in Fig. 2 along with excess, viscous, and experimental values. The variation of ΔF with pressure has been shown in Fig. 1. Considering the uncertainties involved in the assumption of β_r and τ, it is obvious from Table II that the theoretical values of $(\alpha/f^2)_{tot}$ have an excellent agreement with the experimental results. This indicates quite conclusively that the two-state theory can be used to describe the pressure dependence of the compressional losses in methyl alcohol as well. However, the previous concept about the linear variation of ΔF and ΔV with pressure is to be modified. Figure 1 shows that the variation of ΔF with pressure for methyl alcohol is similar to the variation of its velocity, which is nonlinear. Similarly, the variations of ΔF and velocity of water are quite similar and linear with pressure, thereby suggesting that there is a correlation of ΔF with velocity, which seems to be convincing, because β_r, ΔF, and β_0 (and so velocity) depend upon the internal structure and so on the population density of the two states under consideration in the liquid. Hence the variations of ΔF and c are expected to be alike, which is true in our case. This point further supports the assumption for β_r and τ at 1000 kg/cm^2, because the variation of ΔF with pressure, as evaluated using the above assumption, is as expected. So this successful attempt to explain the experimental curve of the ultrasonic absorption in methyl alcohol gives an idea about the variation of ΔF and β_r with pressure in methyl alcohol, in particular, and primary alcohols in general.

[1]L. Hall, Phys. Rev. 73, 772 (1948).
[2]S. K. Kor, Gulshan Rai, and O. N. Awasthi, Phys. Rev. 186, 105 (1969).
[3]E. H. Carnevale and T. A. Litovitz, J. Acoust. Soc. Am. 27, 547 (1955).

PHYSICAL REVIEW A VOLUME 3, NUMBER 1 JANUARY 1971

Ground-State Energy of a Mixture of Fermions and Bosons in One Dimension with a Repulsive δ-Function Interaction*

C. K. Lai and C. N. Yang
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790
(Received 24 August 1970)

The ground-state energy of a mixture of spin-$\frac{1}{2}$ fermions and bosons in one dimension, interacting with a repulsive δ-function potential, is analyzed. The wave function is given by repeated use of a generalized Bethe hypothesis. The "momenta" in the hypothesis are determined by coupled Fredholm integral equations. Numerical solutions are given. No phase separation is found.

I.

Consider a one-dimensional N-body problem:

$$H=-\sum_{1}^{N}\frac{\partial^2}{\partial x_i^2}+2c\sum_{i<j}\delta(x_i-x_j)\,,\quad c>0 \tag{1}$$

for M_1 fermions of species 1, M_2 fermions of species 2, and M_b bosons, where

$$N=M_1+M_2+M_b\,,\quad M_1\geq M_2>0\,. \tag{2}$$

In this paper we shall discuss the ground-state energy of this system, especially in the limit that M_1, M_2, M_b, and the length L of the box go to infinity proportionately. A periodic boundary condition is assumed, and we further assume that

$$M_1=\text{odd},\quad M_2=\text{odd}. \tag{3}$$

II.

Our method of solution starts with first proving the following:

Theorem. The ground state of the system in question has a wave function $\psi(x_1\cdots x_N)$, which belongs to the irreducible representation of the permutation group S_N, characterized by the following partition:

$$(2+M_b,\; 2^{M_2-1},\; 1^{M_1-M_2})\,, \tag{4}$$

as shown in Fig. 1. Furthermore, it is the lowest energy state among wave functions of this symmetry.

(To illustrate the meaning of the question answered by the theorem, consider the case of two fermions of spin up, two fermions of spin down, and two bosons. It is easy, by a computation with group characters, to find that any eigenfunction for such a system belongs to one of the representations as shown in Fig. 2. The question is which of these does the ground-state wave function belong to?)

The proof is similar to that of a theorem due to Lieb and Mattis.[1] Take the case, for example, M_1

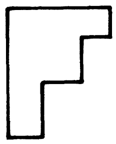

FIG. 1. Diagram of partition (4).

$=5$, $M_2=3$, $M_b=2$, $N=5+3+2=10$. Consider the Schrödinger equation $H\psi=E\psi$ for five fermions x_1, x_2, x_3, x_4, x_5 of spin up, three fermions x_6, x_7, x_8 of spin down, and two bosons x_9, x_{10} in a box so that all x's are between 0 and L, with periodic boundary conditions. Call this problem 1. Next define a region R so that

(i) $x_1 \leqq x_2 \leqq x_3 \leqq x_4 \leqq x_5$ or cyclic permutation,

(ii) $x_6 \leqq x_7 \leqq x_8$ or cyclic permutation,

(iii) all $x_i \geqq 0$ and $< L$ (cyclic).

