The above theory assumes that the thermomagnetic-force effect is an effect over the area of the disk rather than an edge effect. This last statement needs experimental verification and a new apparatus is presently being built which will allow measurements to be made on a number of different size disks with varying edge-to-area ratios. We hope to show conclusively whether our new results are a surface effect or an edge effect.

The exact molecular-collision mechanism giving rise to the thermomagnetic force is not known and theoretical calculations have not yet been reported. We conclude that the thermomagnetic force is closely related to the SB effect. It is hoped that the work reported here on the force effect will contribute to a better understanding of the molecular interactions in polyatomic gases and also to a better understanding of how these molecular interactions are reflected in the transport properties.

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Third Sound in Superfluid Helium Films of Arbitrary Thickness*

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The hydrodynamic equations of a superfluid helium film are solved numerically to yield the dispersion equation of third sound over a wide range of thicknesses, including both unsaturated and saturated films. An analytical approximation to the dispersion equation is found that is good for the entire range where the equations are valid, and which becomes particularly simple in the case of either very thin or very thick films. A clear physical picture is formed of the processes determining the properties of third sound in thick helium films.

I. INTRODUCTION

In a previous article,[1] which will henceforth be referred to as I, we obtained a full linearized hydrodynamic description of the lateral motion that is possible in a thin superfluid film situated upon a flat solid substrate and in equilibrium with its own vapor. The equations obtained there were solved explicitly to determine the dispersion equation for third-sound waves only in the limiting case when

$$\kappa_g\,\omega/\rho_g C_{pg} c_3^2 \ll 1, \quad \eta_g\,\omega/\rho_g c_3^2 \ll 1 ,$$

which means, in practice, very thin films and low frequencies.[1]

Although this case includes all the third-sound experiments conducted on unsaturated helium films by Rudnick's group at UCLA,[2,3] it is also of interest to solve the equations outside this regime. Atkins's original pioneering experiments to detect third sound and measure its properties were in fact made on saturated helium films[4] (i.e., adsorbed films that form upon the walls of a vessel containing liquid helium, at a height of no more than several centimeters above the surface of the liquid), which are considered to be thick films in the context of this article, and he has published some data on both the velocity and the attenuation of third sound in such films.[4]

We have therefore set out to try to extend our solution of the third-sound equations to thicker films and higher frequencies than those allowed in our previous solution.[1] In Sec. II we rewrite these equations in a precise form (i.e., without any approximations that are unjustified for thick films or high frequencies) and then solve them numerically to obtain the dispersion equation for third sound. The solution thus found is exceedingly regular not only for thin films, but for thick films as well, which leads us to attempt, in Sec. III, to fix up the analytical solution of I so that it becomes an exact solution of the revised equations, and to try to identify those parts of it that are of greatest importance. Finally, in Sec. IV, we find, *a posteriori*, what physical assumptions and approximations must be made in the original equations in order to get as an exact solution the approximation obtained in Sec. III. This leaves us with both a set of convenient equations to describe analytically the properties of third sound in thick films, as well as a clearer understanding of the physical processes that determine these properties. Section V is a summary and discussion of this work.

II. THIRD-SOUND EQUATIONS: NUMERICAL SOLUTION AND DISCUSSION

As in I, we shall be discussing lateral motions in a thin superfluid film, adsorbed upon a flat solid substrate and in equilibrium with its own vapor. Our physical system is described schematically in Fig. 1 where we have chosen to include only one lateral direction—the one where the observable motion takes place. In saying this we are not implying that vertical motions do not exist in the film (on the contrary, they must be present too whenever there is any periodic lateral motion) but merely noting that only the lateral motions lead to large-scale observable effects, e.g., superfluid mass transport third-sound waves.

To describe these motions, we derived in I a basic set of linearized hydrodynamic equations of motion for the situation described in Fig. 1,[5] which we reproduce here in a slightly modified form:

$$\rho_f \dot{h} + h\bar{\rho}_s \frac{\partial \bar{v}_s}{\partial x} + J_M = 0 \ , \qquad (1)$$

$$\rho_f \dot{h} + \left(1 + \frac{T\bar{S}}{L}\right) h\bar{\rho}_s \frac{\partial \bar{v}_s}{\partial x} - \frac{\bar{\rho}_f h C_h}{L} \dot{T}_f + \frac{\kappa_f h}{L} \frac{\partial^2 T_f}{\partial x^2} - \frac{1}{L}\left(J_{Q\,\mathrm{sub}} + J_{Qg}\right) = 0 \ , \qquad (2)$$

$$\dot{\bar{v}}_s - \bar{S} \frac{\partial T_f}{\partial x} + f \frac{\partial h}{\partial x} = 0 \ , \qquad (3)$$

where h is the thickness of the film, κ_f is its heat-conductivity coefficient, ρ_f is the total mass density of the film at the liquid-gas interface, $\bar{\rho}_f$ is the film density averaged over the thickness, $\bar{\rho}_s$ is the superfluid mass density averaged over the thickness, $\bar{v}_s$ is the average velocity in the x direction, defined by

$$\bar{v}_s \equiv \frac{1}{h\bar{\rho}_s} \int_0^h \rho_s v_{sx}\, dy \ ,$$

T is the equilibrium temperature, T_f is the instantaneous film temperature, $\bar{S}$ is the partial entropy of the film per unit mass,

$$\bar{S} \equiv \left(\frac{\partial S}{\partial M}\right)_T ,$$

where M stands for the total mass of a constant-area film, L is the latent heat of evaporation from the film per unit mass, f is the van der Waals acceleration acting on atoms of He near the surface of the substrate, C_h is the average specific heat of the film per unit mass, J_M is the net mass current flowing from the film into the gas (i.e., the net rate of evaporation), and $J_{Q\,\mathrm{sub}}$ and J_{Qg} are the net flows of heat from the film to the substrate and to the gas, respectively.

