Permutation-Group-Theoretical Treatment of n-Body Exchange Forces in Atomic and Nuclear Spectra

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An energy sum rule for fermion levels split by n-body exchange forces is derived and expressed in terms of Young tableau "hook lengths." The Franzini-Radicati nuclear mass formula is rederived and generalized to include the effect of n-body operators. The form of a mass splitting by a three-body term is computed in detail.

I. INTRODUCTION

One procedure for dealing with configuration-interaction problems in complex spectra involves construction of an effective Hamiltonian operator on the basis of a single configuration. The perturbation expansion of this operator can be done in terms of Feynman or Goldstone graph theory.[1,2] The terms in the effective Hamiltonian represented by n-line graphs are called n-body operators. In this way it is thought that observed energy-level spacings can be reproduced by some combination on n-body operator spectra obtained wholly within the subspace of the configuration being studied. It is hoped that n-body terms will become smaller for larger n, with $n=3$ and possibly $n=4$ terms giving small corrections to the usual one- and two-body operator spectrum.[3]

In this paper the combinatorial aspects of the n-body problem involving a spin-independent Hamiltonian are solved using group theory of Young tableau. A simple method is given that may allow one to estimate the magnitude of some n-body terms in certain experimental situations.

II. EXCHANGE FORCES

The descriptions of the Pauli principle and so-called "exchange forces" involving a two-body interaction $g(r_1r_2)$ such as Coulomb repulsion, are well known.[4] For example, in an $(ns)(n's)$ configuration, such as that of the first excited electronic states in helium, there are the following energy sum rules for levels of different symmetry:

$$e^{[1,1]}=I-J=e(^3S) \quad \text{(triplet states)},$$

$$e^{[2]}=I+J=e(^1S) \quad \text{(singlet state)},$$

where I includes the *direct integral* of $g(r_1r_2)$ and other integrals of one-body operators, while J is the *exchange integral* of $g(r_1r_2)$. The notation [2] ([1, 1]) indicates that the two-particle spatial wave function is symmetric (antisymmetric) to an interchange of the two particles. The numbers $[\mu_1, \mu_2, \dots]$ indicate the number of boxes in rows of *Young tableau*[5] (Fig. 1) which designate general types of parasymmetry of an n-particle wave function. These tableaux are generally defined in connection with the permutation-group (S_n) irreducible representation (IR), which they label. However, their use as labels for parasymmetry of unitary-group [$SU(m)$] IR is probably more widely known.[6] For example, n particles each with half-integral spin [$SU(2)$] form various states of definite total spin, which are labeled by the tableaux as shown in Fig. 2.

It is now shown how one may derive general sum rules like the above for a configuration involving an arbitrary number of particles interacting through a general n-body interaction. A three-particle configuration will be used to display the preliminary derivation.

III. $(ns)(n's)(n''s)$ CONFIGURATION

Consider a three-fermion configuration with each fermion in an s state of different principal quantum number. The set of all spatial wave functions for this configuration is a six-dimensional space spanned by the permutations of

$$\psi_n(r_1)\psi_{n'}(r_2)\psi_{n''}(r_3)\equiv\psi_1\psi_2\psi_3$$

shown below:

$$(\psi_1\psi_2\psi_3,\ \psi_3\psi_1\psi_2,\ \psi_2\psi_3\psi_1,\ \psi_2\psi_1\psi_3,\ \psi_3\psi_2\psi_1,\ \psi_1\psi_3\psi_2)\ .$$

The 6×6 representation of permutation group S_3 for which the above is a basis can be reduced to two pairs of type [2, 1] IR bases:

$$\psi_1^{[2,1]}=(1/2\sqrt{3})(2\psi_1\psi_2\psi_3-\psi_3\psi_1\psi_2-\psi_2\psi_3\psi_1+2\psi_2\psi_1\psi_3-\psi_3\psi_2\psi_1-\psi_1\psi_3\psi_2),$$

$$\psi_2^{[2,1]}=\tfrac{1}{2}(\psi_3\psi_1\psi_2-\psi_2\psi_3\psi_1+\psi_3\psi_2\psi_1-\psi_1\psi_3\psi_2),$$

$$\psi_1^{[2,1]}=\tfrac{1}{2}(\psi_3\psi_1\psi_2-\psi_2\psi_3\psi_1-\psi_3\psi_2\psi_1+\psi_1\psi_3\psi_2),$$

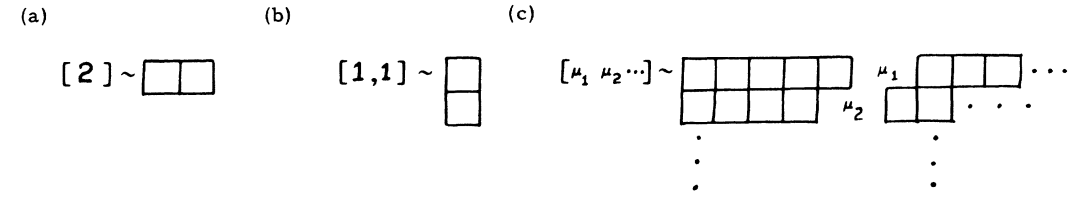


FIG. 1. Young tableaux designating (a) symmetric two-particle state, (b) antisymmetric two-particle state, and (c) general n-particle parasymmetric state.