Consider the Schrödinger equation $H\psi=E\psi$ in R with $\psi=0$ on the boundary of R. Call this problem 2.

The complete space $L \times L \times \cdots \times L = L^{10}$ is bigger than R by a factor of $(4!)(2!)=48$. It is clear that every eigenfunction ψ of problem 1 is also an eigenfunction of problem 2 when restricted to the region R. It can be shown without much difficulty that, furthermore, ψ is

cyclicly symmetrical with respect to x_1, x_2, x_3, x_4, x_5, (5a)

cyclicly symmetrical with respect to x_6, x_7, x_8, (5b)

symmetrical with respect to $x_9 \leftrightarrow x_{10}$. (5c)

Conversely every eigenfunction of problem 2 satisfying (5a)–(5c) can be uniquely extended into the full L^{10} space through the conditions of antisymmetry, so as to form an eigenfunction of problem 1.

Now the ground state ψ_0 of problem 2 evidently can be normalized to satisfy

$$\psi_0 > 0 \text{ inside } R, \quad \psi_0 = \text{nondegenerate}. \tag{6}$$

Symmetrization of ψ_0 with respect to $x_9 \leftrightarrow x_{10}$ leads, because of (6), to the condition

$$\psi_0 = \text{symmetrical with respect to } x_9 \leftrightarrow x_{10}. \tag{7}$$

Similarly cyclic symmetrization with respect to x_1, x_2, x_3, x_4, x_5 and cyclic symmetrization with respect to x_6, x_7, x_8 show that ψ_0 satisfies conditions (5).

Thus ψ_0 is identical, in R, with the ground-state eigenfunction ψ_1 of problem 1. Equation (6) then leads to

$$\psi_1 > 0 \text{ in } R, \quad \psi_1 = \text{nondegenerate}. \tag{8}$$

Next, construct the operator

$$Y = AB,$$

where

$$A = (\textstyle\sum 24 \text{ permutations among } 1, 6, 9, 10) \times (1+P_{2,7})(1+P_{3,8}), \tag{9}$$

$$B = [\textstyle\sum(-1)^P \text{ the } 5! \text{ permutations among } 1, 2, 3, 4, 5] \times [\textstyle\sum(-1)^P \text{ the } 3! \text{ permutations among } 6, 7, 8]$$

for the tableau

$$[4, 2^2, 1^2], \tag{10}$$

as shown in Fig. 3. By a well-known theorem, $Y\psi_1$ is again a wave function for problem 1 and belongs to the partition (10). [To prove $Y\psi_1 = AB\psi_1 \neq 0$, we observe that $B\psi_1 = (5!)(3!)\psi_1$. Now A is a sum of $(24)(2)(2)=96$ permutations. Each of these permutations transforms the region R_1 into itself.

$$0 < x_1,\ x_6,\ x_9,\ x_{10} < \tfrac{1}{5} L,$$
$$\tfrac{1}{5} L < x_2,\ x_7 < \tfrac{2}{5} L,$$
$$\tfrac{2}{5} L < x_3,\ x_8 < \tfrac{3}{5} L,$$
$$\tfrac{3}{5} L < x_4 < \tfrac{4}{5} L,$$
$$\tfrac{4}{5} L < x_5 < L.$$

Now R_1 is entirely within R, in which $\psi_1 > 0$ by (8). Thus $A\psi_1 > 0$ in R_1. Hence $Y\psi_1 = (5!)(3!)A\psi_1 \neq 0$.]

$Y\psi_1$ evidently has the same eigenvalue as ψ_1. Hence they are proportional, and we have established that ψ_1 belongs to the symmetry (10).

Conversely let ψ_2 be an eigenstate belonging to symmetry (10). There exists a permutation of its ten coordinates resulting in an eigenstate ψ_3 so that $\psi_4 = BA\psi_3 \neq 0$. Now $\psi_4 = B$(something) is evidently an eigenstate of problem 1. Hence its energy is not lower than that of ψ_1, i.e., ψ_1 is the lowest energy state among all wave functions with symmetry (10). This completes the proof of the theorem.

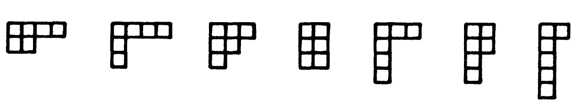

FIG. 2. Representations for the eigenfunctions of a special 6-particle system.

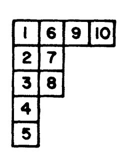

FIG. 3. Diagram of the tableau represented by (10).

III.