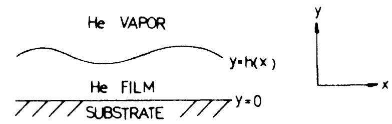

FIG. 1. Schematic drawing of a vertical section of the helium film.

The only real difference between these equations and the corresponding ones of I is in the presence of the term

$$\frac{\kappa_f h}{L} \frac{\partial^2 T_f}{\partial x^2}$$

in (2). This term was discarded in I for being small in very thin films, but we now wish to consider films so thick that it can no longer be automatically neglected.

In order to solve these equations we must first substitute explicit expressions for J_M, $J_{Q\,\mathrm{sub}}$, and J_{Qg}. These depend not only on the variables of the film h, $\bar{v}_s$, and T_f, but also on the variables of the gas and the substrate which are in contact with the film. For example, J_M is given by

$$J_M = \rho_g (v_{gy} - \dot{h}) \ , \qquad (4)$$

where ρ_g is the average gas density, v_{gy} is the y component of the gas velocity at the film-gas interface. Owing to the boundary conditions between the film and its surrounding media, any motion that occurs in the film excites corresponding motions in these media. Therefore the variables of the surrounding media vary with time and we must consider their equations of motion as well as those for the film. All the equations we consider are linearized.

We look for a solution of (1)–(3) that has the form of a traveling wave

$$e^{-i\omega t+ikx}\,, \tag{5}$$

and for solutions of the gas and substrate equations (see I) in the form of a wave having one component traveling in the x direction, together with the wave in the film, and another component traveling in the y direction, away from the film (i.e., a component that is radiated away from the film),

$$e^{-i\omega t+ikx+q_{\mathrm{sub}}y},\quad \mathrm{Im}\,q_{\mathrm{sub}}<0 \text{ in the substrate} \tag{6}$$

$$e^{-i\omega t+ikx-q_g y},\quad \mathrm{Im}\,q_g<0 \text{ in the gas.} \tag{7}$$

As a result of solving the equations for the substrate and taking into account the boundary condition that the heat current flowing from the film to the substrate is proportional to the temperature discontinuity at the interface, we can write[1]

$$J_{Q\,\mathrm{sub}}=B(T_f-T)\,, \tag{8}$$

where

$$\frac{1}{B}\equiv\frac{1}{B_1}+\left[\left(\frac{\kappa_{\mathrm{sub}}\omega}{c_3}\right)^2-i\kappa_{\mathrm{sub}}\omega\rho_{\mathrm{sub}}C_{p\,\mathrm{sub}}\right]^{-1/2},\qquad \mathrm{Im}B<0\,; \tag{9}$$

$1/B_1$ is the Kapitza resistance, κ_{sub} is the heat-conductivity coefficient of the substrate, ρ_{sub} is its mass density, $C_{p\,\mathrm{sub}}$ is its heat capacity at constant pressure, and c_3 is the complex velocity of third sound, defined by

$$c_3\equiv\omega/k\,. \tag{10}$$

As a result of solving the equations for the gas,[1] we can express all the gas variables in terms of the independent amplitudes of the three different wave modes of the form (7) which exist in the gas. Choosing v_{gx}, T_{g2}, and T_{g3}, the x component of the total gas velocity, the temperature amplitude in the second (thermal) mode, and the temperature amplitude in the third (acoustic) mode, as the basic independent amplitudes, we can write the following exact expressions[6]:

$$\frac{\rho_g'}{\rho_g}=-\frac{T_{g2}}{T}+\frac{1}{\gamma-1}\frac{T_{g3}}{T}\,, \tag{11}$$

$$\frac{v_{gy}}{c_3}=\frac{ic_{01}}{c_3}\frac{v_{gx}}{c_3}-\frac{\kappa_g\omega}{\rho_g C_{pg}c_3^2}\left(\frac{c_3}{c_{02}}-\frac{c_{01}}{c_3}\right)\frac{T_{g2}}{T}+\frac{iTC_{pg}}{c_3^2}\left(\frac{c_3}{c_{03}}-\frac{c_{01}}{c_3}\right)\frac{T_{g3}}{T}\,, \tag{12}$$

$$J_{Qg}\equiv-\kappa_g\frac{\partial T_g}{\partial y}=\kappa_g\omega\left(\frac{T_{g2}}{c_{02}}+\frac{T_{g3}}{c_{03}}\right)\,, \tag{13}$$

where ρ_g' is the amplitude of density oscillations in the gas; $\gamma\equiv C_{pg}/C_{vg}$; κ_g is the heat-conductivity coefficient of the gas; C_{pg} and C_{vg} are its constant-pressure and constant-volume heat capacities; c_{01}, c_{02}, and c_{03} are the complex velocities of the three wave modes in the gas:

$$\frac{1}{c_{01}}\equiv\left(\frac{1}{c_3^2}-\frac{i\rho_g}{\omega\eta_g}\right)^{1/2}\qquad\text{(viscous mode)}\,, \tag{14}$$

$$\frac{1}{c_{02}}\equiv\left(\frac{1}{c_3^2}-\frac{i\rho_g C_{pg}}{\kappa_g\omega}\right)^{1/2}\qquad\text{(thermal conduction mode)}, \tag{15}$$

$$\frac{1}{c_{03}}\equiv\left(\frac{1}{c_3^2}-\frac{1}{c^2}\right)^{1/2}\qquad\text{(acoustic mode)}\,, \tag{16}$$

$$\mathrm{Im}1/c_{0i}<0\quad\text{for } i=1,\ 2,\ 3; \tag{17}$$

η_g is the shear viscosity coefficient in the gas; and c is the velocity of sound in the gas. Equations (15) and (16) are exact only up to terms of the order

$$\kappa_g\omega/\rho_g C_{pg}c^2\quad\text{or}\quad\eta_g\omega/\rho_g c^2\,, \tag{18}$$

which are much less than 1 and which we shall consistently neglect, as in I.

