, with the initial conditions such as to produce an (infinitely slow) approach to unstable equilibrium,[5] but it is unnecessary to discuss them for present purposes. We do want to consider, however, for later use, initial conditions that are very close to the unstable equilibrium conditions.

Let the initial conditions be specified by

$$\mathfrak{N}(0)=\epsilon_1, \quad \mathcal{L}(0)=1-\epsilon_2, \quad \theta(0)=\theta_0, \tag{58}$$

where ϵ_1 and ϵ_2 are small positive quantities of first order, and θ_0 is arbitrary. In other words, the initial conditions are those for which the LO is slightly excited and the NO is almost fully excited. We have then

$$\mathcal{E}=\mathfrak{N}(0)+\mathcal{L}(0)=1+\delta, \tag{59a}$$

where

$$\delta=\epsilon_1-\epsilon_2 \tag{59b}$$

and, up to second order,

$$\tfrac{1}{2}\mathcal{K}^2=2\epsilon_1\epsilon_2\cos^2\theta_0 . \tag{60}$$

The expression for $F_c(\mathcal{L})$ becomes

$$F_c(\mathcal{L})=2[(\mathcal{L}-1-\delta)(\mathcal{L}^2-1)-2\epsilon_1\epsilon_2\cos^2\theta_0]. \tag{61}$$

If either δ or $\cos^2\theta_0$ does not vanish, the coincidence of the two upper roots of $F_c(\mathcal{L})$ is removed, and the period becomes finite. It is shown in Appendix A that, to lowest significant order, the roots are

$$\begin{aligned}&\mathcal{L}_1=-1, \quad \mathcal{L}_2=1+\tfrac{1}{2}[\delta-(\delta^2-\mathcal{K}^2)^{1/2}] ,\\ &\mathcal{L}_3=1+\tfrac{1}{2}[\delta+(\delta^2+\mathcal{K}^2)^{1/2}],\end{aligned} \tag{62}$$

and the period is given by

$$\begin{aligned}P&=5\ln 2+\tfrac{1}{2}\ln(\delta^2+\mathcal{K}^2)^{-1}\\ &=5\ln 2+\tfrac{1}{2}\ln[(\epsilon_1-\epsilon_2)^2+4\epsilon_1\epsilon_2\cos^2\theta_0]^{-1} .\end{aligned} \tag{63}$$

We see that as ϵ_1 and ϵ_2 vanish, we recover formally the unstable equilibrium initial conditions, and P does indeed become infinite. The case $\epsilon_1=\epsilon_2 \neq 0$, $\cos^2\theta_0=0$, which also yields an infinite P, describes a situation – to which reference was made above – with just the right initial conditions to reach (after an infinite time) unstable equilibrium. It is important to notice the strong sensitivity of P to the initial conditions when these are near unstable equilibrium.

VI. QUANTUM-MECHANICAL DESCRIPTION

We return now to the general second-order equation for the NO energy, Eq. (26), which, in the quantum-mechanical case, reads

$$L_3''=3L_3^2-(2E+1)L_3-L_0(L_0+1) . \tag{64}$$

The analysis differs in two respects from that of the classical description. First, the formal differential equation is different (that is, the coefficients are different) and, second, L_3 cannot be treated, in general, either as a c number or as a diagonal matrix. The latter difference is due to the fact that L_3' does not commute with L_3 [as is apparent from Eqs. (17)] and a solution to a second-order differential equation must depend, in general, on an initial first derivative.

If we take expectation values of the operators in Eq. (64), we obtain

$$\langle L_3\rangle''=3\langle L_3^2\rangle-(2E+1)\langle L_3\rangle-L_0(L_0+1) . \tag{65}$$

This equation is *not* a differential equation for $\langle L_3\rangle$, since $\langle L_3^2\rangle \neq \langle L_3\rangle^2$, in general, and approximations will be necessary for its solution in the case of arbitrary L_0. We can, however, obtain certain information from this equation without the use of approximation. Consider the NO to be initially in its highest-energy state and the LO to be in an energy state with eigenvalue n_0, so that

$$\langle L_3^2(0)\rangle=\langle L_3(0)\rangle^2=L_0^2, \quad E=L_0+n_0 . \tag{66}$$

Equation (65) yields

$$\langle L_3''(0)\rangle=-2L_0(n_0+1) . \tag{67}$$

Now, if $n_0=0$, we have initial conditions that, classically, are the unstable equilibrium conditions. According to Eq. (67), however, the quantity $\langle L_3(\tau)\rangle$ does not remain constant for these initial conditions since the second derivative does not vanish. We see, therefore, that as far as the quantum-mechanical expectation value is concerned, unstable equilibrium does not exist. This fact, of course, is merely the well-known perturbation-theory result that there exists spontaneous emission when the NO is initially in the highest-energy state and the LO is initially in the ground state. For convenience, we shall continue to refer in the quantum-mechanical discussion to these initial conditions as the "unstable-equilibrium initial conditions," and to the fact that $\langle L_3(\tau)\rangle$ does not remain constant for these initial conditions as the presence of spontaneous emission. In view of the qualitative difference between the quantum-mechanical and classical behavior for these conditions, even for large L_0, they will be of central interest.

A. Two-Level System

Although a detailed discussion of the operator L_3 is, in general, facilitated by an explicit representation, we consider first a special NO, a two-level system, for which an exact solution of the energy equation may be obtained very simply without reference to the representation.[13] For this NO, $L_0=\frac{1}{2}$, L_3^2 is $\frac{1}{4}$ times the unit operator, $L_0(L_0+1)$ is $\frac{3}{4}$ (times the unit operator), the first and third term on the right-hand side of Eq. (64) cancel, and the energy equation becomes

$$L_3'' = -(2E+1)L_3 . \tag{68}$$

Recalling that E may be regarded as a c number, we have as the solution of this equation

$$L_3(\tau) = L_3(0)\cos\Omega\tau + [L_3'(0)/\Omega]\sin\Omega\tau , \tag{69a}$$

where

$$\Omega = (2E+1)^{1/2} . \tag{69b}$$

Substituting for $L_3'(0)$ from Eqs. (17), we can write

$$L_3(\tau) = L_3(0)\cos\Omega\tau - i\,\Omega^{-1}[A(0)L_+(0) - A^\dagger(0)L_-(0)]\sin\Omega\tau . \tag{70}$$

It is not difficult to check, as is shown in Appendix B, that this formal expression for $L_3(\tau)$ leads to $L_3^2(\tau) = \frac{1}{4}$. Equation (70) displays $L_3(\tau)$ as an operator in both the LO and NO space.

B. Representations

It is convenient at this point to discuss explicit representations, which are easily illustrated with the two-level NO. The generalization to a larger number of levels will be obvious, and the pertinent quantitative relationships for making this generalization will be given.