The problem is thus reduced to finding the ground state with the symmetry (4). This is a problem that can be solved by repeated[2,3] use of a generalized Bethe hypothesis. The result[3] for symmetry (4) is given by the solution of

$$e^{ipL}=\prod_{\Lambda'}\frac{ip-i\Lambda'-c'}{ip-i\Lambda'+c'}\,,\qquad \text{number of } p=N \tag{11a}$$

$$\prod_{p'}\left(\frac{i\Lambda-ip'+c'}{i\Lambda-ip'-c'}\right)=-\prod_{\Lambda'}\left(\frac{i\Lambda-i\Lambda'+c}{i\Lambda-i\Lambda'-c}\right)\prod_{A'}\left(\frac{i\Lambda-iA'-c'}{i\Lambda-iA'+c'}\right),\qquad \text{number of } \Lambda=M \tag{11b}$$

$$\prod_{\Lambda'}\left(\frac{iA-i\Lambda'+c'}{iA-i\Lambda'-c'}\right)=-\prod_{A'}\left(\frac{iA-iA'+c}{iA-iA'-c}\right)\prod_{B'}\left(\frac{iA-iB'-c'}{iA-iB'+c}\right),\qquad \text{number of } A=M_b \tag{11c}$$

$$\prod_{A'}\left(\frac{iB-iA'+c'}{iB-iA'-c'}\right)=-\prod_{B'}\left(\frac{iB-iB'+c}{iB-iB'-c}\right)\prod_{C'}\left(\frac{iB-iC'-c'}{iB-iC'+c'}\right),\qquad \text{number of } B=M_b-1 \tag{11d}$$

$$\cdots$$

$$\prod_{X'}\left(\frac{iY-iX'+c'}{iY-iX'-c'}\right)=-\prod_{Y'}\left(\frac{iY-iY'+c}{iY-iY'-c}\right)\prod_{Z'}\left(\frac{iY-iZ'-c'}{iY-iZ'+c'}\right),\qquad \text{number of } Y=2 \tag{11e}$$

$$\prod_{Y'}\left(\frac{iZ-iY'+c'}{iZ-iY'-c'}\right)=-\prod_{Z'}\left(\frac{iZ-iZ'+c}{iZ-iZ'-c}\right),\qquad \text{number of } Z=1 \tag{11f}$$

where $M=M_b+M_2$ and $c'=\frac{1}{2}c$. When L, N, M, $M_b\to\infty$ proportionally, the number of equations contained in (11a)–(11f) becomes infinite. Fortunately, due to the theorem below, Eqs. (11c)–(11f) could be reduced to a single equation.

Theorem. Equations (11c)–(11f) can be reduced to the following equation:

$$\prod_{\Lambda'}\left(\frac{iA-i\Lambda'+c'}{iA-i\Lambda'-c'}\right)=1\,, \tag{11g}$$

so that the A's in (11b) will satisfy (11g).

We prove it by induction. It is obvious that the right-hand side of (11f) equals 1 since there is only one Z' to sum over. Thus (11f) reduces to

$$\prod_{Y'}\left(\frac{iZ-iY'+c'}{iZ-iY'-c'}\right)=1\,. \tag{12}$$

To complete the induction let us suppose that Eqs. (11d)–(11f) can be reduced to the following:

$$\prod_{A'}\left(\frac{iB-iA'+c'}{iB-iA'-c'}\right)=1\,. \tag{13}$$

With the A's considered as given (M_b in number), expansion of (13) into a polynomial equation for B yields an equation of degree M_b-1, which has M_b-1 roots. Since the B's are different from each other, these roots should be exactly the set of B's. It will be shown in Appendix A that from (13) the following identity holds for any A:

$$\prod_{B'}\left(\frac{iA-iB'+c'}{iA-iB'-c'}\right)=-\prod_{A'}\left(\frac{iA-iA'+c}{iA-iA'-c}\right). \tag{14}$$

Substitute (14) into the right-hand side of (11c), and one readily obtains (11g). Thus the theorem is proved.

[One might be tempted to reduce (11b) and (11g) further. But as the degree of the polynomial equation for A obtained from (11g) is $M-1$, it admits more solutions than the collection of A's which number only M_b. The quantum numbers to be discussed below serve to choose the correct A's from among the $(M-1)$ solutions. These quantum numbers will be determined by continuity arguments.]

IV.