We would like to digress at this point in order to draw attention to the fact that the condition that determines which square root is to be taken in (14)–(16), as well as in (9), is that the imaginary part of the root be negative. This condition ensures that any companion waves in the surrounding media always move away from the helium film. In I we used instead the condition that the real part of the same roots be positive. In the case of the acoustic mode [Eq. (16)] this leads to the other root. We believe that the present assignment of sign is the correct one, whereas in I it was in error. When we use condition (17), we find that the acoustic wave excited in the gas has an amplitude that increases exponentially as we move away from the film. This seemingly paradoxical result can be understood once we realize that the waves that are further away from the film were emitted at an earlier point in the film, where the amplitude of third-sound oscillations was correspondingly greater. Since the attenuation rate of third sound is larger than that of ordinary sound in the gas—which is how the acoustic mode propagates—we get an exponentially increasing amplitude for the acoustic mode in the steady state.

When Eqs. (4), (8), (12), and (13) are substituted in Eqs. (1)–(3) we find that there are more independent unknown amplitudes than equations. The three additional equations that are needed to determine them all are the boundary conditions for the film-gas interface[1]:

$$v_{gx}=0, \tag{19}$$

$$J_M = 4A\left(1-\frac{\rho_g}{\rho_f}\right)^{-1}\left[\left(\frac{L}{k_BT/m}-\frac{1}{2}\right)\frac{T_{f_1}}{T} - \frac{1}{2}\frac{T_g'}{T} - \frac{\rho_g'}{\rho_g} + \frac{fh_1}{k_BT/m}\right], \quad (20)$$

$$J_{Qg} = -\frac{k_BT}{m}A\left[\left(\frac{L}{k_BT/m}-\frac{9}{2}\right)\frac{T_{f_1}}{T} + \frac{7}{2}\frac{T_g'}{T} - \frac{\rho_g'}{\rho_g} + \frac{fh_1}{k_BT/m}\right], \quad (21)$$

where

$$A \equiv \tfrac{1}{2}\rho_g(k_BT/2\pi m)^{1/2}; \quad (22)$$

T_{f_1}, h_1 are the amplitudes of oscillation of T_f and h; $T_g' = T_{g2} + T_{g3}$ is the amplitude of the total temperature oscillation in the gas (there is no contribution to this from the viscous mode); and m is the mass of a He^4 atom.

Most of the preceding equations in this section have been discussed and derived in I. But whereas in that reference we only attempted to write down and solve the full equations in a certain limit, we now write down the exact equations, which number five instead of six since we immediately use (19) to substitute for v_{gx} in (12), the only place where it appears. The exact equations, neglecting, as always, terms of order (18), are

$$\left(1-\frac{\rho_g}{\rho_f}\right)\frac{h_1}{h} - v - \frac{i\kappa_g}{h\rho_f C_{pg}}\left(\frac{1}{c_{02}}-\frac{c_{01}}{c_3^2}\right)\frac{T_{g2}}{T} - \frac{\rho_g}{\rho_f}\frac{TC_{pg}}{h\omega}\left(\frac{1}{c_{03}}-\frac{c_{01}}{c_3^2}\right)\frac{T_{g3}}{T} = 0,$$

$$\frac{h_1}{h} - \left(1+\frac{TS}{L}\right)v - \frac{T}{h\rho_f L}\left(\frac{iB}{\omega} + h\bar{\rho}_f C_h + \frac{i\kappa_f\omega h}{c_3^2}\right)\frac{T_{f_1}}{T} - \frac{i\kappa_g T}{h\rho_f L}\left(\frac{1}{c_{02}}\frac{T_{g2}}{T} + \frac{1}{c_{03}}\frac{T_{g3}}{T}\right) = 0,$$

$$\frac{hf\bar{\rho}_s}{c_3^2\rho_f}\frac{h_1}{h} - v - \frac{T\bar{S}\,\bar{\rho}_s}{c_3^2\,\rho_f}\frac{T_{f_1}}{T} = 0, \quad (23)$$

$$\left[2hf\frac{m}{k_BT} - \frac{ih\omega}{a}\left(\frac{\rho_f}{\rho_g}-1\right)\right]\frac{h_1}{h} + \frac{ih\omega}{a}\left(\frac{\rho_f}{\rho_g}-1\right)v + \left(\frac{2L}{k_BT/m}-1\right)\frac{T_{f_1}}{T} + \frac{T_{g2}}{T} - \frac{\gamma+1}{\gamma-1}\frac{T_{g3}}{T} = 0,$$

$$hf\frac{m}{k_BT}\frac{h_1}{h} + \left(\frac{L}{k_BT/m}-\frac{9}{2}\right)\frac{T_{f_1}}{T} + \left(\frac{9}{2} + \frac{2\kappa_g\omega T}{aP_g c_{02}}\right)\frac{T_{g2}}{T} + \left(\frac{7}{2} - \frac{1}{\gamma-1} + \frac{2\kappa_g\omega T}{aP_g c_{03}}\right)\frac{T_{g3}}{T} = 0,$$

where

$$v \equiv (\bar{\rho}_s/\rho_f)(\bar{v}_s/c_3), \quad a \equiv (k_BT/2\pi m)^{1/2}.$$

At this point it would be wise to summarize the limitations on the use of this "exact" set of linearized equations for third sound in thin superfluid films: (i) The quantities in (18) must be much less than 1. (ii) The quantities

$$\frac{\eta_f\,\omega}{\rho_f c_3^2}, \quad \bar{\rho}_s\frac{\zeta_1\omega}{\rho_f c_3^2}, \quad \bar{\rho}_s^{\,2}\frac{\zeta_3\omega}{\rho_f c_3^2}$$

must also be much less than 1. η_f, ζ_1, and ζ_3 are the shear viscosity coefficient and two of the bulk viscosity coefficients of the film.[7] (iii) The film thickness must be much less than the viscous penetration depth in the film, and the mean free path in the gas and in the substrate must be much less than $1/q_g$ and $1/q_{sub}$, respectively. These, in turn, must be much less than the thickness of the gas and the substrate, respectively, so that we may ignore the possibility of reflections.