The most obvious vector space for the representation of $L_3(\tau)$ is constructed by taking the direct product of the uncoupled LO and NO spaces, which yields the basis vectors $|m, n_0\rangle$, where m and n_0 are the eigenvalues of $L_3(0)$ and $n(0)$, respectively. Of particular interest is the subspace defined by $m + n_0 = E$, with E fixed, since in the Schrödinger picture a state vector initially in this subspace remains in this subspace, E being a constant of motion. When the system under discussion has a well-defined E, the only basis vectors we may need for its description are $|m, E-m\rangle$, where $-L_0 \leq m \leq L_0$ and $m \leq E$. For simplicity, we will suppress, in the absence of any possible ambiguity, the second quantum number, writing $|m\rangle$ instead of $|m, E-m\rangle$. Different E's describe different subspaces, of course. The dimensionality of a given subspace is $2L_0+1$ for $E \geq L_0$ and $L_0 + E + 1$ for $E \leq L_0$. It should be noted that the basis vectors defined above are not eigenvectors of the Hamiltonian, but only of E $(= L_3 + n)$ and $L_3(0)$ [or $n(0)$, of course]. We shall refer to this representation as the L_3 representation.

There exists another representation, in which the basis vectors are eigenvectors of the Hamiltonian, which means, in this instance, that they are eigenvectors of both E and K (recalling that $H_{LN} = \frac{1}{2}\hbar\gamma K$). We shall refer to this representation as the K representation. K, as noted previously, is a constant of motion. The matrix elements of K in the L_3 representation, obtained from the well-known harmonic-oscillator and angular momentum matrix elements, are given by (the full notation being used for the L_3 basis vectors)

$$\begin{aligned}\langle m-1, n_0+1|K|m, n_0\rangle &= \langle m-1, n_0+1|A^\dagger L_-|m, n_0\rangle \\ &= \langle m, n_0|K|m-1, n_0+1\rangle \\ &= \langle m, n_0|A\,L_+|m-1, n_0+1\rangle \\ &= \{\tfrac{1}{2}(n_0+1)[L_0(L_0+1) - m(m-1)\ ]\}^{1/2},\end{aligned} \tag{71}$$

all other elements vanishing. For the two-level NO, with $E \geq \frac{1}{2}$, the 2×2 matrix K in the L_3 representation has its diagonal elements equal to zero and its off-diagonal elements equal to $\frac{1}{2}\Omega$. The two basis vectors in the K representation are

$$\Phi_\pm = 2^{-1/2}(|\tfrac{1}{2}\rangle \pm |-\tfrac{1}{2}\rangle), \tag{72}$$

and for these states the coupling energy H_{LN} is well defined with eigenvalues $\pm\frac{1}{4}\hbar\gamma\Omega$. From Eqs. (70)-(72), we obtain immediately $L_3(\tau)$ in either representation. In the L_3 representation (with rows and columns labeled according to $|\frac{1}{2}\rangle$, $|-\frac{1}{2}\rangle$), we have

$$L_3(\tau) = \frac{1}{2}\begin{pmatrix} \cos\Omega\tau & -i\sin\Omega\tau \\ i\sin\Omega\tau & -\cos\Omega\tau \end{pmatrix}, \tag{73}$$

and in the K representation (with rows and columns labeled according to Φ_+, Φ_-), we have

$$L_3(\tau) = \frac{1}{2}\begin{pmatrix} 0 & e^{i\Omega\tau} \\ e^{-i\Omega\tau} & 0 \end{pmatrix} . \tag{74}$$

In the case of a two-level NO, the energy equation was solved easily since it is linear. For higher-level NO's the energy equation is nonlinear, and it will prove simpler to solve the problem in the Schrödinger picture. It is of interest, therefore, to write down the solution in the two-level case, also, in the Schrödinger picture. From the results already derived, we obtain, for the states that are initially $|\frac{1}{2}\rangle$, $|-\frac{1}{2}\rangle$, in obvious notation

$$\begin{aligned}|\tfrac{1}{2}\tau\rangle &= 2^{-1/2}(\Phi_+ e^{(-1/2)i\Omega\tau} + \Phi_- e^{(1/2)i\Omega\tau}) \\ &= |\tfrac{1}{2}\rangle\cos\tfrac{1}{2}\Omega\tau - |-\tfrac{1}{2}\rangle i\sin\tfrac{1}{2}\Omega\tau ,\end{aligned} \tag{75a}$$

$$\begin{aligned}|-\tfrac{1}{2}\tau\rangle &= 2^{-1/2}(\Phi_+ e^{(-1/2)i\Omega\tau} - \Phi_- e^{(1/2)i\Omega\tau}) \\ &= -|\tfrac{1}{2}\rangle i\sin\tfrac{1}{2}\Omega\tau + |-\tfrac{1}{2}\rangle\cos\tfrac{1}{2}\Omega\tau ,\end{aligned} \tag{75b}$$

where the common phase constant $e^{-iE\omega t}$ has been dropped.

The matrix form of $L_3(\tau)$ allows us to read off all the statistical properties of the system. If the NO is initially in the upper state, the expectation value of the NO energy (in units of $\hbar\omega$) is given by the $(\frac{1}{2}, \frac{1}{2})$ element in the L_3 representation, namely,

$\frac{1}{2}\cos\Omega\tau$. We see that $\langle L_3(\tau)\rangle$ oscillates sinusoidally between $+\frac{1}{2}$ and $-\frac{1}{2}$. If the NO is initially in the lower state, the corresponding value is given by the $(-\frac{1}{2}, -\frac{1}{2})$ element, namely, $-\frac{1}{2}\cos\Omega\tau$. Note, however, that for the same eigenvalue n_0, E (and therefore Ω) is different for these two initial states, being $n_0+\frac{1}{2}$ if the NO is initially in the upper state, and $n_0-\frac{1}{2}$ if the NO is initially in the lower state. This fact leads to the well-known relationship between spontaneous and induced emission, or between the Einstein A and B coefficients. The transition rate w when the NO is initially in the state $|\pm\frac{1}{2}\rangle$ is given by

$$w_\pm \equiv \mp\frac{d}{dt}\langle\pm\tfrac{1}{2}|\,L_3(\tau)\,|\pm\tfrac{1}{2}\rangle = \mp\tfrac{1}{2}\gamma\frac{d}{d\tau}\langle\pm\tfrac{1}{2}|\,L_3(\tau)\,|\pm\tfrac{1}{2}\rangle$$

$$= \tfrac{1}{4}\Omega_\pm\sin\Omega_\pm\tau, \qquad (76a)$$

where

$$\tfrac{1}{2}\Omega_\pm{}^2 = \pm\tfrac{1}{2}+n_0+\tfrac{1}{2}. \qquad (76b)$$

For $\tau \ll \Omega$, we have

$$w_\pm \approx \tfrac{1}{4}\gamma\Omega^2{}_\pm\tau, \qquad (77)$$

so that

$$w_+/w_- = (n_0+1)/n_0. \qquad (78)$$

It is to be noted that the initial conditions specified by the statement that the system is initially in the state $|\frac{1}{2}, 0\rangle$ are the unstable-equilibrium initial conditions. Equation (75a) shows that the NO oscillates under these conditions from the upper energy state to the lower energy state, with $\Omega=\sqrt{2}$.