Thus (11a) and (11b) together with (11g) are the equations we need to solve. Taking the logarithm, we have

$$pL=2\pi I_p+\textstyle\sum_\Lambda\theta(2p-2\Lambda)\,, \tag{15a}$$

$$\textstyle\sum_{\Lambda'}\theta(\Lambda-\Lambda')=2\pi J_\Lambda+\sum_p\theta(2\Lambda-2p)+\sum_A\theta(2\Lambda-2A)\,, \tag{15b}$$

$$0=2\pi K_A+\textstyle\sum_\Lambda\theta(2A-2\Lambda)\,, \tag{15c}$$

where $\theta(x)=-2\tan^{-1}(x/c)$, $|\theta|<\pi$, and I_p, J_Λ, and K_A are quantum numbers resulting from the term (mod$2\pi i$) in taking logarithms. For the case when all M_1, M_2, M_b are odd, I_p, $J_\Lambda+\frac{1}{2}$, and K_A are integers.

We shall now show that for $c>0$ and for the ground state with symmetry (4), the quantum numbers I, J, and K are given by

$$I_p=\text{successive integers from } -\tfrac{1}{2}(N-1) \text{ to } \tfrac{1}{2}(N-1)\,, \tag{16a}$$

$$\tfrac{1}{2}+J_\Lambda = \text{successive integers from } 1-\tfrac{1}{2}M \text{ to } \tfrac{1}{2}M\,, \tag{16b}$$

$$K_A = \text{successive integers from } -\tfrac{1}{2}(M_b-1) \text{ to } \tfrac{1}{2}(M_b-1)\,. \tag{16c}$$

To do this we substitute (16) into (15) and go to the limit $c \to 0+$. For concreteness, take $M_1 = 5$, $M_2 = 3$, $M_b = 3$. Now we have $\theta(x) = \frac{1}{2}\pi$ or $-\frac{1}{2}\pi$ as $c \to 0+$. One can readily show that the p, Λ, and A values shown in Fig. 4(a) satisfy (15) and (16) in that limit. A different set of quantum numbers I, J, K is illustrated in Fig. 4(b). As explained in the caption, the energy for the case of Fig. 4(a) is lower. Generalizing this example, one obtains (16).

V.

Now one can go to the limit N, M_1, M_b, $L \to \infty$ proportionally in the case $c > 0$ and obtain integral equations in the standard fashion:

$$\begin{pmatrix}\rho\\ \sigma\\ \tau\end{pmatrix} = \frac{1}{2\pi}\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0 & X_1 & 0\\ X_1 & -X_2 & X_1\\ 0 & X_1 & 0\end{pmatrix} \times \begin{pmatrix}B_0 & & \\ & B_1 & \\ & & B_2\end{pmatrix}\begin{pmatrix}\rho\\ \sigma\\ \tau\end{pmatrix}\,, \tag{17}$$

where X_n is an integral operator:

$$\langle k|X_n|k'\rangle = \frac{nc}{2\pi}\frac{4}{n^2c^2+4(k-k')^2} \tag{18}$$

and B_0, B_1, B_2 are projection operators, i.e.,

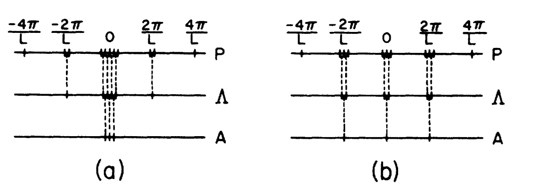

FIG. 4. Distribution of p, Λ, and A as $c \to 0$ in the case $N=11$, $M=6$, $M_b=3$ (i.e., $M_1=5$, $M_2=3$, $M_b=3$). (a) For closed-packed quantum numbers (16): $I_p = (-5, -4, \cdots 4, 5)$, $J_\Lambda = (-\frac{5}{2}, -\frac{3}{2}\cdots\frac{3}{2}, \frac{5}{2})$, $K_A = (-1,0,1)$. Five of the p's, four of the Λ's, and three of the A's are crowded near the origin. Notice that each Λ is in between two p's, and each A is in between two Λ's. (b) For the same set of I_p and J_Λ as above but for $K_A = (-2, 0, 2)$. One p jumps from the origin to $2\pi/L$ and another from the origin to $-2\pi/L$. It is clear that the energy $(=\sum p^2)$ is smaller for (a).

$$\langle k|B_j\rho\rangle = \begin{cases}\rho(k) & \text{for } |k| \le b_j\\ 0 & \text{for } |k| > b_j\end{cases}\,. \tag{19}$$

Here b_0, b_1, b_2 are real positive numbers. In addition, we have

$$\frac{M_1+M_2+M_b}{L} = \frac{N}{L} = \int_{-b_0}^{b_0}\rho(k)\,dk\,, \tag{20}$$

$$\frac{M_2+M_b}{L} = \frac{M}{L} = \int_{-b_1}^{b_1}\sigma(k)\,dk\,,\quad \frac{M_b}{L} = \int_{-b_2}^{b_2}\tau(k)\,dk\,,$$

and the energy E is given by

$$E/L = \int_{-b_0}^{b_0} k^2\rho(k)\,dk\,. \tag{21}$$

VI.