Keeping these limitations in mind, we have solved this set of equations numerically to obtain the complex velocity of third sound c_3 over a rather wide range of film thicknesses and for several frequencies and temperatures. The results for the real velocity are identical to the results of I in the case of thin films. For the thicker films that we considered, the real velocity is always identical to c_{30}, defined by

$$(hf\bar{\rho}_s/c_{30}^2\rho_f)(1+T\bar{S}/L)^2 = 1.$$

We have therefore not plotted out these results.

The results for the coeffficent of attenuation of third sound, α, derived from the imaginary part of $1/c_3$, show an interesting behavior. Consequently, in Fig. 2 we have plotted the values of α as a function of the thickness h for $T = 1.3\,^\circ$K and for frequencies $\nu = 1$ kHz and $\nu = 5$ kHz.[8] For comparison we have also plotted on the same figure the results of the approximate analytical solution for $\nu = 5$ kHz obtained in I and expected to be valid in the limit

$$\kappa_g\,\omega/\rho_g C_{pg} c_3^2 \ll 1; \quad \eta_g\,\omega/\rho_g c_3^2 \ll 1,$$

i.e., for small thicknesses and low frequencies. We have also plotted in the same figure the predictions of Atkins's theory[9] for α.

The general behavior of α is as follows: For small thicknesses (but not too small) it decreases with increasing h, approximately as $h^{-5/2}$, and is proportional to $\omega^{1/2}$. In this region the approximate results of I are completely in agreement with our present computation. Then α passes through a minimum and thereafter increases with increasing

h as $h^{11/2}$, while also being proportional to ω^2. In this region it has the same behavior as a function of h and ω as predicted by Atkins's theory,[9] but the values differ by a factor of order 1. Also contradicted by our numerical results is Atkins's basic assumption[9] that T_g remains fixed while only T_f oscillates. We find that, in fact, that T_g oscillates with an amplitude comparable to that of T_f, even though they are no longer identical, as was the case in the very thin films.

In any case, the great regularity of the results for thick films and their similarity to Atkins's old results[9] prompted us to try to obtain a useful analytical approximation for the entire region described in Fig. 2. This is done in Sec. III.

III. THIRD-SOUND EQUATIONS: ANALYTICAL SOLUTION FOR ARBITRARY FILM THICKNESSES

In order to get an analytic solution, we return to the approximate equations (44)–(48) of I and fix them so as to be equivalent to (23). This is done by redefining the J coefficients of I and defining two new coefficients N_1 and N_2. The equations then become

$$\left(1-\frac{\rho_g}{\rho_f}\right)h_1-\frac{h\bar{\rho}_s}{c_3\rho_f}v_{s_1}-\frac{e^{i\pi/4}}{TC_{pg}\rho_f}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}N_1T'_g-\frac{u^{1/2}C_{pg}}{\omega}\frac{\rho_g}{\rho_f}J_1T_{g3}=0\ ,$$

$$h_1-\frac{h\bar{\rho}_s}{c_3\rho_f}\left(1+\frac{T\bar{S}}{L}\right)v_{s_1}-\frac{1}{\rho_f L}\left(\frac{iB}{\omega}+\bar{\rho}_f hC_h+\frac{i\kappa_f\omega h}{c_3^2}\right)T_{f_1}-\frac{e^{i\pi/4}}{\rho_f L}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}(N_2T'_g-J_2T_{g3})=0\ ,$$

$$\frac{f}{c_3}h_1-v_{s_1}-\frac{\bar{S}}{c_3}T_{f_1}=0\ , \tag{24}$$

$$\left[1-\frac{i\omega P_g}{4Af}\left(1-\frac{\rho_g}{\rho_f}\right)\right]fh_1+\left(\frac{L}{T}-\frac{k_B}{2m}\right)T_{f_1}+\frac{k_B}{2m}\left[1-\frac{i\omega}{2AC_{pg}}e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}N_1\left(1-\frac{\rho_g}{\rho_f}\right)\right]T'_g$$
$$-\frac{iu^{1/2}P_gC_{pg}}{4A}\left[J_1\left(1-\frac{\rho_g}{\rho_f}\right)-\frac{4iA}{P_g u^{1/2}}\right]T_{g_3}=0\ ,$$

$$fh_1+\left(\frac{L}{T}-\frac{9}{2}\frac{k_B}{m}\right)T_{f_1}+\frac{9}{2}\frac{k_B}{m}J_6T'_g-C_{pg}J_7T_{g_3}=0\ ,$$

while their (exact) solution becomes

$$1-\frac{hf\bar{\rho}_s}{c_3^2\rho_f}\left(1+\frac{T\bar{S}}{L}\right)^2-\left(\frac{T\bar{S}}{L}\right)^2+\left(1-\frac{J_3}{J_4}\right)\left(\frac{T\bar{S}}{L}-\frac{\rho_g}{\rho_f}\right)\left(1+\frac{T\bar{S}}{L}\right)+\frac{J_3}{J_4}\frac{\rho_g}{\rho_f}\frac{T\bar{S}}{L}\left(1+\frac{T\bar{S}}{L}\right)+\frac{hf\bar{\rho}_s}{c_3^2\rho_f}\frac{T\bar{S}}{L}\left(1+\frac{T\bar{S}}{L}\right)$$
$$\times\frac{9}{2}\frac{k_BT/m}{L}(J_5-1)\left(1+\frac{9}{2}\frac{k_BT/m}{L}(J_5-1)\right)^{-1}+\frac{J_3}{J_4}\left(1+\frac{T\bar{S}}{L}\right)\left(1+\frac{9}{2}\frac{k_BT/m}{L}(J_5-1)\right)^{-1}\frac{Tf}{\rho_f L^2}$$
$$\times\left\{\frac{iB}{\omega}+\bar{\rho}_f hC_h+\frac{i\kappa_f\omega h}{c_3^2}+\frac{J_5}{J_6}e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}\left[N_2-\frac{LN_1}{TC_{pg}}\left(1+\frac{T\bar{S}}{L}-\frac{J_4}{J_3}\right)\right]\right\}=0\ . \tag{25}$$