C. Three-Level System

We consider next a three-level system. Here, Eq. (64) no longer reduces to a linear equation in L_3, and there appears to be no simple method of solving the Heisenberg equation formally. Our vector space in this instance, constructed in the same manner as that for the two-level system, consists of three basis vectors, which, in the L_3 representation, are $|1, E-1\rangle$, $|0, E\rangle$, $|-1, E+1\rangle$, provided $E\geq 1$. For $E=0$, the space is two dimensional, with basis vectors $|0,0\rangle$, $|-1,1\rangle$, and for $E=-1$, it is one dimensional (and trivial) with basis vector $|-1,0\rangle$. We will consider only the first case, $E\geq 1$; the case $E=0$ is less interesting for our purposes and can be studied by the same methods. As previously, we will suppress the second quantum number and use the simpler notation $|1\rangle$, $|0\rangle$, $|-1\rangle$ for the basis vectors in the L_3 representation.

From Eq. (71), the K matrix in the L_3 representation is given by

$$K = \begin{pmatrix} 0 & E^{1/2} & 0 \\ E^{1/2} & 0 & (E+1)^{1/2} \\ 0 & (E+1)^{1/2} & 0 \end{pmatrix}, \qquad (79)$$

where rows and columns are labeled according to the states $|1\rangle$, $|0\rangle$, $|-1\rangle$. The three eigenvalues of this matrix are easily seen to be $\pm\Omega$, 0, where, as previously, $\Omega=(2E+1)^{1/2}$. Labeling the basis vectors of the K representation (the eigenvectors of K) with the subscripts +, 0, −, corresponding to the eigenvalues of K that are $+\Omega$, 0, $-\Omega$, respectively, we obtain

$$\Phi_+ = 2^{-1/2}\Omega^{-1}\left[E^{1/2}|1\rangle+\Omega|0\rangle+(E+1)^{1/2}|-1\rangle\right], \qquad (80a)$$

$$\Phi_0 = \Omega^{-1}\left[(E+1)^{1/2}|1\rangle-E^{1/2}|-1\rangle\right], \qquad (80b)$$

$$\Phi_- = 2^{-1/2}\Omega^{-1}\left[E^{1/2}|1\rangle-\Omega|0\rangle+(E+1)^{1/2}|-1\rangle\right]. \qquad (80c)$$

(These are all also eigenstates, of course, of H_L+H_N, with eigenvalue $\hbar\omega E$.)

We can begin to discern here the meaning of the eigenstates of K in terms of the classical description of the coupled oscillators, or more simply yet, in terms of the spherical pendulum. The state Φ_0 corresponds qualitatively to the simple pendulum with zero angular momentum about the z axis, which, for the oscillators, implies a large interchange of energy between the two oscillators (we recall that NO energy corresponds to pendulum potential energy, and LO energy corresponds to pendulum kinetic energy), while the states $\Phi_\pm$ correspond, in the same manner, to the conical pendulum with equal and opposite angular momenta about the z axis, which implies a small interchange of energy between the two oscillators. For the conical pendulum the potential energy is somewhat below the middle (the middle being zero), while for the plane pendulum it oscillates between the two extremes and spends more time on top than on bottom. We have, indeed,

$$(\Phi_\pm, L_3\Phi_\pm) = -[2(2E+1)]^{-1} \qquad (81)$$

and

$$(\Phi_0, L_3\Phi_0) = (2E+1)^{-1}, \qquad (82)$$

which is qualitatively consistent with the classical time averages for the conical pendulum and simple pendulum, respectively. The probabilities of finding the system in one of the states $|-1\rangle$, $|0\rangle$, $|+1\rangle$, which can be read off from Eqs. (80), are also qualitatively consistent with the classical picture. One would expect, of course, that the similarity to the classical picture becomes more quantitative as the number of levels increases.

Viewing the system in the Schrödinger picture, our main interest lies in the time variation of the states that are initially $|1\rangle$, $|0\rangle$, $|-1\rangle$. Using the same method as in the case of the two-level NO, we obtain

$$|1\tau\rangle = \Omega^{-2}\{[E(1+\cos\Omega\tau)+1]|1\rangle - iE^{1/2}\Omega\sin\Omega\tau|0\rangle - [E(E+1)]^{1/2}(1-\cos\Omega\tau)|-1\rangle\}, \quad (83a)$$

$$|0\tau\rangle = -iE^{1/2}\Omega^{-1}\sin\Omega\tau|1\rangle + \cos\Omega\tau|0\rangle, - i(E+1)^{1/2}\Omega^{-1}\sin\Omega\tau|-1\rangle, \quad (83b)$$

$$|-1\tau\rangle = \Omega^{-2}\{-[E(E+1)]^{1/2}(1-\cos\Omega\tau)|1\rangle - i(E+1)^{1/2}\Omega\sin\Omega\tau|0\rangle + [E+(E+1)\cos\Omega\tau]|-1\rangle\}, \quad (83c)$$

where, again, the common phase constant $e^{-iE\omega t}$ has been dropped. By means of these expressions we can write down the matrix elements of all the Heisenberg operators, of course, and it can be shown that the L_3 matrix satisfies, indeed, the operator equation (64). In particular, let us consider the expectation value of the NO energy when the initial NO energy is a maximum. One obtains

$$\langle 1\tau|L_3|1\tau\rangle = E(E+1)\Omega^{-4}\{4\cos\Omega\tau + (E+1)^{-1} \times (1-\cos^2\Omega\tau) + [E(E+1)]^{-1}\}. \quad (84)$$

We see that the expectation value of the energy oscillates, as in the two-level case, with a frequency Ω, but, in contrast with the two-level case, it is no longer a sinusoidal oscillation between the two extreme eigenvalues of L_3. First, the oscillation contains a harmonic, and second, the energy expectation value (which starts at $+1$) does not reach the lowest eigenvalue -1. The minimum value reached is

$$\langle 1\tau|L_3|1\tau\rangle_{\min} = -1 + 2\Omega^{-4}. \quad (85)$$

[As E becomes large, however, the oscillation becomes sinusoidal in accordance with Eq. (35).] The behavior of $\langle L_3\rangle$ for the three-level oscillator is thus qualitatively different from that for the two-level oscillator, in which $\langle L_3\rangle$ oscillates sinusoidally from L_0 to $-L_0$. The three-level oscillator which starts in the highest state does not go completely into the lowest state. This is also exhibited by an inspection of the probability of finding the NO in any one of the three levels when it is initially in the highest level. Each of the probabilities is given by the square of the absolute value of the corresponding coefficient in Eq. (83a), and we see that the probability of finding the NO in the highest state does not reach zero at any time, the minimum value being (when $\cos\Omega\tau = -1$)

$$|\langle 1|1\tau\rangle|^2_{\min} = \Omega^{-4}. \quad (86)$$

[The other two probabilities at the same time are $|\langle 0|1\tau\rangle|^2 = 0$ and $|\langle -1|1\tau\rangle|^2 = 4E(E+1)\Omega^{-4}$.] If the LO is initially in the ground state and the three-level oscillator is initially in the highest state, it can be said that there is a finite probability of $\frac{1}{9}$, statistically speaking, of finding unstable equilibrium in the classical sense, that is, of finding that the initial energy has not changed even though the expectation value of the energy oscillates.