Note that

$$X_nX_m = X_{n+m}\,, \tag{22}$$

$$X_n\psi = e^{-nc|a|}\psi \text{ if } \langle k|\psi\rangle = e^{2iak}\,. \tag{23}$$

By integrating the second or third equations of (17) over all real k, one obtains the following:

(i) $b_2 = 0 \to M_b = 0$ (i.e., no bosons, agrees with Ref. 2),

(ii) $b_2 = \infty \to M_2 = 0$ (i.e., no spin-down fermions),

(iii) $b_1 = \infty \to M_1 = M_2$ (i.e., spin-up and -down fermions equal in number),

(iv) $b_1 = b_2 = \infty \to M_1 = M_2 = 0$ (i.e., no fermions, agrees with Ref. 4).

Now assume b_1 or b_2 or both to be $\neq \infty$. One can eliminate ρ and τ from (17) and obtain

$$\sigma = XB_0(1/2\pi) + (X_1B_0X_1 - X_2 + X_1B_2X_1)B_1\sigma\,. \tag{24}$$

Using $B_1^2 = B_1$, we obtain

$$B_1\sigma = B_1X_1B_0(1/2\pi) + (\Omega_- + \Omega_+)(B_1\sigma)\,, \tag{25}$$

where

$$\Omega_- = B_1X_1(B_0-1)X_1B_1\,, \tag{26}$$

$$\Omega_+ = B_1X_1B_2X_1B_1\,. \tag{27}$$

Now Ω_- is a nonpositive operator. The maximum eigenvalue of Ω_+ is 1 if $b_2 = b_1 = \infty$ (for which $\Omega_+ = X_2$). For b_1 or b_2 or both $\neq\infty$, the maximum eigenvalue of Ω_+ is < 1. Thus in this case $\Omega_- + \Omega_+$ has a maximum eigenvalue < 1, and (25) is a nonsingular Fredholm equation. Thus (17) has a unique solution.

VII.

In the following, we will take $b_1=\infty$ so that spin-up and spin-down particles are equal in number. For a fixed total number M_1+M_2 of fermions and a fixed number M_b of bosons, this system gives[1] the lowest energy. When $b_2\cong 0$, one can expand quantities in power of M_b/L which is of the order of b_2; the energy thus obtained gives the energy of mixing of bosons in the fermion system. It will be shown in Appendix B that one can readily obtain, for a fixed density,

$$\frac{E}{L}=\left(\frac{E}{L}\right)_0-\frac{M_b}{L}\left(\frac{d(E/L)_0}{dr_0}\int_{-b_0}^{b_0}\rho_1(k)\,dk\right.$$
$$\left.\times-\int_{-b_0}^{b_0}k^2\rho_1(k)\,dk\right)+\cdots, \quad (28)$$

where $(E/L)_0$ is the energy of the pure fermion system at density $r_0=N/L$ and $\rho_1(k)$ is given by (B9). [It is clear that $\rho_1(k)$ is proportional to the change of distribution function of the fermions when bosons are added to the system.]

As for the case $b_2\cong\infty$ so that $N\cong M_b$, the energy in the expansion of $(N-M_b)/L$ gives the energy of mixing of fermions in a boson system. At a fixed density, one readily obtains (see Appendix C)

$$\frac{E}{L}=\left(\frac{E}{L}\right)_0+\left(\frac{N-M_b}{L}\right)^3\frac{\pi^2}{12c}$$
$$\times\left[\frac{1}{r_0}\frac{d(E/L)_0}{dr_0}-\frac{1}{r_0^2}\left(\frac{E}{L}\right)_0\right]+\cdots, \quad (29)$$

where $(E/L)_0$ and r_0 refer now to the energy and density of the pure boson system. That the first two powers of $(N-M_b)/L$ are absent from this formula is easy to understand: At fixed N a substitution of one (or two) fermions for one (or two) bosons in a pure boson system cannot change the ground-state energy.

VIII.

We solve (24) numerically by applying Simpson's rule to the integrals on a sufficiently fine grid and then iterating the equation to yield approximate values of $\sigma(k)$ on the grid. The functions $\rho(k)$ and $\tau(k)$ are then computed from the first and second equations of (17), and E/L, N/L, M_b/L are computed from (20) and (21). By using different values of b_0 and b_1 each time, fixing $b_1=\infty$, one obtains the ground-state energy E/L for various values of N/L and M_b/L. It takes about ten iterations of (24) to get three-digit accuracy. The energy E/N plotted in Fig. 5 is for a fixed density and is obtained by interpolation from calculated data. The smooth concave nature of the curve indicates no phase separation, in contrast to the case of real He^3-He^4 mixture which shows[5] a phase separation at 0 °K.