$$\psi_2^{[2,1]'} = (1/2\sqrt{3})(-2\psi_1\psi_2\psi_3 + \psi_3\psi_1\psi_2 + \psi_2\psi_3\psi_1 + 2\psi_2\psi_1\psi_3 - \psi_3\psi_2\psi_1 - \psi_1\psi_3\psi_2),$$

and one each of type [1, 1, 1] and type [3] IR base:

$$\psi^{[1,1,1]} = (1/\sqrt{6})(\psi_1\psi_2\psi_3 + \psi_3\psi_1\psi_2 + \psi_2\psi_3\psi_1 - \psi_2\psi_1\psi_3 - \psi_3\psi_2\psi_1 - \psi_1\psi_3\psi_2),$$
$$\psi^{[3]} = (1/\sqrt{6})(\psi_1\psi_2\psi_3 + \psi_3\psi_1\psi_2 + \psi_2\psi_3\psi_1 + \psi_2\psi_1\psi_3 + \psi_3\psi_2\psi_1 + \psi_1\psi_3\psi_2). \quad (3.1b)$$

A desired completely antisymmetric atomic (nuclear) wave function is constructed from products of each of the above spatial bases $\psi^{[\mu]}$[(3. 1)] with spin (spin-isospin) bases $\Theta^{[\tilde{\mu}]}$ of opposite or *conjugate* symmetry. For example $\psi^{[3]}\Theta^{[1,1,1]}$, $\psi^{[1,1,1]}\Theta^{[3]}$, and

$$\psi_1^{[2,1]}\Theta_2^{[2,1]} + \psi_2^{[2,1]}\Theta_1^{[2,1]}$$

each satisfy the Pauli antisymmetry requirement. Each of these belong to different degenerate energy levels of definite total spin (spin-isospin) for a Hamiltonian $\mathcal{H}$ that does not involve spin or isospin, as indicated in Fig. 3. To compute a formula for these different energies, imagine the matrix representation $S(\mathcal{H})$ of the Hamiltonian $\mathcal{H}$ in the symmetrized spatial basis (3. 1). In this basis the matrix representing the permutation operators has

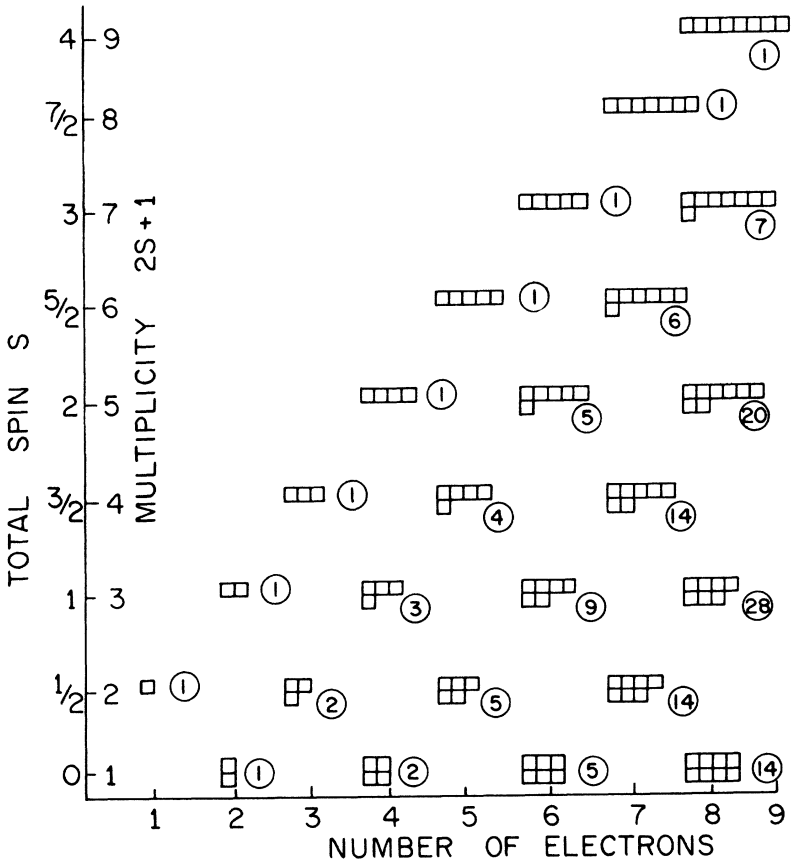


FIG. 2. Young tableaux associated with products of one to eight single-electron spinors. They represent functions that have definite total spin. The circled numbers are dimensions of respective S_m IR (see Appendix A).