D. Four-Level System

We will now consider only one feature of a four-level system, the energy eigenvalues, since a complete treatment such as that of the three-level system becomes quite complicated. From Eq. (71), we obtain for the matrix K in the L_3 representation, with rows and columns labeled according to $|\frac{3}{2}\rangle, |\frac{1}{2}\rangle, |-\frac{1}{2}\rangle, |-\frac{3}{2}\rangle$:

$$K = \begin{pmatrix} 0 & [\frac{3}{4}(\Omega^2-2)]^{1/2} & 0 & 0 \\ [\frac{3}{4}(\Omega^2-2)]^{1/2} & 0 & \Omega & 0 \\ 0 & \Omega & 0 & [\frac{3}{4}(\Omega^2+2)]^{1/2} \\ 0 & 0 & [\frac{3}{4}(\Omega^2+2)]^{1/2} & 0 \end{pmatrix}, \quad (87)$$

where, as previously, $\Omega = (2E+1)^{1/2}$. We have assumed here that $E \geq \frac{3}{2}$. Lower values of E will yield a matrix of smaller dimension, which we will not consider. The eigenvalues of the above matrix are

$$\kappa = \pm\tfrac{1}{2}\Omega[1-4\beta]^{1/2}, \quad \pm\tfrac{3}{2}\Omega[1+\tfrac{4}{9}\beta]^{1/2}, \quad (88a)$$

where

$$\beta = (1 + 9/4\,\Omega^4)^{1/2} - 1. \quad (88b)$$

Now, for any multilevel system, we have

$$|m\tau\rangle = \textstyle\sum_\kappa A_\kappa \Phi_\kappa e^{-i\kappa\tau}, \quad (89)$$

where the A_κ's are appropriate superposition coefficients, so that

$$\langle m\tau|L_3|m\tau\rangle = \sum_{\kappa,\kappa'} A^*_{\kappa'} A_\kappa (\Phi_{\kappa'}, L_3\Phi_\kappa) e^{i(\kappa'-\kappa)\tau}. \quad (90)$$

Thus, $\langle L_3(\tau)\rangle$ will, in general, contain terms that oscillate with frequencies equal to the differences between the eigenvalues of K. As E becomes large β approaches zero, and the four eigenvalues become equally spaced with spacing Ω, a result consistent with previous discussion since it will yield an oscillation of $\langle L_3(\tau)\rangle$ with the frequency Ω. However, for E sufficiently small so that β may not be neglected, the spacing between the eigenvalues gives us slightly different numbers which are incommensurable, and the expectation value of L_3 will oscillate aperiodically. We see here a new qualitative feature in the oscillation of the energy expectation value, an imprecise period, only approximately equal to $2\pi/\Omega$.

E. Higher-Level Systems

One can write down the K matrix in the L_3 representation for arbitrary L_0 from Eq. (71). It can be shown that its eigenvalues occur symmetrically about zero. This fact is to be expected from the classical meaning of K since a change in sign of K corresponds merely to a change of initial phase angle (from θ to $\pi-\theta$) between the oscillations of the two oscillators, or an opposite rotation about the z axis in the case of the spherical pendulum. Numerical calculations for certain combinations of L_0 and E have been made[8-10] and show the same qualitative features as those exhibited by the four-level system, in particular, slightly unequal spacing between the eigenvalues of K or what amounts to the same result, aperiodic oscillation of $\langle L_3(\tau)\rangle$.

As we have seen in the present discussion, the highest absolute eigenvalue of K designates the state that corresponds most closely to the classical "conical pendulum" solution of the coupled oscillators, and the lowest absolute eigenvalue of K designates the state that corresponds most closely to the classical "simple-pendulum" solution, with appropriate intermediate correspondences. It is interesting to compare the classical value of L_3 corresponding to $|K|_{\max}$ (the "conical-pendulum" value) to the numerical calculation of $\langle L_3\rangle$ for the state of highest $|\kappa|$, where κ is an eigenvalue of K. One finds, for instance, a numerical calculation[9] of $\langle L_3\rangle$ for the case $E=L_0=25$, the value being -8.210. If we set $\mathcal{E}=1$ (that is $E=L_0$) in our Eq. (54), which gives the value of $\mathcal{L}$ when $|K|$ is a maximum, we see that

$$\mathcal{L}=-\tfrac{1}{3} \quad\text{or}\quad L_3=L_0\mathcal{L}=-\tfrac{25}{3};$$

this result differs from the computed value by approximately 1%.

One can also approximate the largest eigenvalue of K by the classical $K_{\max}$, and then obtain $\Delta\kappa$, the *average* spacing between the eigenvalue of K, by dividing $2K_{\max}$ by the number of intervals between the eigenvalues. For $E\geq L_0$, the number of such intervals is $2L_0$, and we have

$$\Delta\kappa=K_{\max}/L_0=L_0^{1/2}\mathcal{K}_{\max}$$
$$=\tfrac{2}{3}(\tfrac{1}{3}L_0)^{1/2}[(\mathcal{E}^2+3)^{3/2}-\mathcal{E}^3+9\mathcal{E}]^{1/2}\,, \tag{91}$$

where Eq. (56) has been used for $\mathcal{K}_{\max}$. For the unstable equilibrium initial conditions, $\mathcal{E}=1$, and the above equation yields

$$\Delta\kappa=\tfrac{8}{3}(\tfrac{1}{3}L_0)^{1/2}\,. \tag{92}$$

If we approximate by ignoring the unequal spacing between the eigenvalues of K, we have here a value for the frequency of oscillation of $\langle L_3(\tau)\rangle$. It is interesting to note that the frequency of this "quantum-mechanical" oscillation (L_3 does not oscillate, classically, for the unstable equilibrium initial conditions) is obtained by using the classical result for $K_{\max}$. For $\mathcal{E}\gg 1$, we obtain from Eq. (91)

$$\Delta\kappa\approx(2E)^{1/2}\,, \tag{93}$$

in agreement with previous results.

Besides a listing of computer calculations of the eigenvalues of K, one finds (as mentioned previously) computer calculations of $\langle L_3(\tau)\rangle$ [or rather $\langle n(\tau)\rangle$, which is $E-\langle L_3(\tau)\rangle$] plotted against τ for the unstable-equilibrium initial conditions.[8,10] In Ref. 8, the case $L_0=40$ is plotted and exhibits the following features: (i) The expectation value of the NO energy, initially at L_0, oscillates; (ii) the minima and maxima fluctuate and do not all reach $\pm L_0$; (iii) the oscillation is only approximately periodic; (iv) the period (in units of τ) that occurs most frequently, and is also the largest period, is 0.9. [Note that our Eq. (92) yields a value of 0.7 in this instance.] We have already observed the first three qualitative features in some of the above special cases for small L_0. We will now consider approximate methods for the study of $\langle L_3(\tau)\rangle$ for large L_0.

F. Large L_0: Semi-Quantum-Mechanical Approximation

We return to the quantum-mechanical second-order energy equation for a general L_0, Eq. (64), and examine the possibility of an approximate solution for large L_0. As observed previously, the problem differs in two respects from the classical treatment; the coefficients in the differential equation are different, and the dependent variable is an operator (or matrix). (Note that if the equation were linear, the second difference would be of no consequence for the solution of the energy equation since the differential equation would apply to each matrix element separately.)