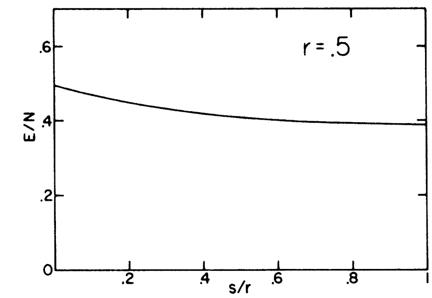

FIG. 5. Ground-state energy E/N plotted as function of s/r at density $r=0.5$, $s=M_b/L$. We take $c=2$ and $b_1=\infty$ (i.e., $M_1=M_2$) in the numerical computation.

APPENDIX A

Write

$$iA-c'=a,\quad iA+c'=\tilde{a}.$$

Equation (13) then becomes, as a polynomial equation for B:

$$(iB)^\alpha-(\sum a)(iB)^{\alpha-1}+(\sum aa)(iB)^{\alpha-2}-\cdots$$
$$=(\text{same with } a\to\tilde{a}),$$

where $(\sum aa)$ means sum over all pairs of *different* a's, etc., and $\alpha=M_b$, i.e.,

$$(iB)^{\alpha-1}(\sum a-\sum\tilde{a})-(iB)^{\alpha-2}(\sum aa-\sum\tilde{a}\tilde{a})+\cdots=0.$$

From the relationship between the roots and the coefficients of a polynomial equation, one obtains

$$\sum(iB)=(\sum aa-\sum\tilde{a}\tilde{a})(\sum a-\sum\tilde{a})^{-1},$$
$$\sum(iB)(iB)=(\sum aaa-\sum\tilde{a}\tilde{a}\tilde{a})(\sum a-\sum\tilde{a})^{-1},\quad\text{etc.}$$

We can now evaluate the numerator of the left-hand side of (14):

$$\Pi_B(iA-iB+c')=\Pi_B(\tilde{a}-iB)=\tilde{a}^{\alpha-1}-\tilde{a}^{\alpha-2}\sum(iB)+\tilde{a}^{\alpha-3}\sum(iB)(iB)-\cdots$$
$$=(\sum a-\sum\tilde{a})^{-1}[(\sum a-\sum\tilde{a})\tilde{a}^{\alpha-1}-(\sum aa-\sum\tilde{a}\tilde{a})\tilde{a}^{\alpha-2}+\cdots]$$
$$=(\sum a-\sum\tilde{a})^{-1}\{[\tilde{a}^\alpha-\Pi_{a'}(\tilde{a}-a')]-[\tilde{a}^\alpha-\Pi_{a'}(\tilde{a}-\tilde{a}')]\}=-(\sum a-\sum\tilde{a})^{-1}\Pi_{a'}(\tilde{a}-a').$$

Similarly the denominator of the left-hand side of (14) is

$$\Pi_B (a-iB) = (\sum a - \sum \tilde{a})^{-1} \Pi_{\tilde{a}'} (a-\tilde{a}') .$$

Taking the ratio one obtains (14).

APPENDIX B

When $b_2 \cong 0$, the second equation in (17) can be expanded as follows (one takes $b_1 = \infty$ here):

$$\sigma = \frac{1}{2\pi}\int_{-b_0}^{b_0} \frac{c\rho(k)\,dk}{c^2/4+(k-k')^2} - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2c\sigma(k)\,dk}{c^2+(k-k')^2}$$

$$+\frac{1}{2\pi}\frac{c}{c^2/4+k^2}\,2b_2\tau(0)+O(b_2^3). \tag{B1}$$

One also has

$$s = M_b/L = \int_{-b_2}^{b_2} \tau(k)\,dk = 2b_2\tau(0)+O(b_2^3). \tag{B2}$$

If one writes ρ and σ in the power expansion of s,

$$\rho=\rho_0+s\rho_1+s^2\rho_2+\cdots, \qquad \sigma=\sigma_0+s\sigma_1+s^2\sigma_2+\cdots, \tag{B3}$$

then by substituting (B3) into (17) and by using (B1) and (B2), one obtains the following in operator forms:

$$\rho_0 = 1/2\pi + X_1\sigma_0, \quad \sigma_0 = X_1B_0\rho_0 - X_2\sigma_0 , \tag{B4}$$