the form shown below:

$$S(p)=\begin{array}{|cc|cc|cc|c|}\hline \multicolumn{2}{|c|}{\mathfrak{D}^{[2,1]}(p)} & \multicolumn{2}{c|}{0\ \ 0} & \multicolumn{2}{c|}{0} & 0\\ \multicolumn{2}{|c|}{} & \multicolumn{2}{c|}{0\ \ 0} & \multicolumn{2}{c|}{0} & 0\\ \hline \multicolumn{2}{|c|}{0\ \ 0} & \multicolumn{2}{c|}{\mathfrak{D}^{[2,1]}(p)} & \multicolumn{2}{c|}{0} & 0\\ \multicolumn{2}{|c|}{0\ \ 0} & \multicolumn{2}{c|}{} & \multicolumn{2}{c|}{0} & 0\\ \hline \multicolumn{2}{|c|}{0\ \ 0} & \multicolumn{2}{c|}{0\ \ 0} & \multicolumn{2}{c|}{\mathfrak{D}^{[1,1,1]}(p)} & 0\\ \hline \multicolumn{2}{|c|}{0\ \ 0} & \multicolumn{2}{c|}{0\ \ 0} & \multicolumn{2}{c|}{0\ \ 0} & \mathfrak{D}^{[3]}(p)\\ \hline\end{array}$$

Since the Hamiltonian is symmetric, one has

$$S(\mathcal{H})S(p) = S(p)S(\mathcal{H})$$

and Schur's lemmas demand that $S(\mathcal{H})$ be almost diagonal as shown below:

$$S(\mathcal{H}) = \begin{array}{|cc|cc|cc|c|}\hline a & 0 & b & 0 & 0 & 0 & \\ 0 & a & 0 & b & 0 & 0 & \\ \hline c & 0 & d & 0 & 0 & 0 & \\ 0 & c & 0 & d & 0 & 0 & \\ \hline \multicolumn{4}{|c|}{0\ \ 0\ \ 0\ \ 0} & e & 0 & \\ \hline \multicolumn{5}{|c|}{0\ \ 0\ \ 0\ \ 0\ \ 0} & f\\ \hline\end{array} .$$

Now introduce the operators[7]

$$P^{(\mu)} = \frac{l^{(\mu)}}{n!}\sum_{[p]}\chi^{(\mu)}(p)[p], \qquad (3.2)$$

where $l^{(\mu)}$ and $\chi^{(\mu)}(p)$ are the dimensions and IR characters, respectively, which are easily computed (see Appendix B). The representations of these are shown below:

$$S(P^{[2,1]}) = \begin{vmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{vmatrix},$$

$$S(P^{[1,1,1]}) = \begin{vmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{vmatrix},$$

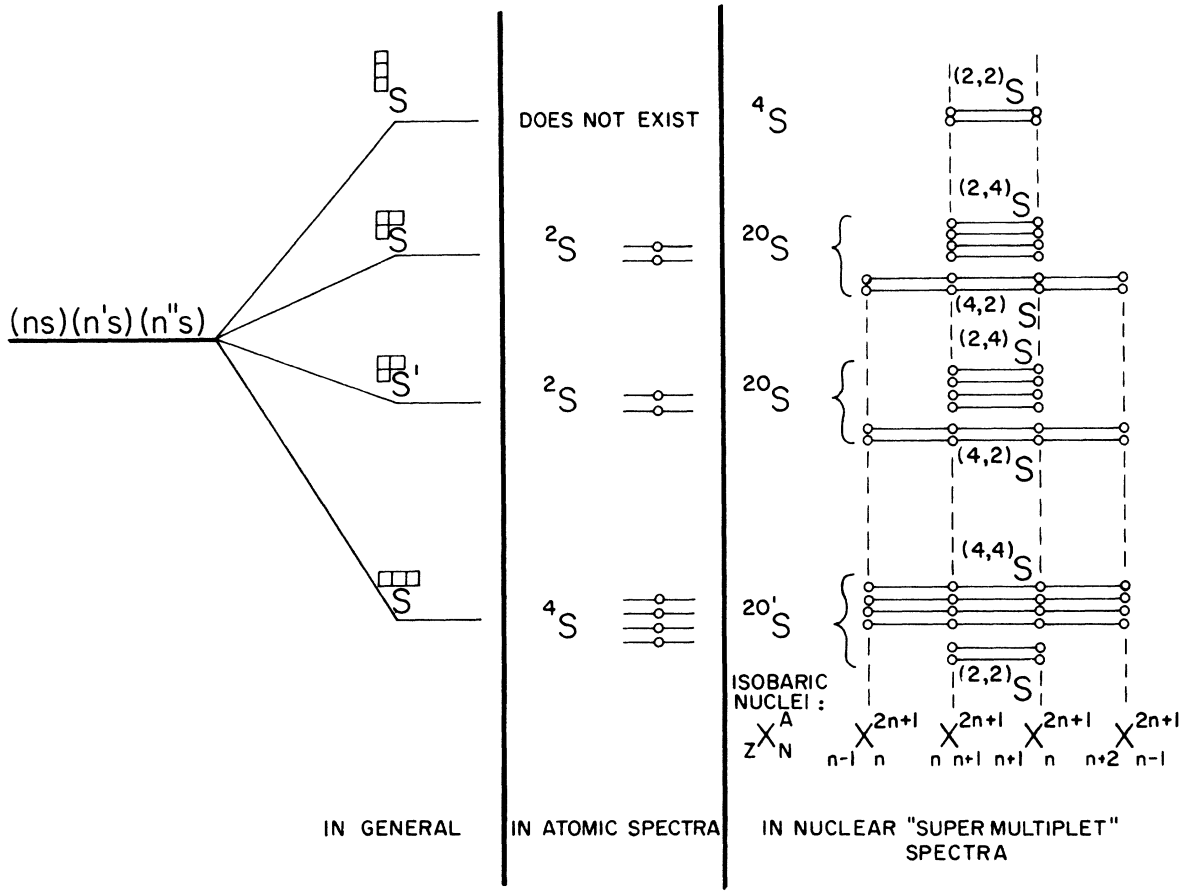


FIG. 3. Atomic and nuclear multiplets associated with Young tableaux involving products of three single-particle bases.