We now make the approximation

$$\langle L_3^2(\tau)\rangle\approx\langle L_3(\tau)\rangle^2\,, \tag{94}$$

so that Eq. (65) becomes a differential equation for $\langle L_3(\tau)\rangle$. This does not destroy all the quantum-mechanical properties of Eq. (65), since the quantum-mechanical coefficients are still there. We may call this approximation the "semi-quantum-mechanical" (SQM) approximation. For simplicity of notation we drop the expectation-value brackets, remembering, however, that L_3 stands for $\langle L_3 \rangle$ in the present discussion.

One can proceed to solve Eq. (65) with the approximation of Eq. (94) exactly as in the procedure for the classical case, merely taking note of the fact that the coefficients are different. It is convenient to use again the normalized quantity $\mathcal{L}$, which is now an expectation value and therefore a c number. The differential equation for $\mathcal{L}$ is Eq. (29) with $\epsilon = L_0^{-1}$. Multiplying both sides of Eq. (29) by $2\mathcal{L}_s$ – also an expectation value – and integrating, we obtain for general initial conditions[14]

$$\begin{aligned} \mathcal{L}_s^2 = 2\{&[\mathcal{L} - (\mathcal{E} + \tfrac{1}{2}\epsilon)][\mathcal{L}^2 - (1+\epsilon)] \\ &- [\mathcal{L}(0) - (\mathcal{E} + \tfrac{1}{2}\epsilon)][\mathcal{L}^2(0) - (1+\epsilon)]\} + \mathcal{L}_s^2(0) \\ \equiv{}& F_q(\mathcal{L}) \; . \end{aligned} \tag{95}$$

The only difference between this first-order energy equation and the corresponding classical equation [Eq. (45)] is the presence of ϵ. One has here, as in the classical case, an equation that leads to an oscillatory periodic solution for $\mathcal{L}$, the details of which are determined by the roots of the polynomial $F_q(\mathcal{L})$ in accordance with Eqs. (48) and (49), where now $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ are the roots of $F_q(\mathcal{L})$ rather than $F_c(\mathcal{L})$. We see immediately that one consequence of the approximation of Eq. (94) is the elimination of the aperiodicity in the expression for the expectation value that was present in the exact (for $L_0 \geq \frac{3}{2}$) and numerical solutions considered previously.

We examine the solution of Eq. (95) only for the case $L_0 \gg 1$. It is clear that this equation will yield a solution that differs significantly from the classical solution only when the roots of $F_q(\mathcal{L})$ differ significantly from the roots of $F_c(\mathcal{L})$ or when the presence of ϵ in $F_q(\mathcal{L})$ affects the roots significantly. Since $\epsilon \ll 1$ and the presence of ϵ can only produce a shift in the roots of the order of magnitude of ϵ, the roots of $F_q(\mathcal{L})$ will differ significantly from those of $F_c(\mathcal{L})$, as far as the effect on the solution is concerned, only when F_c has a double root (or almost a double root) at $\mathcal{L} = 1$ (that is, $\mathcal{L}_2 = \mathcal{L}_3$), and the presence of ϵ produces a separation of these roots. (Separation of a double root at $\mathcal{L} = -1$, on the other hand, has little effect on the solution.) In accordance with the previous discussion of the classical case, a double root at $\mathcal{L} = 1$ leads to unstable equilibrium either as a condition that exists initially or that is approached after an infinite time. Under both conditions $\mathcal{E} = 1$. The unstable-equilibrium initial conditions are of greater present interest[15] and we consider only these. We therefore set

$$\mathcal{L}(0) = \mathcal{E} = 1, \quad \mathcal{L}_s(0) = 0 \; . \tag{96}$$

The vanishing of $\mathcal{L}_s(0)$ follows from the fact that under the present initial conditions both the NO and LO are in energy states, for which the expectation value indicated by $\mathcal{L}_s(0)$ vanishes.

Equation (95) now reads

$$\mathcal{L}_s^2 = F_q(\mathcal{L}) = 2[f(\mathcal{L}) - \tfrac{1}{2}\epsilon^2] \; , \tag{97a}$$

where

$$f(\mathcal{L}) \equiv [\mathcal{L} - (1 + \tfrac{1}{2}\epsilon)][\mathcal{L}^2 - (1+\epsilon)] \; . \tag{97b}$$

We recall that $\mathcal{L}$ oscillates between $\mathcal{L}_1$ and $\mathcal{L}_2$, where $\mathcal{L}_1 \leq \mathcal{L}_2 \leq \mathcal{L}_3$ are the three roots of $F_q(\mathcal{L})$. The roots of $F_q(\mathcal{L})$ can be visualized in terms of the roots of $f(\mathcal{L})$ exactly in the same way as the roots of $F_c(\mathcal{L})$ were visualized in terms of the roots of $\varphi(\mathcal{L}, 0)$; we merely think of the graph of $f(\mathcal{L})$ as being lowered by an amount $\frac{1}{2}\epsilon^2$. The roots of $f(\mathcal{L})$ are, in increasing order, $-(1+\epsilon)^{1/2}$, $(1 + \frac{1}{2}\epsilon)$, $(1+\epsilon)^{1/2}$. Up to the lowest order of ϵ, these roots are $-(1 + \frac{1}{2}\epsilon)$, $(1 + \frac{1}{2}\epsilon)$, and $(1 + \frac{1}{2}\epsilon)$. Lowering the curve of $f(\mathcal{L})$ by an amount $\frac{1}{2}\epsilon^2$ will affect the first root negligibly, and – as can easily be ascertained – separate the second and third roots, shifting them in opposite directions by $\frac{1}{2}\epsilon$, respectively, so that

$$\mathcal{L}_1 = -(1 + \tfrac{1}{2}\epsilon), \quad \mathcal{L}_2 = 1, \quad \mathcal{L}_3 = 1 + \epsilon \; . \tag{98}$$

The solution $\mathcal{L}(s)$ is given by Eq. (49) with this set of roots. We see immediately that $\mathcal{L}$ will oscillate between 1 (its initial value) and $-(1 + \frac{1}{2}\epsilon)$. One observes here another error introduced by the approximation of Eq. (94), the dipping of $\mathcal{L}$ below -1, which, however, is slight.