The various coefficients are now given by

$$J_1\equiv\frac{1}{c_3u^{1/2}}\left(\frac{c_3}{c_{03}}-\frac{c_{01}}{c_3}\right)-\frac{i\kappa_g\omega}{\rho_g C_{pg}^2Tu^{1/2}c_3}\left(\frac{c_3}{c_{02}}-\frac{c_{01}}{c_3}\right)\ ,\qquad N_1\equiv e^{i\pi/4}\left(\frac{\kappa_g\omega}{\rho_g C_{pg}c_3^2}\right)^{1/2}\left(\frac{c_3}{c_{02}}-\frac{c_{01}}{c_3}\right)\ ,$$

$$J_2\equiv e^{i\pi/4}\left(\frac{\kappa_g\omega}{\rho_g C_{pg}c_3^2}\right)^{1/2}\left(\frac{c_3}{c_{02}}-\frac{c_3}{c_{03}}\right)\ ,\qquad N_2\equiv e^{i\pi/4}\left(\frac{\kappa_g\omega}{\rho_g C_{pg}c_3^2}\right)^{1/2}\frac{c_3}{c_{02}}\ ,$$

$$J_3\equiv J_1-\frac{\lambda\omega J_7}{u^{1/2}f}\frac{\rho_f}{\rho_g}\ ,\qquad J_4\equiv J_8+\frac{\omega J_7}{u^{1/2}f}\left(-\frac{\rho_f}{\rho_g}\frac{T\bar{S}}{L}+1+\frac{T\bar{S}}{L}\right)\ ,$$

$$J_5\equiv J_6\left\{1+\left(1-\frac{9}{2}\frac{k_BT/m}{L}\right)\frac{i\omega\rho_g L}{16Af}\left[\frac{J_9}{J_4}\frac{\rho_f}{\rho_g}\frac{T\bar{S}}{L}\left(1-\frac{\rho_g}{\rho_f}\right)-\frac{J_9}{J_4}+1-\frac{\rho_g}{\rho_f}\right]-\frac{i\omega T}{16AL}\frac{J_9}{J_4}\left(\frac{iB}{\omega}+\bar{\rho}_f hC_h+\frac{i\kappa_f\omega h}{c_3^2}\right)\right\}$$
$$\times\left(1-\frac{9}{32}\frac{k_BT/m}{L}\frac{i\omega\rho_g LJ_6}{Af}\left[\frac{J_9}{J_4}\frac{\rho_f}{\rho_g}\frac{T\bar{S}}{L}\left(1-\frac{\rho_g}{\rho_f}\right)-\frac{J_9}{J_4}+1-\frac{\rho_g}{\rho_f}\right]+\frac{i\omega T}{16AL}e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}\right.$$

$$\times\left\{\left[N_2 - \frac{LN_1}{TC_{pg}}\left(1+\frac{T\bar{S}}{L}\right)\right]\frac{J_9}{J_4} + \frac{L}{TC_{pg}}\left[\left(1-\frac{\rho_g}{\rho_f}\right)N_1 - \frac{4C_{pg}N_2}{k_B/m}\right]\right\}\Bigg)^{-1},$$

$$J_6 \equiv 1 - \frac{2}{9}N_2\,\frac{i\omega m}{Ak_B}\,e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}, \qquad J_7 \equiv 1 - \frac{i\omega J_2}{AC_{pg}}\,e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2},$$

$$J_8 \equiv J_1\left(1+\frac{T\bar{S}}{L}\right) + \frac{\omega J_2}{\rho_g C_{pg} u^{1/2} L}\,e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}, \quad J_9 \equiv \left(1-\frac{\rho_g}{\rho_f}\right)\left(J_1 + \frac{\omega J_7}{u^{1/2} f}\right) + \frac{4\omega J_2}{P_g C_{pg} u^{1/2}}\,e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2},$$

$$u^{1/2} \equiv 1/c_{03}, \qquad \lambda \equiv 1 - \frac{hf\bar{\rho}_s}{c_3^2\rho_f} - \frac{\rho_g}{\rho_f}. \tag{26}$$

If $\kappa_f\omega/\rho_g C_{pg} c_3^2$ and therefore also $\eta_f\omega/\rho_g c_3^2$, are at most not much greater than 1, while $\kappa_f\omega/\rho_g C_{pg} c c_3 \ll 1$, the following approximations can be made:

$$J_6 \approx J_7 \approx 1, \quad J_4 \approx J_8 \approx J_1(1+T\bar{S}/L), \quad J_9 \approx J_1(1-\rho_g/\rho_f), \quad J_3 \approx J_1, \quad \lambda \ll 1,$$

$$J_5 - 1 \approx \frac{i\omega\rho_f L}{16Af}\left(1-\frac{\rho_g}{\rho_f}\right)\frac{T\bar{S}}{L}\left[1+\frac{T\bar{S}}{L} - \frac{9}{32}\,\frac{k_B T}{m}\,\frac{i\omega\rho_f}{Af}\,\frac{T\bar{S}}{L}\left(1-\frac{\rho_g}{\rho_f}\right)\right]^{-1}, \tag{27}$$

$$1+\frac{9}{2}\,\frac{k_B T/m}{L}(J_5-1) \approx \left[1 - \frac{9}{32}\,\frac{k_B T}{m}\,\frac{i\omega\rho_f}{Af}\,\frac{T\bar{S}}{L}\left(1-\frac{\rho_g}{\rho_f}\right)\left(1+\frac{T\bar{S}}{L}\right)^{-1}\right]^{-1} \equiv J_{10}.$$

