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# Note on Fluctuations*

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We treat three standard problems of the nonrelativistic quantum theory by a Brownian-motion method devised by Sokolov and Tumanov. In all three cases, the commutation (Poisson bracket) relations for the particle's dynamical variables are deduced from the corresponding relations for the components of the fluctuating electromagnetic field. In addition, the particle ground-state energy is determined by the zero-temperature vacuum fluctuations of the field. The ground-state energy includes the lowest-order radiative correction, although the calculation makes no use of traditional perturbation theory or mass renormalization. If the vacuum radiation temperature is high, the energy of the particle goes to its equipartition value.

## I. INTRODUCTION

Every physical system (object) is exposed to a variety of fluctuating forces and consequently will absorb energy from its surroundings (reservoir). If the system is to achieve even temporary stability, it must lose energy, and, in general, this loss will manifest itself in some form of friction, viscosity, or radiation damping. The object's properties will be determined by the balance between its natural motion and the competing forces of fluctuation and dissipation. Of course, these observations represent the usual view of a macroscopic object coupled to a thermal reservoir. However, in light of the fact that elementary particles, even at zero temperature, are permanently immersed in the radiation baths of their associated fields, it seems reasonable to ask if one could not fit the description of both macroscopic and microscopic objects into a more unified scheme.

Our views are linked to the traditional analysis of fluctuation-dissipation phenomena in statistical mechanics. There one examines a nonrelativistic system possessing a continuum of energy states and subject to fluctuations not necessarily of thermodynamic origin. If the fluctuation forces are treated as small perturbations of the original motion, simple thermodynamic assumptions incorporated into ordinary perturbation theory give rise to an explicit expression for the force of friction in terms of the fluctuations.[1–4] The system may then be described by the theory of Brownian motion, i.e., by a Langevin equation with frictional loss consistent with that derived from perturbation theory.

The Langevin equation can be a classical equation or an operator, i.e., Heisenberg, equation of the quantum theory. The latter must certainly be the case for fluctuating forces of known quantum-mechanical origin.

If the fluctuating forces are uncorrelated in space and time, the system's variables will be uncorrelated in similar fashion. On the other hand, if the fluctuations obey strict dynamical laws, and therefore are correlated, the system's variables will exhibit a corresponding connectedness. Thus, if the correlated fluctuations are given in terms of classical (quantum) variables, we might anticipate that Poisson bracket (commutation) relations for the system's variables would follow from those of the fluctuating field.

Sokolov and Tumanov[5] have demonstrated the validity of this contention for a single problem, the radiation-damped harmonic oscillator in electrodynamics. They have also shown that with an appropriate definition of the system's energy, and without the need for mass renormalization, the quantized vacuum fluctuations of the electromagnetic field determine the ground-state energy of the oscillator, including the lowest-order radiative corrections. In an independent analysis of the same problem, Bourret[6] extended the example of Sokolov and Tumanov to radiation baths at finite temperatures. In the high-temperature limit of this theory, the energy of the system approaches its usual classical value at thermal equilibrium. Furthermore, the detailed calculation of the energy coincides with the corresponding average values derived from the fluctuation-dissipation theory of statistical mechanics.[3]

This present analysis and the original work of Sokolov and Tumanov are restricted to electromagnetic fluctuations. However, the approach is clearly more general than might be inferred from this restriction. The problems treated in this paper are linear, or are linearized, and they include the harmonic oscillator, a charge in a uniform and constant magnetic field, and a neutral spinning particle in the same field. In each case, we show that the classical (quantized) fluctuations determine the Poisson brackets (commutators) for the system's variables. We also calculate the particle's ground-state energy for the first two of these three problems.

Our Eq. (2.20) for the oscillator is not the traditional one given by Sokolov and Tumanov [see our Eq. (2.11)]. In their theory, the lowest-order radiative correction to the ground-state energy of the oscillator differs from the corresponding term calculated in quantum field theory. On the other hand, their analysis agrees with Kramers's Lamb-shift calculation[7] for the oscillator as given in his classic paper on mass renormalization. For Sokolov and Tumanov and for Kramers, the effective cutoff frequency in the logarithmic correction to the ground-state energy is associated with the classical electron radius and not with the correct value, the electron's Compton wavelength. We have modified the usual theory to include spin-orbit coupling [our Eq. (2.20)], and this simple relativistic extension leads to the correct logarithmic dependence as well as other relativistic corrections to the oscillator's ground-state energy.