$$S(P^{[3]}) = \begin{vmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 1 \end{vmatrix}.$$

Clearly the average energy of levels belonging to IR $[\mu]$ of S_3 is given by $\langle e^{[\mu]}\rangle$ below:

$$\begin{aligned}\langle e^{[\mu]}\rangle &= [1/(l^{[\mu]})^2]\ \mathrm{tr}\, S(\mathcal{H})S(P^{[\mu]}) \\ &= [1/(l^{[\mu]})^2]\ \mathrm{tr}\, S(\mathcal{H}P^{[\mu]}) \\ &= \frac{1}{n!}\sum_{[p]}\frac{\chi^{[\mu]}(p)}{l^{[\mu]}}\,\mathrm{tr}\, S(\mathcal{H}[p]).\end{aligned}$$

One now computes the trace in the original basis

$$\psi_i\psi_j\psi_k \equiv \psi_n(r_i)\psi_{n'}(r_j)\psi_{n''}(r_k)\ ,$$

$$\begin{aligned}\langle e^{(\mu)}\rangle &= \frac{1}{n!}\sum_{[p]}\frac{\chi^{[\mu]}(p)}{l^{[\mu]}}\sum_{\substack{\text{all permuta-}\\ \text{tions of 123}}}\int dr_1\int dr_2\int dr_3 \\ &\quad\times \psi_1^*\psi_2^*\psi_3^*\mathcal{H}\,[p]\,\psi_1\psi_2\psi_3 \\ &= \frac{1}{n!}\sum_{\substack{\text{classes}\ j\\ \text{of } S_n}}\frac{\chi_1^{[\mu]}}{l^{[\mu]}}\sum_{\substack{\text{all permuta-}\\ \text{tions of 123}}}\int dr_1\int dr_2\int dr_3 \\ &\quad\times \psi_1^*\psi_2^*\psi_3^*\mathcal{H}\kappa_j\,\psi_1\psi_2\psi_3\,.\end{aligned}$$

In the above a class κ_j is just the sum of permutations having a given cycle structure:

$$\begin{aligned}\kappa_1 &= [1], \\ \kappa_2 &= [12]+[13]+[23], \\ \kappa_3 &= [123]+[132].\end{aligned}$$

The property of a class is that every permutation commutes with it. Using this and the fact that every permutation commutes with $\mathcal{H}$, we have the following:

$$\langle e^{[\mu]}\rangle = \sum_{\text{classes}\ j}\frac{\chi_j^{[\mu]}}{l^{[\mu]}}\int dr_1\int dr_2\int dr_3\,\psi_1^*\psi_2^*\psi_3^*\mathcal{H}\kappa_j\psi_1\psi_2\psi_3.$$

Now the coefficients $\chi_j^{[m]}/l^{[m]}$ are computed by methods given in Appendix B, and the following sum rule results:

$$\begin{aligned}e^{[1,1,1]} &= I - J + K, \\ \langle e^{[2,1]}\rangle &= I - \tfrac{1}{2}K, \\ e^{[3]} &= I + J + K.\end{aligned} \qquad (3.3)$$

For notational convenience let

$$\int dr_1\int dr_2\int dr_3\,\psi_1^*\psi_2^*\psi_3^*\hat{O}\,\psi_i^*\psi_j\psi_k \equiv \langle 123\,\hat{O}\,ijk\rangle$$

for any operator $\hat{O}$. Then the I, J, and K above are given by the following:

$$\begin{aligned}I &= \langle 123\mathcal{H}\,123\rangle\,, \\ J &= \langle 123\mathcal{H}\,([12]+[13]+[23])\,123\rangle \\ &= \langle 123\mathcal{H}\,213\rangle + \langle 123\mathcal{H}\,321\rangle + \langle 123\mathcal{H}\,132\rangle\,, \\ K &= \langle 123\mathcal{H}\,([123]+[132])\,123\rangle \\ &= \langle 123\mathcal{H}\,312\rangle + \langle 123\mathcal{H}\,231\rangle\,.\end{aligned}$$

Suppose that $\mathcal{H}$ contains only symmetric sums of one-body and two-body operators:

$$\begin{aligned}\mathcal{H} &= \sum_j^3 f(i) + \sum_{i>j} g(ij) \\ &= f(1)+f(2)+f(3)+g(12)+g(13)+g(23).\end{aligned}$$