$$\rho_1 = X_1\sigma_1, \quad \sigma_1 = X_1B_0\rho_1 - X_2\sigma_1 + X_1\xi, \tag{B5}$$

where $\xi(k)$ is defined as a δ function,

$$\langle k|\xi\rangle = \delta(k). \tag{B6}$$

Note that (B4) are the integral equations for the fermion system. Eliminating ρ_1 in (B5), one obtains

$$\sigma_1 = -X_1(1-B_0)X_1\sigma_1 + X_1\xi. \tag{B7}$$

Thus one has

$$\sigma_1 = [1+X_1(1-B_0)X_1]^{-1}X_1\xi, \tag{B8}$$

$$\rho_1 = X_1[1+X_1(1-B_0)X_1]^{-1}X_1\xi. \tag{B9}$$

The density $r=N/L$ and the energy E/L are given by

$$r = \int_{-b_0}^{b_0}\rho_0\,dk + s\int_{-b_0}^{b_0}\rho_1\,dk = r_0 + sf, \tag{B10}$$

$$E/L = \int_{-b_0}^{b_0} k^2\rho_0\,dk + s\int_{-b_0}^{b_0} k^2\rho_1\,dk = (E/L)_0 + sg, \tag{B11}$$

where r_0 and $(E/L)_0$ denote quantities with respect to the fermion system. At fixed density r, one has

$$\Delta r = \frac{dr_0}{db_0}\Delta b_0 + \Delta s\cdot f = 0, \tag{B12}$$

$$\frac{E}{L} = \left(\frac{E}{L}\right)_0 + \frac{d(E/L)_0}{db_0}\Delta b_0 + \Delta s\cdot g$$

$$= \left(\frac{E}{L}\right)_0 + \Delta s\left(g - f\cdot\frac{d(E/L)_0}{dr_0}\right). \tag{B13}$$

The expansion (28) for fixed density immediately follows from the above. Note that from (B4),

$$\rho_0 = \frac{1}{2\pi} + X_1\frac{1}{1+X_1(1-B_0)X_1}X_1B_0\left(\frac{1}{2\pi}\right), \tag{B14}$$

and by (B9) one has

$$f = \int_{-b_0}^{b_0}\rho_1(k)\,dk = 2\pi\rho_0(0) - 1. \tag{B15}$$

APPENDIX C

When $b_2 \cong \infty$, it is more convenient to use the following equations derived from (17):

$$\rho = 1/2\pi + \tau, \tag{C1}$$

$$\tau = X_2B_0\rho - X_2(1-B_2)\tau. \tag{C2}$$

Let us write $\rho=\rho_0+\rho_1$, $\tau=\tau_0+\tau_1$, such that

$$\rho_0 = 1/2\pi + \tau_0, \tag{C3}$$

$$\tau_0 = X_2B_0\rho_0; \tag{C4}$$

$$\rho_1 = \tau_1, \tag{C5}$$

$$\tau_1 = X_2B_0\rho_1 - X_2(1-B_2)\tau_1 - X_2(1-B_2)\tau_0. \tag{C6}$$

(C3) and (C4) give the equations for a boson system:

$$\rho_0 = 1/2\pi + X_2B_0\rho_0, \tag{C7}$$

with solution

$$B_0\rho_0 = (1-B_0X_2B_0)^{-1}B_0 1/(2\pi)^{-1}, \tag{C8}$$

and (C5) and (C6) give a single equation for τ_1,

$$\tau_1 = X_2B_0\tau_1 - X_2(1-B_2)(\tau_1+\tau_0). \tag{C9}$$

Integration of (C2) over all real k gives

$$(N-M_b)/L = 2\int_{|k|>b_2}\tau(k)\,dk. \tag{C10}$$

The energy E/L and density $r=N/L$ are given by

$$r = \int_{-b_0}^{b_0}(\rho_0+\rho_1)\,dk = r_0 + \int_{-b_0}^{b_0}\tau_1\,dk, \tag{C11}$$

$$\frac{E}{L} = \int_{-b_0}^{b_0} k^2(\rho_0+\rho_1)\,dk = \left(\frac{E}{L}\right)_0 + \int_{-b_0}^{b_0} k^2\tau_1\,dk. \tag{C12}$$

As b_2 is very large, we can make the following approximation in (C11) and (C12). From (C9), we have

$$B_0\tau_1 = B_0X_2B_0\tau_1 - B_0X_2(1-B_2)(\tau_1+\tau_0), \tag{C13}$$

and thus

$$B_0\tau_1 = -(1-B_0X_2B_0)^{-1}B_0X_2(1-B_2)(\tau_1+\tau_0). \tag{C14}$$

Since

$$\langle k|B_0X_2(1-B_2)|k'\rangle = \frac{1}{2\pi}\left(\frac{2c}{k'^2}+\frac{4ck}{k'^3}+\cdots\right) \text{ as } k'\to\infty, \tag{C15}$$