A major problem in this paper, as well as in the work of Sokolov and Tumanov, is the arbitrariness in the definition of the system's energy. In actuality, the energies of the system and the field (reservoir) are inextricably interwoven, and the difficulty in the theory is to isolate the finite energy of the system from the total energy. We, and Sokolov and Tumanov, have accomplished this by intuitive guesswork.

On the other hand, the views expressed here and in the work of Sokolov and Tumanov are implicitly contained in Kramers's unconventional treatment of mass renormalization in quantum electrodynamics.[7] His field-theoretical approach, although suffering from the usual divergence malady, is more general and carries greater authority than our calculations. Yet, for the problems we have treated, similar if not identical conclusions may be drawn from Kramers's work. Thus our calculations should be viewed as preliminary, pending a more comprehensive investigation as suggested by Kramers's theory.

Finally, when the system and reservoir are described by more conventional stochastic variables, our analysis reduces to the standard theory of Brownian motion.

## II. HARMONIC OSCILLATOR

Sokolov and Tumanov[5] have treated the problem of the harmonic oscillator in interaction with the vacuum electromagnetic field. They assume as Langevin equation the extended Lorentz force law,

$$m\ddot{\vec{r}} = -mk_0^2\vec{r} + e(\vec{E}^h + \vec{E}^s) . \tag{2.1}$$

Here

$$\vec{E}^s = -\dot{\vec{A}}^s = \frac{2}{3}e\frac{d^3\vec{r}}{dt^3} , \tag{2.2}$$

$$\vec{E}^h = -\dot{\vec{A}}^h = i(2\pi)^{-3/2}\int kZ(k)[\vec{a}(k)\,e^{-ikt} - \text{c.c.}]d^3k, \tag{2.3}$$

with $Z(k)^2 = 2\pi\hbar/k$. The force of radiation damping is given by $e\vec{E}^s$, while $\vec{E}^h$ is the field of the vacuum oscillations. $\vec{a}^\dagger$ and $\vec{a}$ are either classical variables or the creation and destruction operators of the vacuum field.

If one ignores the damped and runaway solutions of the homogeneous equation, the solution for $\vec{r}$ is

$$\vec{r} = ie(2\pi)^{-3/2}m^{-1}\int kZ(k)[W(k)\vec{a}(k)\,e^{-ikt} - \text{c.c.}]d^3k , \tag{2.4}$$

with

$$W(k) = (k_0^2 - k^2 - i\gamma k^3)^{-1}, \quad \gamma = 2e^2/3m. \tag{2.5}$$

In addition, from the customary definition of the momentum,

$$\vec{p} = m\dot{\vec{r}} + e(\vec{A}^h + \vec{A}^s), \tag{2.6}$$

they find

$$\vec{p} = ek_0^2(2\pi)^{-3/2}\int Z(k)[W(k)\vec{a}(k)\,e^{-ikt} + \text{c.c.}]d^3k . \tag{2.7}$$

Furthermore, given the transverse-field commutation (Poisson bracket) relations,

$$(i\hbar)^{-1}(a_i(k), a_j^\dagger(k')) = (i\hbar)^{-1}(\delta_{ij} - \hat{k}_i\hat{k}_j')\delta(\vec{k} - \vec{k}') , \quad \hat{\vec{k}} = \vec{k}/|k| \tag{2.8}$$

the corresponding commutation (Poisson bracket) relations for the system's variables are then deduced from (2.4), (2.7), and (2.8),

$$(i\hbar)^{-1}(r_i , p_j) = \delta_{ij} + O_{ij}(\gamma k_0). \tag{2.9}$$

The radiative correction $O_{ij}$ to the usual commutator (Poisson bracket) values in (2.9) should be anticipated, since the Langevin equation (2.1) implies a coupling between particle and field variables not present in the traditional classical or quantum theory where radiation damping is ignored.