Then we have

$$\begin{aligned}I &= \langle 1\,f(1)\,1\rangle + \langle 2f(2)\,2\rangle + \langle 3f(3)\,3\rangle \\ &\quad + \langle 12g(12)\,12\rangle + \langle 13g(13)\,13\rangle + \langle 23g(23)\,23\rangle, \qquad (3.4) \\ J &= \langle 12g(12)\,21\rangle + \langle 13g(13)\,31\rangle + \langle 23g(23)\,32\rangle, \\ K &= 0\,.\end{aligned}$$

The notation in (3.4) is the following:

$$\begin{aligned}\langle \alpha\beta g(\gamma\delta)\,\epsilon\rangle &= \int dr_\alpha\int dr_\beta\,\psi_n^*(r_\alpha)\psi_{n'}^*(r_\beta)g(\gamma\delta)\psi_n(r_\epsilon)\psi_{n'}(r_\iota). \\ &\quad\text{The } K \text{ term will not be zero if in } \mathcal{H} \text{ there is a}\end{aligned}$$

three-body operator $h(123)$ which cannot be decomposed into a sum of the two-body operators. For the Hamiltonian

$$\mathcal{H} = f(1)+f(2)+f(3)+g(12)+g(13)+g(23)+h(123),$$

the I, J, and K terms are changed to the following:

$$\begin{aligned}I &= \langle f(1)\,1\rangle + \langle 2f(2)\,2\rangle + \langle 3f(3)\,3\rangle \\ &\quad + \langle 12g(12)\,12\rangle + \langle 13g(13)\,13\rangle + \langle 23g(23)\,23\rangle \\ &\quad + \langle 123\,h(123)\,123\rangle\,, \\ J &= \langle 12g(12)\,21\rangle + \langle 13g(13)\,31\rangle + \langle 23g(23)\,32\rangle \\ &\quad + \langle 123\,h(123)\,213\rangle + \langle 123\,h(123)\,321\rangle \\ &\quad + \langle 123\,h(123)\,132\rangle, \\ K &= \langle 123\,h(123)\,312\rangle + \langle 123\,h(123)\,231\rangle\,.\end{aligned}$$

Using the $\chi_j^{[\mu]}/l^{[\mu]}$ for S_4, which are calculated by methods given in Appendix B, four-particle sum rules can be written as follows:

$$\begin{aligned}e^{[4]} &= I + J + K + L + M, \\ \langle e^{[3,1]}\rangle &= I + \tfrac{1}{3}J - \tfrac{1}{3}L - \tfrac{1}{3}M, \\ \langle e^{[2,2]}\rangle &= I - \tfrac{1}{2}K + L, \\ \langle e^{[2,1,1]}\rangle &= I - \tfrac{1}{3}J - \tfrac{1}{3}L + \tfrac{1}{3}M, \\ \langle e^{[1,1,1,1]}\rangle &= I - J + K + L - M.\end{aligned} \qquad (3.5)$$

Of course if the Hamiltonian contains no four-body operators, then L and M vanish, and K is zero again if, also, there are no three-body operators. Now, a positive J results from a repulsive two-body force,

and one derives the well-known result that higher energy is found in the more symmetric spatial wave functions in atomic spectra. This implies that the highest total spin will be found in the ground state.

IV. PURE CONFIGURATIONS

Sum rules like (3.3) and (3.5) are exact for mixed configurations like $nl\,n'l'n''l''\ldots$ but are only approximations for pure configurations $(nl)^r$.

To show the nature of this approximation an $(np)^3$ configuration will be discussed. The three $l=1$ orbitals can be coupled to give an F septet ($L=3$), two D quintets ($L=2$), three triplets ($L=1$), and a singlet ($L=0$).

The S_3 permutation projectors (3.3) can be used to help obtain these states. Applying $P^{[1,1,1]}$ to

$$\psi^1_m(r_1)\,\psi^1_{m'}(r_2)\,\psi^1_{m''}(r_3)\equiv(m\,m'\,m'')$$

with $m=1$, $m'=-1$, $m''=0$, one obtains (4.1) which is the S singlet:

$$\psi^{[1,1,1]}(1/\sqrt{6})\,[(1\,-1\,0)+(0\,1\,-1)+(-1\,0\,1)-(-1\,1\,0)-(0\,-1\,1)-(1\,0\,-1)]. \tag{4.1}$$

Similarly (4.2) and (4.3) are two P states with $M=0$ which together span the IR basis for a parasymmetric IR of permutations:

$$\psi_1^{[2,1]\,P}=(1/2\sqrt{3})[2(1\,-1\,0)-(0\,1\,-1)-(-1\,0\,1)+2(-1\,1\,0)-(0\,-1\,1)-(1\,0\,-1)], \tag{4.2}$$

$$\psi_2^{[2,1]\,P}=\tfrac{1}{2}[(0\,1\,-1)-(-1\,0\,1)+(0\,-1\,1)-(1\,0\,-1)]. \tag{4.3}$$

An orthogonal set of parasymmetric base vectors (4.4) and (4.5) below turn out to be D states with $M=0$:

$$\psi_1^{[2,1]\,D}=\tfrac{1}{2}[(0\,1\,-1)-(-1\,0\,1)-(0\,-1\,1)+(1\,0\,-1)], \tag{4.4}$$

$$\psi_2^{[2,1]D}=(1/2\sqrt{3})\,[-2(1\,-1\,0)+(0\,1\,-1)+(-1\,0\,1)+2(-1\,1\,0)-(0\,-1\,1)-(1\,0\,-1)]. \tag{4.5}$$