(C14) becomes

$$B_0\tau_1 = -\mu\left(\frac{1}{1-B_0X_2B_0}\,B_0\,\frac{1}{2\pi}\right)+\cdots \cong -\mu B_0\rho_0\,, \tag{C16}$$

where μ is the integral

$$\mu = \int_{|k'|>b_2} (2c/k'^2)(\tau_1+\tau_0)\,dk'. \tag{C17}$$

Then (C11) and (C12) become

$$r \cong r_0 - \mu r_0\,, \tag{C11$'$}$$

$$\frac{E}{L} \cong \left(\frac{E}{L}\right)_0 - \mu\left(\frac{E}{L}\right)_0. \tag{C12$'$}$$

Now let us write $\tau_1 = \phi + \psi$ in (C9) such that

$$\phi = -X_2(1-B_2)\phi - X_2(1-B_2)\tau_0\,, \tag{C18}$$

$$\psi = -X_2(1-B_2-B_0)\psi + X_2B_0\phi\,. \tag{C19}$$

By arguments similar to those in deriving (C16) and by (C4), which implies that

$$\tau_0(k) = \frac{r_0c}{\pi}\,\frac{1}{k^2} + O\left(\frac{1}{k^3}\right) \quad \text{as } k\to\infty, \tag{C20}$$

we have

$$\phi = -[1+X_2(1-B_2)]^{-1}X_2(1-B_2)\tau_0\,, \tag{C21}$$

$$X_2B_0\phi = -X_2B_0[1+X_2(1-B_2)]^{-1}X_2(1-B_2)\tau_0$$

$$\cong \text{const}\times X_2B_0\times 1\,, \tag{C22}$$

where

$$\text{const} \sim \int_{|k|>b_2} \frac{\tau_0(k')}{k'^2}\,dk' = O\left(\frac{1}{b_2^3}\right). \tag{C23}$$

From the above and (C19), it will readily be seen that we can neglect the contribution of ψ to (C10) and (C17) in our lowest-order calculation. Then (C18) and (C20) gives

$$\phi(k) = -\frac{r_0c}{2\pi}\,\frac{1}{k^2}+\cdots \quad \text{as } k\to\infty. \tag{C24}$$

Using the above and (C20) to evaluate (C10) and (C17), we obtain

$$(N-M_b)/L \cong 2cr_0/\pi b_2\,, \tag{C25}$$

$$r \cong r_0 - r_0^2\,(2c^2/3\pi b_2^3)\,, \tag{C26}$$

$$\frac{E}{L} \cong \left(\frac{E}{L}\right)_0 - r_0\left(\frac{E}{L}\right)_0\frac{2c^2}{3\pi b_2^3}\,. \tag{C27}$$

The expansion (29) for a fixed density immediately follows.

---

*Work supported in part by AEC Contract No. AT(30-1)-3668B.

[1]E. H. Lieb and D. Mattis, Phys. Rev. 125, 164 (1962).
[2]C. N. Yang, Phys. Rev. Letters 19, 1314 (1967).
[3]B. Sutherland, Phys. Rev. Letters 20, 98 (1968).
[4]E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).
[5]D. O. Edwards *et al.*, Phys. Rev. Letters 15, 773 (1965).

PHYSICAL REVIEW A VOLUME 3, NUMBER 1 JANUARY 1971

Wave-Mechanical Formulation of Many-Fluid Plasma Theory in Electromagnetic Fields*

H. E. Wilhelm
Colorado State University, Fort Collins, Colorado 80521
(Received 1 June 1970)

The macroscopic dynamics of classical many-component plasmas in electric and magnetic fields is formulated in terms of scalar complex wave equations which contain pressure and electromagnetic potentials. The complex wave equation of each particle component is shown to be mathematically equivalent to the nonlinear conservation equations of mass, vector momentum, and energy. Thus, the description of a plasma by wave equations leads to a considerable mathematical simplification compared to the conventional many-fluid electromagnetohydrodynamics. As an elementary illustration of the wave-mechanical formalism, the dispersion of longitudinal and transverse electromagnetic waves in an electron plasma is treated.

I. INTRODUCTION

The macroscopic dynamics of classical plasmas is determined by the complete set of conservation equations for mass, vector momentum, and energy of each component. In the electromagnetohydrodynamic approximation for a dissipation-free plasma consisting of r components, these field equa-