Equation (27) also serves to define J_{10}. A good approximation for the dispersion equation (25) then becomes

$$\frac{hf\bar{\rho}_s}{c_3^2\rho_f}\left(1+\frac{T\bar{S}}{L}\right)^2 = \left\{1 + \left[1 - \frac{9}{32}\,\frac{k_B T}{m}\,\frac{i\omega\rho_f}{Af}\,\frac{T\bar{S}}{L}\left(1-\frac{\rho_g}{\rho_f}\right)\left(1+\frac{T\bar{S}}{L}\right)^{-1}\right]\frac{Tf}{\rho_f L^2}\left[\frac{iB}{\omega} + e^{i\pi/4}\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2}\right]\right\}$$
$$\times\left[1 - \frac{9}{32}\,\frac{k_B T}{m}\,\frac{i\omega\rho_f}{Af}\left(\frac{T\bar{S}}{L}\right)^2\left(1-\frac{\rho_g}{\rho_f}\right)\left(1+\frac{T\bar{S}}{L}\right)^{-2}\right]^{-1}. \tag{28}$$

In the case of very thin films this equation reduces to

$$\frac{hf\bar{\rho}_s}{c_3^2\rho_f}\left(1+\frac{T\bar{S}}{L}\right)^2 = 1 + \frac{Tf}{L^2\rho_f}\,e^{i\pi/4}\left\{\left(\frac{\rho_g C_{pg}\kappa_g}{\omega}\right)^{1/2} + \left(\frac{\rho_{sub}C_{psub}\kappa_{sub}}{\omega}\right)^{1/2} B_1\left[B_1 - i\omega e^{i\pi/4}\left(\frac{\rho_{sub}C_{psub}\kappa_{sub}}{\omega}\right)^{1/2}\right]^{-1}\right\}, \tag{29}$$

whereas in the case of very thick films it reduces to

$$\frac{hf\bar{\rho}_s}{c_3^2\rho_f}\left(1+\frac{T\bar{S}}{L}\right)^2 = \left[1 - \frac{9}{32}\,\frac{k_B T}{m}\,\frac{i\omega\rho_f}{Af}\left(\frac{T\bar{S}}{L}\right)^2 \times\left(1-\frac{\rho_g}{\rho_f}\right)\left(1+\frac{T\bar{S}}{L}\right)^{-2}\right]^{-1}. \tag{30}$$

The results of calculating the attenuation of third sound from each of these formulas (28)-(30) are plotted in Fig. 2. The results obtained using (28) are in full agreement with the results of the numerical solution of the third-sound equations described in Sec. II. In fact, to the accuracy shown in the graph they are identical: They never differ by more than 2%, and in fact are usually much closer than that, except near the minimum of the attenuation curve. The graphs show that, except for the region around the minimum, it suffices to use the simpler dispersion equations (29) and (30) to get the correct attenuation. These statements hold for the velocity of third sound as well.

In I we worked out the relative amplitudes of oscillating quantities for very thin films, where (29) is valid. We now write these amplitudes for the opposite extreme case where (30) is valid:

$$\frac{T'_g - T_{f1}}{T} = \frac{T_{f1}}{T}\,\frac{i\omega\rho_f T\bar{S}}{16Af}\left(1-\frac{\rho_g}{\rho_f}\right)\left(1+\frac{T\bar{S}}{L}\right)^{-1}J_{10}, \tag{31}$$

$$\frac{T_{g3}}{T} = -\frac{T_{f1}}{T}\,\frac{\rho_f}{\rho_g}\,\frac{\omega c_3\bar{S}}{fC_{pg}J_1}\left[\left(1+\frac{T\bar{S}}{L}\right)^{-1} - \frac{L}{T\bar{S}}\,\frac{\rho_g}{\rho_f}\right]J_{10}, \tag{32}$$

$$\frac{h_1}{h} = -\frac{T_{f1}}{T}\,\frac{L}{hf}\,J_{10}\left\{1 + \frac{\omega c_3}{fJ_1}\left[\frac{T\bar{S}}{L}\,\frac{\rho_f}{\rho_g}\left(1+\frac{T\bar{S}}{L}\right)^{-1} - 1\right]\right\}$$

$$\approx -\frac{T_{f1}}{T}\,\frac{L}{hf}\,J_{10}, \tag{33}$$

$$v = -\frac{T_{f1}}{T}\,\frac{L}{hf}\,J_{10}\left(1+\frac{T\bar{S}}{L}\right)^{-1}. \tag{34}$$

Since we are talking about the region where $\omega\rho_f T\bar{S}/16Af$ is not small compared to 1, we find that $T'_g - T_{f1}$ is not negligible compared to T_{f1}. This fact is borne out by the numerical work de-