The completely symmetric base states of definite total orbital angular momentum are given below as (4.6) and (4.7):

$$\psi^{[3]F}=(1/\sqrt{10})[(1\,-1\,0)+(0\,1\,-1)+(-1\,0\,1)+(-1\,1\,0)+(0\,-1\,1)+(1\,0\,-1)+2(0\,0\,0)], \tag{4.6}$$

$$\psi^{[3]P}=(1/\sqrt{15})[(1\,-1\,0)+(0\,1\,-1)+(-1\,0\,1)+(-1\,1\,0)+(0\,-1\,1)+(1\,0\,-1)-3(0\,0\,0)]. \tag{4.7}$$

We have now accounted for the $M=0$ states for all the levels that can arise from a $(np)^3$ configuration of a general fermion system. In Fig. 4 these levels are drawn as dark lines. The dotted lines denote levels that would arise in a completely mixed $np\,n'p\,n''p$ configuration.

The Young tableau superscript indicates the permutation symmetry with respect to the "internal" quantum numbers involving spin or spin-isospin. Clearly the P-F orbital decapulet could never occur in atoms.

The energy formulas are found by an essentially identical procedure as was used to obtain (3.3). As is seen below, the new formulas (4.8) differ from the old by a correction term which is added to the average energy of the P-F decapulet:

$$\begin{aligned} &e([3]S)=I-J+K,\\ &\tfrac{1}{2}[e([2,1]P)+e([2,1]D)]=I-\tfrac{1}{2}K,\\ &\tfrac{1}{2}[e([1,1,1]P)+e([1,1,1]F)]=I+J+K+\langle 0001\,\mathcal{K}\,1000\rangle. \end{aligned} \tag{4.8}$$

The formulas (3.3) would be exactly correct for a completely mixed $(np\,n'p\,n''p)$ configuration provided one considers only the energy of the F multiplets.

V. WIGNER SUPERMULTIPLET MODEL

The convenience of the Young tableaux methods will be demonstrated now in connection with nuclear-mass spectra. Formulas for mass spectra have been written and tested by Franzini and Radicati,[8] and readers familiar with their work may find it instructive to derive the connection between their methods and the Young tableaux theory here (Appendix D). The advantage of the tableaux method is that it is simple to explain and use and is quite physical. Furthermore, it can immediately be used to derive sum rules for n-body operator spectra for arbitrary n.

For example, consider the three known isobars ${}_{46}Pd^{105}_{59}$, ${}_{47}Ag^{105}_{58}$, and ${}_{48}Cd^{105}_{57}$. The total isotopic spin T_3 is half the difference between the number of neutrons and number of protons. For Pd it is $\frac{13}{2}$, for Ag it is $\frac{11}{2}$, and for Cd it is $\frac{9}{2}$.

The idea of supermultiplet theory is that these nuclei are just different (spin-isospin) multiplets of single configuration. Figure 5(a) is the spectrum diagrammed as it would appear without the Coulomb repulsion of the protons. The stable nuclear ground states are thought to lie at the extreme right of each multiplet diagram because the Coulomb energy will split the multiplets leaving the greatest isotopic spin states of a given supermultiplet the lowest in energy, as shown in Fig. 5(b).

Now the most difficult question is: What tableaux describe the supermultiplets to which Pd, Ag, and Cd belong? We do not know what the configuration is so we do not know how many boxes in total the