We have in the set of roots given by Eqs. (98) a solution that is similar to the classical solution for initial conditions that are slightly different from those of unstable equilibrium. As can be seen from Eqs. (49)–(51), the solution is approximately (that is, up to lowest significant order in ϵ) the same as for the set of roots,

$$\mathcal{L}_1 = -1, \quad \mathcal{L}_2 = 1 - \tfrac{1}{2}\epsilon \; , \quad \mathcal{L}_3 = 1 + \tfrac{1}{2}\epsilon \; , \tag{99}$$

since the significant aspects of the roots are the position of the first root (approximately -1), the position of the remaining pair (approximately $+1$), and the separation between the members of this pair (ϵ). Comparison of Eq. (99) with Eq. (62) shows that we now have in the SQM treatment for unstable-equilibrium initial conditions exactly the same set of roots as in the classical treatment for off-unstable-equilibrium initial conditions, if we set in the latter treatment

$$\delta = 0, \quad \mathcal{K} = \epsilon \; . \tag{100}$$

Therefore, the period is given, from Eqs. (63) and (100), by

$$P = 5 \ln 2 + \ln \epsilon^{-1} = 5 \ln 2 + \ln L_0 \; . \tag{101}$$

We recall that this is the magnitude of the period measured in terms of s. In terms of τ it is given by

$$P^{(\tau)}=L_0^{-1/2}P=L_0^{-1/2}(5\ln 2+\ln L_0)\ ,\qquad(102)$$

and in terms of t, it is given by $2\gamma^{-1}P^{(\tau)}$. It is seen that one important qualitative difference between the quantum-mechanical and classical descriptions, the presence of spontaneous emission for unstable-equilibrium initial conditions, is contained in the SQM approximation. For $L_0=40$, as mentioned previously, the numerical results of Abate and Haken[8] yield a fluctuating $P^{(\tau)}$ that occurs most frequently with the value 0.9. Equation (102) yields the value 1.1.

It is illuminating to examine the mathematical significance of the approximation used in the SQM treatment. We note, first, that quantum mechanically (we return to the explicit use of expectation-value brackets)

$$\langle L_3^2\rangle \geq \langle L_3\rangle^2\ .\qquad(103)$$

Initially, for the unstable-equilibrium condition, the equality sign holds. Subsequently, however, the error committed is that of replacing $\langle L_3^2\rangle$ by a smaller value (except, perhaps, for some discrete values of τ). Since the curvature of the graph of $\langle L_3\rangle$ versus τ is proportional to $\langle L_3\rangle''$, we see from Eq. (65) that a correct value of $\langle L_3^2\rangle$ will decrease the (absolute) curvature of $\langle L_3(\tau)\rangle$ when the curve is concave downwards and increase the curvature when the curve is concave upwards. Qualitatively speaking, this amounts to a decrease in the curvature of the upper portion and an increase in the curvature of the lower portion of the graph. Such a correction will have two effects; it will raise the minimum of $\langle L_3\rangle$ and thus prevent the (unphysical) dipping of $\langle L_3\rangle$ below $-L_0$, and it may also, unless some special relationship exists between $\langle L_3^2\rangle-\langle L_3\rangle^2$ and the period, introduce some aperiodicity. One sees, therefore, that the qualitative aspects of the difference between the correct solution and the SQM solution are consistent with an inspection of the error introduced in the SQM approximation.

G. Large L_0: Statistical Approximation

The SQM solution gave no indication of aperiodicity. One might argue, on physical grounds, that since the classical solution for $L_3(\tau)$ displays no aperiodicity, the aperiodicity exhibited by the quantum-mechanical solution for $\langle L_3(\tau)\rangle$ disappears as L_0 becomes macroscopically large, that is, in the classical limit.[10] On the other hand, the mathematical plausibility argument for the aperiodicity, presented in the above discussion of the error introduced in the SQM approximation, is independent of L_0. Also, the computational results,[8-10] as far as they go, indicate no trend toward exact periodicity as L_0 increases. It is conceivable, however, that the aperiodicity disappears only when L_0 reaches much larger orders of magnitude than those considered, say, orders of magnitude of 10^{10}. In order to obtain a more complete understanding of the aperiodicity and, in particular, to see how it is affected by an increase in L_0 far beyond the values used for numerical calculation, we seek its physical explanation.

We will discuss a classical system but introduce, or take into account, the uncertainty principle. Although such a combination does not necessarily lead to rigorous quantum mechanics, it does in many instances give a qualitative, or even an approximate quantitative, explanation of the difference between the classical and quantum-mechanical results. Let us consider the initial conditions for unstable equilibrium, namely, $L_3(0)=L_0$, $n(0)=0$. Classically, $L_+(0)$ and $L_-(0)$ are well defined as zero, and $A(0)$ and $A^\dagger(0)$ are also well defined as zero. According to the uncertainty principle, however, $L_+(0)$, $L_-(0)$, $A(0)$, and $A^\dagger(0)$ cannot be well defined since, as operators, the first two variables do not commute with $L_3(0)$, and the last two variables do not commute with $n(0)$. In order to satisfy the uncertainty prinicple, we introduce a random amplitude of oscillation (that is, random with respect to an ensemble of systems) for both the LO and NO. For the LO, we assume an oscillation amplitude described by

$$A(0)=\tilde{A}e^{-i\theta_1},\quad A^\dagger(0)=\tilde{A}e^{i\theta_1}\ ,\qquad(104a)$$

where

$$A^\dagger(0)A(0)=\tilde{A}^2=\tfrac{1}{2}\ ,\qquad(104b)$$

θ_1 being a random variable (with all values equally probable); and for the NO we assume an oscillation amplitude described by

$$L_+(0)=\tilde{L}e^{i\theta_2},\quad L_-(0)=\tilde{L}e^{-i\theta_2},\qquad(105a)$$

where, noting that

$$2L_+(0)L_-(0)=2\tilde{L}^2=L_1^2+L_2^2\ ,\qquad(105b)$$

we set

$$2\tilde{L}^2=L_0(L_0+1)-L_3^2(0)=L_0,\qquad(105c)$$

θ_2 also being a random variable (with all values equally probable). It will be recognized that the magnitudes and phases of these random oscillations are chosen to give the same first and second moments for the oscillation amplitudes as those yielded by a quantum-mechanical description when the LO is in the ground state and the NO is in the highest state. These random oscillations require that we add explicitly to our previous classical E the quantity $\frac{1}{2}$ (the "zero-point" energy), so that $E_{\text{cl}}\to E+\frac{1}{2}$, and take account explicitly of the increase in L^2 so that $L_{\text{cl}}^2\to L_0(L_0+1)$.

Now, the classical equation, in terms of E and

L^2, is [from Eq. (23)]

$$L_3'' = 3L_3^2 - 2EL_3 - L^2 . \qquad (106)$$

The "classical" equation that takes into account the uncertainty principle becomes, therefore,

$$L_3'' = 3L_3^2 - (2E+1)L_3 - L_0(L_0+1) . \qquad (107)$$

Formally, this looks identical either to the quantum-mechanical equation [Eq. (26)] in which L_3 is an operator, or to the equation of the SQM approximation in which L_3 is an expectation value. Actually, however, it is neither of these. L_3, here, is not an operator but a c number which commutes with L_3', and it is also not an expectation value but a random variable which will be averaged only after the solution is obtained. The essential difference, for our purposes, between the treatment of L_3 as a random variable and the treatment of L_3 as an expectation value is the fact that, with the two oscillators in energy states initially, $L_3'(0)$ vanishes in the latter treatment while it does not necessarily vanish in the former treatment; from Eqs. (17), (106), and (107), we have, in fact, for the unstable equilibrium initial conditions

$$\begin{aligned} L_3'(0) &= -i[A(0)L_+(0) - A^\dagger(0)L_-(0)] \\ &= -i\tilde{A}\tilde{L}\,[e^{i(\theta_2-\theta_1)} - e^{-i(\theta_2-\theta_1)}] \\ &= L_0^{1/2}\sin\theta , \end{aligned} \qquad (108)$$

also a random variable. In normalized notation (recalling that $s = L_0^{1/2}\tau$) we can write

$$\mathcal{L}_s^2(0) = L_0^{-2}\sin^2\theta = \epsilon^2\sin^2\theta . \qquad (109)$$