Sokolov and Tumanov define the quantized ground-state energy of the oscillator to be

$$E_0 = \langle 0|p^2/2m + \tfrac{1}{2}mk_0^2r^2|0\rangle . \tag{2.10}$$

The particle energy is a function of the ground-state vacuum field oscillations.

To generalize their argument, take the ground-state particle energy as the thermodynamic average over the vacuum fluctuations,

$$E_0 = \langle p^2/2m + \tfrac{1}{2}mk_0^2r^2\rangle_{av} . \tag{2.11}$$

We substitute (2.4) and (2.7) in (2.11) and assume random phases so that only diagonal terms appear in $E_0$. We also choose the usual thermodynamic average for a Bose-Einstein oscillator,

$$\langle \vec{a}^\dagger(k)\cdot\vec{a}(k)+\vec{a}(k)\cdot\vec{a}^\dagger(k)\rangle_{av}=2\coth(\hbar k/2\alpha T)\ , \tag{2.12}$$

with $\alpha$ the Boltzmann constant. The factor 2 on the right-hand side arises from the two degrees of freedom of the vacuum field. Inserting (2.12) in (2.11), we find

$$E_0=(3k_0^2\hbar\gamma/2\pi)\int_0^\infty |W(k)|^2 k(k^2+k_0^2)\,J(k)\,dk, \tag{2.13}$$

with $J(k)$ the Bose-Einstein distribution function,

$$J(k)=\coth(\hbar k/2\alpha T)\ . \tag{2.14}$$

For zero temperature, the integration of (2.13) yields

$$E_0=3\hbar k_0/2[1-(\gamma k_0/\pi)\ln\gamma k_0],\quad \gamma=2e^2/3m\ . \tag{2.15}$$

This last result is in agreement with Kramers,[7] but disagrees with the calculations of field theory. There, the Compton wavelength replaces the classical electron radius $\gamma$ in the argument of the log term.

We now go over to the high-temperature limit of this theory. For each term in the energy the result is the same: Quadratic terms contribute $\frac{1}{2}\alpha T$ with $\alpha$ the Boltzmann constant. Thus, we find for any component of $\vec{p}$,

$$\langle p^2/2m\rangle_{av}=\tfrac{1}{2}\,\alpha T$$

when we use (2.12) and let $J(k)=2\alpha T/\hbar k$. Our results are in detailed agreement with the fluctuation-dissipation theories of statistical mechanics.[3] In the remainder of this article we shall ignore the finite temperature properties of the reservoir, since the results so closely parallel those recorded in the literature.[3]

We return to the zero-temperature reservoir and note that we may secure better agreement with the results of quantum electrodynamics if we modify the Langevin equation (2.1). We assume that our equations of motion are to be derived from the following Hamiltonian:

$$H=(\vec{p}-e\vec{A})^2/2m+\tfrac{1}{2}mk_0^2\vec{r}^{\,2}-(e/2m^2)\vec{s}\cdot(\vec{E}\times\vec{p}). \tag{2.16}$$

Here $\vec{s}$ is a constant spin vector with components $(0, 0, h)$ and $\vec{E}$ is the electric field acting on the charge. The Hamiltonian (2.16) is the nonrelativistic approximation to the Dirac equation, where the spin dynamics is neglected together with higher-order relativistic terms.[8]

The Heisenberg equations of motion deduced from (2.16) are

$$\vec{p}=m\vec{v}+e\vec{A}+(e/2m)\,\vec{s}\times\vec{E}\ , \tag{2.17}$$

$$\dot{\vec{v}}=-k_0^2\vec{r}+(e/m)\,\vec{E}-(e/2m^2)\,\vec{s}\times\dot{\vec{E}}\ . \tag{2.18}$$

If, in the first approximation, we have

$$(e/m)\,\vec{E}=\dot{\vec{v}}+k_0^2\vec{r}\,, \tag{2.19}$$

Eq. (2.18) becomes

$$\dot{\vec{v}}=-k_0^2\vec{r}+(e/m)\,\vec{E}-(2m)^{-1}\vec{s}\times\ddot{\vec{v}}-(k_0^2/2m)\,\vec{s}\times\vec{v}\ . \tag{2.20}$$

The electric field $\vec{E}$ in (2.20), $\vec{E}=\vec{E}^h+\vec{E}^s$, is given by Eqs. (2.2) and (2.3).