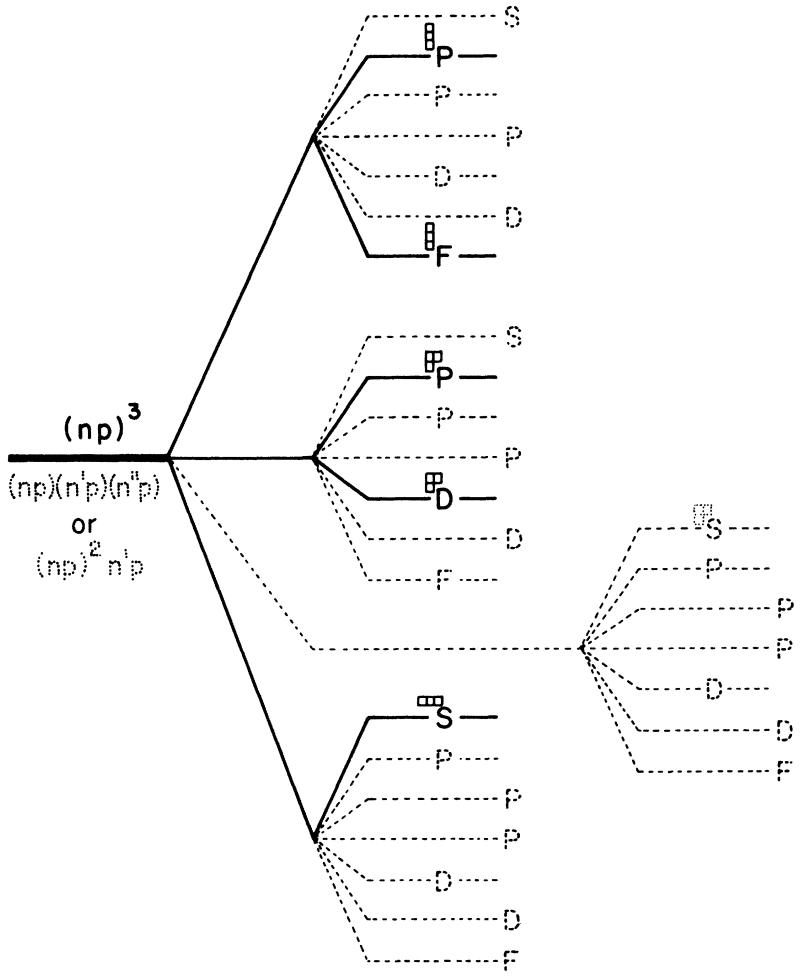


FIG. 4. p^3 configuration shown for the cases in which the radial quantum numbers are all three the same (dark lines), two the same, and none the same.

tableau has. But this does not matter since every spin-isospin tableau has two parts, the inactive part and the active part. The inactive part contains any number of four-box columns, each representing a quadruply antisymmetric combination of a neutron and proton with spin up and down, and we are not interested in it. The active part dictates what states of total charge (isospin) and spin are allowed, in a manner described in Appendix C.

Consider Pd with $T_3=\frac{13}{2}$. This means the active part of Pd's tableau must have 13 boxes. Since Pd is stable with respect to γ decay, it must belong to the lowest-energy supermultiplet capable of producing a $T_3=\frac{13}{2}$ state. Clearly, the active part is the most antisymmetric combination of two rows of 13 boxes. Now according to Appendix C the tableau [7,6] has a ([7,6] ~ $S=\frac{1}{2}$, [13] ~ $T_3=\frac{13}{2}$) state while the tableau [6,6,1] does not. The reason for wanting antisymmetric spin-isospin states is that presumably the nuclear forces are attractive, so the most symmetric spatial state will be lowest in energy. Finally, the Ag and Cd tableaux follow from the choice of the Pd and all are shown in Fig. 5(a).

Disregarding the inactive parts, we write mass sum rules using S_n characters with $n=13$, assuming only two-body forces:

$$m(\mathrm{Pd}) = I + (\chi_{1^1 1_2}^{[7,6]}/l^{[7,6]})(\tfrac{1}{2}n(n-1)J) ,$$

$$m(\mathrm{Ag}) = I + (\chi_{1^1 1_2}^{[6,6,1]}/l^{[6,6,1]})(\tfrac{1}{2}n(n-1)J) ,$$

$$m(\mathrm{Cd}) = I + (\chi_{1^1 1_2}^{[6,5,1,1]}/l^{[6,5,1,1]})(\tfrac{1}{2}n(n-1)J) .$$

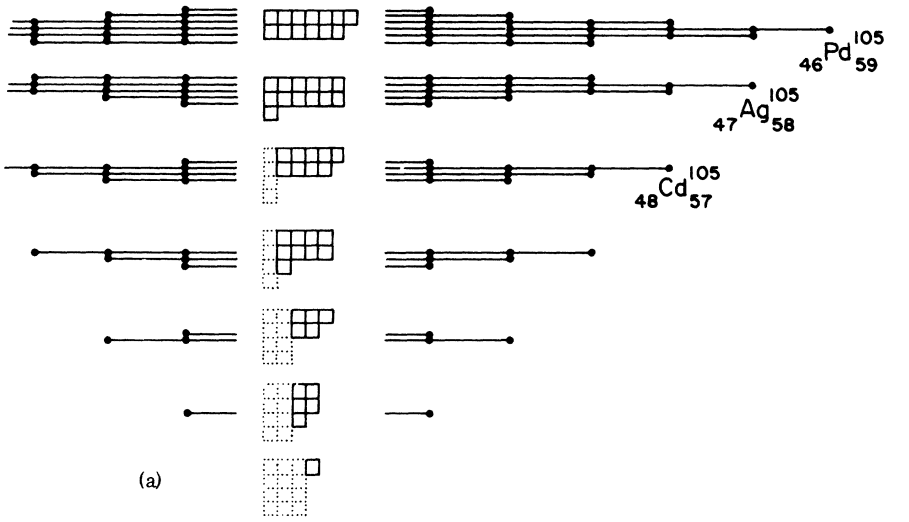
The factor $\frac{1}{2}n(n-1)$ with $n=13$ was factored so that J now represents the "average exchange integral."