One can arrive at Eqs. (107) and (108) from another viewpoint. Instead of starting with a classical description and introducing the uncertainty principle, one can start with a formal quantum-mechanical description and convert it into a purely statistical description in terms of classical random variables having the same first and second moments as the quantum-mechanical variables. In either case there is an approximation involved, of course, since quantum mechanics, although involving statistics in an essential manner, cannot be converted into pure statistics.[16]

Since Eq. (107) is formally identical to that of the SQM approximation it can be integrated to yield Eq. (95), with $\mathcal{L}_s^2(0)$ now given by Eq. (109). Substituting the unstable-equilibrium initial conditions $\mathcal{L}(0) = \mathcal{E} = 1$, we obtain

$$\mathcal{L}_s^2 = 2[\mathcal{L} - (1+\tfrac{1}{2}\epsilon)][\mathcal{L}^2 - (1+\epsilon)] - \epsilon^2\cos^2\theta . \qquad (110)$$

The only difference between this equation and Eq. (97) for the expectation value of $\mathcal{L}$ is that the ϵ^2 term in Eq. (97a) is replaced by the random variable $\epsilon^2\cos^2\theta$, so that the curve for $f(\mathcal{L})$ is lowered by the amount $\frac{1}{2}\epsilon^2\cos^2\theta$ instead of by the amount $\frac{1}{2}\epsilon^2$. The solution of Eq. (107) is therefore a periodic oscillation, the period (measured in terms of s) being given by Eq. (101) with the above replacement, namely,

$$\begin{aligned} P &= 5\ln 2 + \ln(\epsilon\,|\cos\theta|)^{-1} \\ &= 5\ln 2 + \ln L_0 + \ln(|\cos\theta|^{-1}) . \end{aligned} \qquad (111)$$

The function $\ln(|\cos\theta|^{-1})$ varies from 0 to ∞ as θ varies from 0 to $\frac{1}{2}\pi$, but is weighted for the low values since[17]

$$\langle \ln|\cos\theta|^{-1}\rangle_{\text{av}} = -\frac{2}{\pi}\int_0^{\pi/2} d\theta\,\ln(\cos\theta) = \ln 2 \qquad (112\text{a})$$

and

$$\langle(\ln|\cos\theta|^{-1})^2\rangle_{\text{av}} - (\ln 2)^2 = \tfrac{1}{12}\pi^2 . \qquad (112\text{b})$$

We see, therefore, that P is a random variable which assumes values from $5\ln 2 + \ln L_0$ to ∞, the average being

$$\langle P\rangle_{\text{av}} = 6\ln 2 + \ln L_0 , \qquad (113)$$

and the mean square deviation being

$$\langle P^2\rangle_{\text{av}} - \langle P\rangle_{\text{av}}^2 = \tfrac{1}{12}\pi^2 . \qquad (114)$$

If now we consider the random variable $\mathcal{L}$ and take its average, which corresponds, in the present approximation, to the quantum-mechanical expectation value, we will be averaging over a range of periods and obtain an aperiodic function.

H. Aperiodicity

The realization that the expectation value constitutes an ensemble average over a range of periods is the physical explanation of the aperiodicity in the quantum-mechanical expectation value of the NO energy (and, therefore, also in the LO energy). When the initial conditions are those for unstable equilibrium, that is, when the LO is in the ground state and the NO is in the highest state, there are uncertainties in the oscillation of both oscillators of a coupled pair; these uncertainties are usually referred to as quantum fluctuations, the fluctuations being the random differences among members of an ensemble in which each member is a pair of coupled oscillators. As we have seen in the classical analysis [Eq. (63)], the period for a given pair is extremely sensitive, for initial conditions near those of unstable equilibrium, to slight differences in initial conditions. In taking expectation values we average over the quantum fluctuations, that is, over all members of the ensemble, and thus average over a range of periods, obtaining aperiodicity. One would also expect that for those initial conditions for which slight uncertainties do not affect the period significantly the aperiodicity of the expectation value is negligible; as seen in the discussion of the SQM approximation, and in numerical treatments of certain special cases,[8-10] this is indeed true.

The above explanation of the aperiodicity is conceptually somewhat similar to that of the spread of the wave packet in the motion of a free particle. If the initial position of a free particle is reasonably well defined there is an initial spread in velocities among members of an ensemble, and a purely statistical (classical) analysis of the spread in position at a later time based on the initial velocity spread derived from the uncertainty principle will approximate the main features of the spread of the wave packet. In the present instance there exists an initial spread of periods (or frequencies) which produces an aperiodicity in the oscillation of the average.

We are now in a position to examine the aperiodicity as L_0 becomes macroscopically large, say, 10^{10}. Expressing the period T in terms of ordinary time t, we have

$$T = 2\gamma^{-1} L_0^{-1/2}\,[5\ln 2 + \ln L_0 + \ln(|\cos\theta|^{-1})] \qquad (115)$$

and

$$\langle T\rangle_{\text{av}} = 2\gamma^{-1} L_0^{-1/2}(6\ln 2 + \ln L_0). \qquad (116)$$

A measure of the spread, as given by the rms deviation ΔT is, from Eq. (114),

$$\Delta T = 3^{-1/2}\pi\gamma^{-1} L_0^{-1/2}. \qquad (117)$$

A measure of the aperiodicity introduced by this spread may be obtained by considering two periodic oscillations with periods T_+ and T_-, where

$$T_\pm = \langle T\rangle_{\text{av}} \pm \tfrac{1}{2}\Delta T,$$

assuming them to be in phase at $t = 0$, and finding the time t_0 at which they become completely out of phase. It is clear that the two oscillations will be out of phase after r periods, where

$$r = \tfrac{1}{2}(\langle T\rangle_{\text{av}}/\Delta T) = \sqrt{3}\,\pi^{-1}(6\ln 2 + \ln L_0), \qquad (118)$$

and the time t_0 at which they will be out of phase is

$$t_0 = r\langle T\rangle_{\text{av}} = 2\sqrt{3}\,\pi^{-1}\gamma^{-1} L_0^{-1/2}(6\ln 2 + \ln L_0)^2. \qquad (119)$$

In terms of periods of oscillation the aperiodicity decreases with increasing L_0, but very slowly – logarithmically, to be precise. In units of time, the aperiodicity increases with increasing L_0. Even when measured in periods, however, the aperiodicity can hardly be said to disappear as L_0 becomes macroscopically large. Thus, as L_0 changes from 10 to 10^{10}, r changes only from 3.5 to 15, so that, even in the classical limit, the aperiodicity persists.