To describe the motion in the $xy$ plane, we introduce the rotating operator (variable) $\xi=x+iy$. The solution to (2.20) is then

$$\xi=-i(e/m)(2\pi)^{-3/2}$$

$$\times\int kZ[(a_x+ia_y)\,U(k)-i(a_x^\dagger+ia_y^\dagger)\,V(k)]d^3k, \tag{2.21}$$

with

$$Z^2=2\pi\hbar/k,\quad U(k)=[k_0^2-k^2-(\hbar k^3+k_0^2k)/2m-i\gamma k^3]\,,$$

$$V(k)=-[k_0^2+k^2-(\hbar k^3-k_0^2k)/2m-i\gamma k^3]\,.$$

The solution for $p_-=p_x-ip_y$ is

$$p_-=ek_0^2(2\pi)^{-3/2}$$

$$\times\int Z[(a_x-ia_y)V^*(k)-(a_x^\dagger-ia_y^\dagger)\,U^*(k)]d^3k\ . \tag{2.22}$$

With (2.21) and (2.22), we find the commutator (Poisson bracket) relations,

$$(i\hbar)^{-1}(\xi_+,p_-)=2+\Delta\ , \tag{2.23}$$

where $\Delta$ is a small radiative correction similar to $O_{ij}$ in (2.9). The commutator properties of the vacuum electric field determine the commutator properties of the particle variables. If we define the ground-state energy of the oscillator to be the same as in (2.10),[9]

$$E_0=(2m)^{-1}[\langle 0|\,(p_x^2+p_y^2)+m^2k_0^2\,(x^2+y^2)\,|0\rangle\,], \tag{2.24}$$

we find for $E_0$,

$$E_0=\hbar k_0\{1+(\gamma k_0/\pi)\,[\ln(m/\hbar k_0)+0.38\,]\}+(\hbar^2/4m)k_0^2\ . \tag{2.25}$$

The first term in the energy is the usual ground-state energy for the two-dimensional oscillator; the second term is the lowest-order Lamb-shift correction; and the well-known Darwin term is last. The result (2.25) is totally unexpected in view of the definition of $E_0$ [Eq. (2.24)]. The "success" of (2.25), and the arbitrariness in the definition of $E_0$, should be viewed in the light of our comments in the final two paragraphs of the Introduction.

## III. CHARGE IN MAGNETIC FIELD

A charged particle moves in a constant uniform magnetic field. The particle interacts with its own radiation field, so that the Langevin equation reads

$$\dot{\vec{v}} - \vec{v}\times\vec{k}_0 - \gamma\ddot{\vec{v}} = (e/m)\vec{E}^h, \quad (3.1)$$

with $\vec{k}_0 = e\vec{B}/m$. For $\vec{B} = (0, 0, B)$, we have the equations of motion in the $xy$ plane,

$$\dot{v}_x - \gamma\ddot{v}_x - k_0 v_y = (e/m)E_x^h, \quad (3.2a)$$

$$\dot{v}_y - \gamma\ddot{v}_y + k_0 v_x = (e/m)E_y^h. \quad (3.2b)$$

The solution for $\xi = x + iy$ is

$$\xi = -i(e/m)(2\pi)^{-3/2}\int Z[(a_x + ia_y)U - (a_x^\dagger + ia_y^\dagger)V]d^3k\ , \quad (3.3)$$

with

$$U = [k - k_0 + i\gamma k^2]^{-1} \text{ and } V = (k + k_0 - i\gamma k^2)^{-1}\ . \quad (3.4)$$