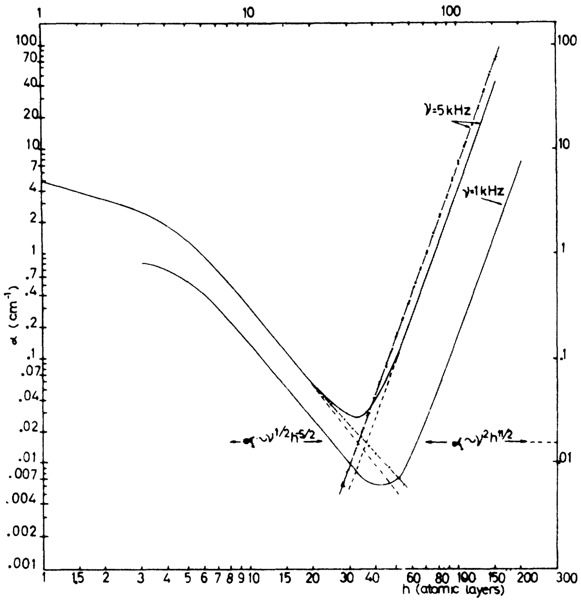

FIG. 2. Attenuation coefficient α vs film thickness h for $T = 1.3$ °K. The two solid lines are the results of a numerical solution of the third-sound equations at frequencies of $\nu = 1$ kHz and $\nu = 5$ kHz. For thick films α is proportional to ν^2 and $h^{11/2}$. For thin films there is a whole region where α is proportional to $\nu^{1/2}$ and also, approximately, to $h^{-5/2}$. The same solid lines also represent, to within the accuracy of the drawing, the results of Eq. (28) for α. The two dashed lines represent the results of the two approximate dispersion equations (29) and (30) for thin films and for thick films, respectively. The dot-dashed line represents what purported to be (see I) a better approximation than (29) for thin films. However, where it deviates appreciably from (29), it evidently does not suffice to give the correct behavior, so its usefulness is therefore curtailed. The dash-dot-dot-dash line represents the prediction of Atkins's theory (Ref. 9) for α. The calculated values of α depend on various experimental coefficients of the gas and the substrate. We have used the following values: $\rho_{sub} = 5$ g/cm^3; $C_{p\,sub} = 31.4 \times T^3$ erg/g deg; $\kappa_{sub} = 3.5 \times 10^3$ erg/cm deg sec; $C_{pg} = \frac{5}{2} k_B/m$; $\kappa_g = 286$ erg/cm deg sec; $\eta_g = 8.3\ (T/1.5)^{1/2}$ μP; $hf = 261\ (k_B/m)(1/h^3)$ erg/g (h in atomic layers) [see Ref. 2 and W. D. McCormick, D. L. Goodstein, and J. G. Dash, Phys. Rev. 168, 249 (1968)]; $1/B_1 = 1.9 \times 10^{-6}\ (1.51/T)^{2.4}$ erg/cm^2 sec deg (the Kapitza resistance) [see Gerald Pollack, Rev. Mod. Phys. 41, 48 (1969)]. We also assumed that $\bar{\rho}_s/\rho_f$ was equal to its value for bulk He^4 at the same temperature and at zero pressure. This should be remembered if one tries to apply these results to very thin films, where this is known to be untrue.

scribed in Sec. II. On the other hand, $T_{f_1} - T_g'$ is never simply equal to T_{f_1}, as Atkins in his original paper on third sound assumed[9] in order to enable him to separate the motion in the film from motion in the gas. T_{g_3}, however, does come out to be much smaller than T_{f_1}. v and h_1/h are nearly the same in this region, and are both much greater than T_{f_1}/T. All these conclusions are borne out by the numerical results of Sec. II, which are in quantitative agreement with the approximate results obtained here for thick films.

IV. APPROXIMATE EQUATIONS OF MOTION AND PHYSICAL INTERPRETATION OF RESULTS FOR THICK FILMS

The conclusions that we reached at the end of Sec. III seem to hint that it might be possible to get the results for thick films by neglecting certain parts of the exact equations (23) or (24) which are evidently not important in the final result. In order to do this we note the following.

(a) Looking at earlier versions of Eqs. (20) and (21),[10] namely,

$$J_M\left(1 - \frac{\rho_g}{\rho_f}\right) = \frac{4mA}{k_B T}\left[\mu_f - \mu_g + \left(s_g - \frac{k_B}{2m}\right)(T_f - T_g)\right], \tag{35}$$

$$J_{Qg} = -A\left[\mu_f - \mu_g + \left(s_g - \frac{9}{2}\frac{k_B}{m}\right)(T_f - T_g)\right], \tag{36}$$

where μ_f and μ_g are the chemical potentials of the film and gas per unit mass and s_g is the entropy per unit mass of the gas, we can easily convince ourselves that a large value of $T_f - T_g$, if not canceled by $\mu_f - \mu_g$, would lead to a prohibitively large thermal flux J_{Qg}. To maintain such a large thermal flux we would need a value of $\partial T_g/\partial y$ so large as to cause T_g to change by something like its entire amplitude of oscillation over one mean free path in the gas! In practice, the role of Eq. (36) is to ensure that this cancellation takes place. We can therefore replace it by

$$\mu_f - \mu_g = -\left(s_g - \frac{9}{2}\frac{k_B}{m}\right)(T_f - T_g).$$

This we immediately substitute into (35) to get

$$J_M = \frac{16A}{T}\left(1 - \frac{\rho_g}{\rho_f}\right)^{-1}(T_f - T_g). \tag{37}$$

We note that this expression is different both in form and in derivation from the expression found by Atkins *et al.*[11] and used by him in his theory of third sound.[9] Nevertheless, quantitatively the two expressions lead to results which do not differ very much, i.e., only by a factor of order 1.

(b) Heat conduction is unimportant as a source of attentuation in the regime of thick films. The main source is the evaporation and condensation of helium atoms on the film, as shown by the appearance of the coefficient A in (30). We consequently ignore the heat-conduction terms as well as the heat capacity of the film in the equations of motion. Lastly, we ignore the contribution T_{g_3} to the temperature oscillations in the gas as compared to T_{g_2}.

These approximations allow us to write the equations of motion in the following form:

$$\rho_f \dot{h} + h\bar{\rho}_s \frac{\partial \bar{v}_s}{\partial x} + J_M = 0, \tag{38}$$

$$\rho_f \dot{h} + h\bar{\rho}_s \left(1 + \frac{T\bar{S}}{L}\right)\frac{\partial \bar{v}_s}{\partial x} = 0$$

or (39)

$$-h\bar{\rho}_s T\bar{S}\ \frac{\partial \bar{v}_s}{\partial x} + LJ_M = 0,$$

$$\dot{v}_s - \bar{S}\,\frac{\partial T_f}{\partial x} + f\,\frac{\partial h}{\partial x} = 0, \tag{40}$$

$$J_M = \frac{16A}{T}\left(1 - \frac{\rho_g}{\rho_f}\right)^{-1}(T_{f_1} - T_g'), \tag{41}$$

$$J_{Qg} = 0 = -A\left(\frac{L}{T}\,T_{f_1} - \frac{9}{2}\frac{k_B}{m}\,(T_{f_1} - T_g') + fh_1\right). \tag{42}$$

The second form of Eq. (39) is especially useful for comparison with Atkins's equations.[9] These equations can easily be solved and shown to yield exactly the results (30), (31), (33), and (34) which were obtained in Sec. III as an approximation.