The fact that the aperiodicity, which is of quantum-mechanical origin, manifests itself in the interaction between two oscillators of which each may be considered macroscopic, with the energy of the coupled system also being macroscopic, may appear to be in conflict with the correspondence principle. The apparent conflict can be explained away, however. The aperiodicity is significant only for initial conditions close to the unstable-equilibrium conditions (or conditions that, classically, lead to unstable equilibrium after an infinite time and can be treated in an essentially similar manner). For these conditions, the LO is in a low quantum-number state and may not necessarily be regarded as classical. It so happens, because of the particular nonlinear properties of the NO, that the period of oscillation of the NO energy is extremely sensitive to the above initial conditions – conditions under which the LO is not classical. Thus, the quantum-mechanical aspects appear in the oscillation of energy of each of the oscillators. This phenomenon indicates that conditions for the application of the correspondence principle must be carefully examined, under some circumstances, in cases involving coupled systems.

Note added in proof. The present problem is also treated by R. Bonifacio and G. Preparata [Phys. Rev. A 2, 336 (1970)], who obtain a periodic solution for the expectation value of the NO energy in the classical limit. A brief discussion of some of the ideas of the present paper, especially with reference to the aperiodicity, is given by I. R. Senitzky *ibid.* 2, 2046 (1970)].

APPENDIX A

We consider here, in connection with Eqs. (58)–(61), the solution of the equation

$$\mathcal{L}_s^2 = F_c(\mathcal{L}) \equiv 2[(\mathcal{L} - 1 - \delta)(\mathcal{L}^2 - 1) - \tfrac{1}{2}\mathcal{K}^2], \qquad (\text{A1})$$

where δ and $\mathcal{K}$ are small quantities of first order. The solution is given by Eq. (49) in terms of the roots of the cubic polynomial $F_c(\mathcal{L})$. In zeroth order, the roots are

$$\mathcal{L}_1^{(0)} = -1, \quad \mathcal{L}_2^{(0)} = \mathcal{L}_3^{(0)} = 1. \qquad (\text{A2})$$

It is easy to see that the presence of δ and $\mathcal{K}$ produce a second-order shift in $\mathcal{L}_1^{(0)}$ and first-order shifts in $\mathcal{L}_2^{(0)}$ and $\mathcal{L}_3^{(0)}$, and, particularly, a separation of $\mathcal{L}_2^{(0)}$ and $\mathcal{L}_3^{(0)}$. Furthermore, a small change in $\mathcal{L}_1^{(0)}$ produces an insignificant change in the solution while a separation of $\mathcal{L}_2^{(0)}$ and $\mathcal{L}_3^{(0)}$ produces a qualitative change, altering the solution from a nonoscillatory to an oscillatory function. We therefore ignore the change in $\mathcal{L}_1^{(0)}$ produced by the presence of $\mathcal{K}$. The first-order change in $\mathcal{L}_2$ and $\mathcal{L}_3$ is obtained by setting

$$\mathcal{L} = 1 + x \qquad (\text{A3})$$

and finding the roots of $F_c(1+x)$, to lowest order, in the neighborhood of 1. These are given by

$$x = \tfrac{1}{2}\delta \pm \tfrac{1}{2}(\delta^2 + \mathcal{K}^2)^{1/2}, \qquad (\text{A4})$$

which leads to the set of roots

$$\mathcal{L}_1 = -1, \quad \mathcal{L}_2 = 1 + \tfrac{1}{2}[\delta - (\delta^2 + \mathcal{K}^2)^{1/2}],$$

$$\mathcal{L}_3 = 1 + \tfrac{1}{2}[\delta + (\delta^2 + \mathcal{K}^2)^{1/2}]. \tag{A5}$$

To lowest order, p and k, in Eq. (49), are given by

$$p = 1, \quad k^2 = 1 - \tfrac{1}{2}(\delta^2 + \mathcal{K}^2)^{1/2}. \tag{A6}$$

According to Jahnke and Emde[18] [where the notation K is used for the complete elliptic integral which we designate by I in Eq. (51)], for k^2 close to unity we have

$$I \approx \ln \frac{4}{(1-k^2)^{1/2}} = \ln 4 \frac{\sqrt{2}}{(\delta^2 + \mathcal{K}^2)^{1/4}}$$

$$= \tfrac{5}{2} \ln 2 + \tfrac{1}{4} \ln \frac{1}{\delta^2 + \mathcal{K}^2}, \tag{A7}$$

which leads to

$$P = (2/p) I = 5 \ln 2 + \tfrac{1}{2} \ln(\delta^2 + \mathcal{K}^2)^{-1}, \tag{A8}$$

the result of Eq. (63).

APPENDIX B

We prove here formally that $L_3^2(\tau) = \frac{1}{4}$, where, from Eq. (70),

$$L_3(\tau) = L_3(0) \cos \Omega \tau$$

$$- i \Omega^{-1} [A(0) L_+(0) - A^\dagger(0) L_-(0)] \sin \Omega \tau, \tag{B1}$$

and the NO is a two-level system.

The square of this expression is

$$L_3^2(\tau) = L_3^2 \cos^2 \Omega \tau - \Omega^{-2} (A L_+ + A^\dagger L_-)^2 \sin^2 \Omega \tau$$

$$- i \Omega^{-1} \{L_3, (A L_+ - A^\dagger L_-)\} \sin \Omega \tau \cos \Omega \tau, \tag{B2}$$

where the notation $\{A, B\} \equiv AB + BA$ is used, and where the operators without argument refer to $\tau = 0$. Noting that for a two-level system

$$\{L_3, L_+\} = \{L_3, L_-\} = L_+^2 = L_-^2 = 0, \tag{B3}$$

we obtain

$$L_3^2(\tau) = L_3^2 \cos^2 \Omega \tau + \Omega^{-2} (A A^\dagger L_+ L_- + A^\dagger A L_- L_+) \sin^2 \Omega \tau. \tag{B4}$$

Since

$$A A^\dagger = n + 1, \quad \{L_+, L_-\} = L^2 - L_3^2,$$

and

$$L_+ L_- = \tfrac{1}{2}(L^2 - L_3^2 + L_3),$$

we have

$$L_3^2(\tau) = L_3^2 \cos^2 \Omega \tau + \Omega^{-2} [n(L^2 - L_3^2)$$

$$+ \tfrac{1}{2}(L^2 - L_3^2 + L_3)] \sin^2 \Omega \tau. \tag{B5}$$

Now, for a two-level system

$$L^2 = \tfrac{3}{4}, \quad L_3^2(0) = \tfrac{1}{4}. \tag{B6}$$

Substituting these values into Eq. (B5), and noting that

$$\Omega^2 = 2E + 1 = 2(n + L_3) + 1,$$

we obtain the desired result.

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[14] Equation (95) can also be obtained as an approximation from Eq. (32); obtaining it in this manner, however, would require, formally, more statements of approximation than that of Eq. (94).

[15] These are also the initial conditions for the study of the onset of oscillation in an idealized maser or laser where the atomic systems are fully excited initially.

[16] For instance, higher moments than the second, in the above statistical description, may not be equal to the corresponding quantum-mechanical expectation values. An illustration of this point is the expression on the right-hand side of Eq. (30), $2(AA^\dagger L_+ L_- + A^\dagger A L_- L_+)$. According to the purely statistical description of the unstable-equilibrium initial conditions, Eqs. (104) and (105), this quantity is initially equal to L_0, while Eq. (31) shows that the quantum-mechanical expectation value of this quantity, for the same initial conditions, is initially equal to $2L_0$.

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