As usual, the momentum $\vec{p}$ is defined as $\vec{p} = m\vec{v} + e\vec{A}$, where $\vec{A}$ is the vector potential for the total field. For example,

$$p_x = mv_x - mk_0 y - m\gamma\dot{v}_x + eA_x^h\ . \quad (3.5)$$

We find for $p_- = p_x - ip_y$,

$$p_- = -ek_0(2\pi)^{-3/2}\int Z[(a_x - ia_y)V^* - (a_x^\dagger - ia_y^\dagger)U^*]d^3k\ . \quad (3.6)$$

As expected, the derived commutator (Poisson bracket) is

$$(i\hbar)^{-1}(\xi, p_-) = 2 + \Delta\ . \quad (3.7)$$

For the ground-state energy defined as

$$E_0 = (1/4m)\langle 0|p_+^2 + p_-^2|0\rangle\ , \quad (3.8)$$

we have

$$E_0 = \tfrac{1}{2}\hbar k_0 + \text{radiative corrections.} \quad (3.9)$$

The radiative corrections in (3.9) are similar to those found for the oscillator in (2.15).

## IV. MAGNETIC DIPOLE

In classical electrodynamics, a magnetic dipole in a magnetic field satisfies the following equation of motion[10]:

$$\dot{\vec{s}} - g(\vec{s}\times\vec{B}) - \tfrac{2}{3}g^2(\vec{s}\times\dddot{\vec{s}}\ ) = g\ \vec{s}\times[\vec{B}_0\ e^{-i\omega t} + \text{c. c.}]. \quad (4.1)$$

In (4.1), $\vec{s}$ is the spin vector, $\vec{B}$ the constant magnetic field, and $\vec{B}_0$ is the constant vector amplitude associated with the vacuum magnetic field fluctuations.

We linearize (4.1) as follows: We assume that the constant magnetic field $\vec{B}$ lies along the $z$ axis with $B_z \equiv B$ and that the large component of the spin vector $\vec{s}$ is $s_z$. In our approximation we shall treat $s_z$ as a constant. We introduce the small fluctuating components of the spin $\xi = s_x + is_y$, and find that $\xi$ satisfies the equation

$$\dot{\xi} + igB\xi - i\gamma s_z\dddot{\xi} = igs_z(H\ e^{-ikt} + H_+^\dagger e^{ikt}), \quad (4.2)$$

where $H = B_{0x} + iB_{0y}$, and $H_+^\dagger = B_{0x}^\dagger + iB_{0y}^\dagger$. The solution for (4.2) is

$$\xi = gs_z\ [He^{-ikt}\ U(k) + H_+^\dagger\ e^{ikt}\ V(k)], \quad (4.3)$$

with

$$U(k) = (k_0 + k - i\gamma k^3 s_z)^{-1},$$

$$V(k) = (k_0 - k + i\gamma k^3 s_z)^{-1}, \quad \text{and } k_0 = gB.$$

If we make use of the commutation (Poisson bracket) relations

$$(i\hbar)^{-1}(B_{0x}, B_{0x}^\dagger) = (i\hbar)^{-1}(2\hbar/3\pi)\int_0^\infty k^3\ dk, \quad (4.4)$$

we find that the commutation (Poisson bracket) relations for $\xi$ are

$$(2i\hbar)^{-1}(\xi^\dagger, \xi) = (i\hbar)^{-1}(s_x, s_y) = s_z + \Delta, \quad (4.5)$$

with $\Delta$ a small radiative correction to the usual angular momentum relations. Once again we find that the commutation (Poisson bracket) relations for the fluctuations determine the corresponding relations for the particle's dynamical variables.

## V. COMMENT

The derived commutation (Poisson bracket) relations and ground-state energies are critically dependent on the form of the damping term in Langevin equation, e.g., Eq. (2.2). The damping terms arise in the classical theory from the laws of conservation of energy and momentum for both particle and field. The fact that the Poisson bracket relations for the particle are determined by the bracket relations for the vacuum field fluctuations is a sign that particle and field theories are even more intimately linked than previously believed. The same link seems to hold in the quantum theory where the damping term arises from the perturbation theory.

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