These results verify the assumptions we made above as well as the physical picture that they imply: In the regime of thick films (not too thick though, as we pointed out near the end of Sec. II), the attenuation of third sound is governed mainly by the processes of evaporation and condensation of helium atoms between the film and the gas. Made as an assumption, this statement formed the starting point in Atkins's original theory.[9] His other assumption—that T_g does not oscillate and consequently that one does not have to consider equations of motion for the gas—turns out to be wrong in all the regions we investigated.

Despite this incorrect assumption, and the different way of calculating the rate of evaporation, Atkins obtained results not too different from ours for the thick films.[9] This can be understood if we use (31) to rewrite J_M of (37) in terms of T_{f_1} alone:

$$J_M = -\frac{T_f}{T}\ \frac{i\omega\rho_f T\bar{S}}{f}\left(1 + \frac{T\bar{S}}{L}\right)^{-1} J_{10}\,.$$

When J_{10} is much larger than 1, as it will be for very thick films or high frequencies, this can be written as

$$J_M \approx \frac{T_{f_1}}{T} \times \frac{16}{9}\ \frac{L}{k_B T/m}\ \rho_g\left(\frac{k_B T}{2\pi m}\right)^{1/2}.$$

By the Clausius-Clapeyron equation this can be seen to differ from Atkins's expression[9] only by the factor $\frac{16}{9}$. This is the main cause of the difference between our results and his.

V. SUMMARY AND DISCUSSION

We have worked out the solution of the third-sound equations, which were derived in I, for ranges of film thickness and sound frequency far beyond those discussed in I. This was done in three different ways, each of which contributed its share to the understanding of the physical properties of third sound. We obtained an analytical expression that is a very good approximation of the dispersion equation over the entire region where the equations are valid. This expression simplifies at the two extremes of very thin films and low frequencies, and thick films and high frequencies, leading in the former case to the expression that was obtained in I. The validity and quality of these approximations was checked by an exact numerical solution of the equations.

From our investigations, there emerges the result that the attenuation of third sound in thick films is governed by the evaporation and condensation of helium, as originally assumed by Atkins.[9] His other assumptions are not verified, however, and as a consequence, our result for the attenuation in thick films differs slightly from his (for thin films it is completely different; see I).

The only published experimental data on attentuation in thick films were gathered by Atkins in some early experiments.[4] His measured values were larger then his theoretical predictions[9] by 2–3 orders of magnitude. Since our predictions only differ from his by a factor of order 1, the discrepancy remains. In view of the rather poor reproducibility in these experiments,[4] we believe that a new set of measurements of α is required in order to decide whether the discrepancy is real. If so, mechanisms contributing to the attenuation will have to be looked for outside the framework of the equations that we have been using.

Finally, we wish to remark that, just as measurements of the velocity and attenuation of third sound in thin films could be devised so as to supply us with information about the various physical phenomena on which they depend, such as the Kapitza resistance, so we could presumably devise experiments on thick films to teach us about the evaporation and condensation of helium from such films.

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[6]Note that Eq. (31) of I is wrong and that the correct form for v_{gy} is Eq. (12) of this paper. The approximate expression for v_{gy} [Eq. (36) of I] is, however, correct, as are (we hope) all the other equations in that paper.

[7]For the precise definition of these coefficients, see, e.g., I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965), Chap. 9.

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Turbulence Theory for the Current Carriers in Solids and a Theory of $1/f$ Noise*

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The theory of homogeneous isotropic turbulence of current carriers in metals and semiconductors is developed in terms of the magnetic field fluctuations. A quasinormality hypothesis is used to truncate the chain of equations for the correlation functions. A k^{-3} spectrum in wave numbers and a $1/f$ spectrum in frequencies are obtained. The $1/f$ spectrum is also obtained for fluctuations of current and voltage in a ring-shaped circuit embedded in the turbulent field. These results are used to construct a universal theory of $1/f$ noise in electronic circuits.

I. INSTABILITIES AND FLUCTUATIONS WHICH GENERATE TURBULENCE: BASIC CONCEPTS

The instability which might be responsible for turbulence in the plasma of current carriers in electronic circuits has been described[1-3] as a "magnetic barrier instability" (MBI). This is a new type of current instability which has been shown to occur only in the presence of a transversal potential barrier such as a semiconductor junction,[1,3] a bad electric contact, or the surface of the cathode[2] in vacuum tubes and gas discharges. The nature of this instability is related to that of the pinch instability inasmuch as in both cases the self-magnetic-field is important. The difference is that the MBI is a linear effect which is proportional to both the self-magnetic-field and the internal electric equilibrium field of the barrier. The latter is usually very strong. The critical current of the MBI is very low compared with the critical pinch current, and may be taken to be actually zero for systems in which transversal inhomogeneities are properly taken into account.

The MBI should occur only in systems with current carriers of both signs present in the barrier region. This is obvious from the theory of the MBI,[1-3] in which the possibility of carrier accumulation without appreciable space charge is essential. A strong argument for the MBI as a source of the actually observed $1/f$ noise is its universality. The same MBI has been shown to occur in the Debye sheet at the surface of the cathode in a gas discharge or vacuum tube,[2] as well as in a metal-semiconductor contact, in a semiconductor junction, in a bad contact involving an oxide layer, and in the contact between two carbon grains or between two grains in an evaporated thin layer.

Other possible sources of turbulence are the generation-recombination and shot noise fluctuations. Although these fluctuations do not have a