The group theoretical factors are easily computed using the methods of Appendix B. The answers are most simply expressed by products and quotients of hook lengths, as shown below:

$$\frac{n(n-1)}{2}\,\frac{\chi_{1^1 1_2}^{[76]}}{l^{[76]}} = \frac{1}{2}\left[\text{product}\begin{Bmatrix}8&7&6&5&4&3&1\\6&5&4&3&2&1\end{Bmatrix} \Big/ \text{product}\begin{Bmatrix}8&7&6&5&3&2&1\\4&3&2&1\end{Bmatrix}\right] = 30 ,$$

$$\frac{n(n-1)}{2}\,\frac{\chi^{[6,6,1]}_{1^{11}_2}}{l^{[6,6,1]}} = \frac{1}{2}\left(\text{product}\begin{Bmatrix}8&6&5&4&3&2\\7&5&4&3&2&1\\1\end{Bmatrix}\Big/\text{product}\begin{Bmatrix}8&6&5&4&2&1\\5&3&2&1\\1\end{Bmatrix} - \begin{Bmatrix}8&6&5&4&3&2\\7&5&4&3&2&1\\1\end{Bmatrix}\Big/\begin{Bmatrix}7&5&4&3&2\\6&4&3&2&1\\1\end{Bmatrix}\right) = 22\,,$$

$$\frac{n(n-1)}{2}\,\frac{\chi^{[6,5,1,1]}_{1^{11}_2}}{l^{[6,5,1,1]}} = \frac{1}{2}\begin{Bmatrix}9&6&5&4&3&1\\7&4&3&2&1\\2\\1\end{Bmatrix}\Big/\begin{Bmatrix}9&6&5&3&2&1\\5&2&1\\2\\1\end{Bmatrix} - \frac{1}{2}\begin{Bmatrix}9&6&5&4&3&1\\7&4&3&2&1\\2\\1\end{Bmatrix}\Big/\begin{Bmatrix}7&6&5&4&3&1\\5&4&3&2&1\end{Bmatrix} = 15\,.$$



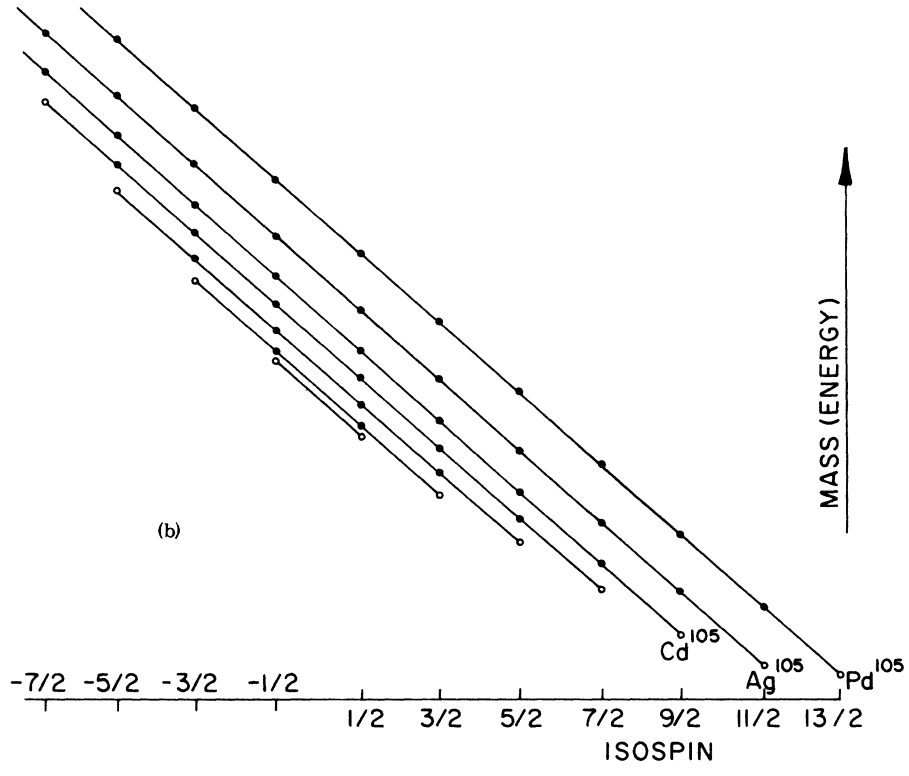


FIG. 5. (a) Schematic representation of the supermultiplet levels associated with Pd^{105}, Ag^{105}, and Cd^{105}. Here it is imagined that Coulomb repulsion and spin-orbit interactions are "turned off." (b) Effect of Coulomb repulsion is sketched where states with more protons are shown with higher energy.

Hence the sum rule is the following:

$$m(\mathrm{Pd}) = I + 30J\ ,$$

$$m(\mathrm{Ag}) = I + 22J\ ,$$

$$m(\mathrm{Cd}) = I + 15J\ .$$

The experimental values[9] of the nuclear masses must have the Coulomb energies,[10] given below, subtracted:

$$\Delta M = \frac{0.00075Z^2}{(A)^{1/3}}\ \mathrm{amu} = \begin{cases} 0.3364 & \text{for Pd} \\ 0.3512 & \text{for Ag} \\ 0.3663 & \text{for Cd} \end{cases}\ .$$

The resulting "bare" masses are given below:

$$m^{\mathrm{expt}}(\mathrm{Pd}) = 104.5763\ ,$$

$$m^{\mathrm{expt}}(\mathrm{Ag}) = 104.5630\ ,$$

$$m^{\mathrm{expt}}(\mathrm{Cd}) = 104.5513\ .$$

A ratio of differences between adjacent mass values is computed and compared with the same ratio predicted by the two-body sum rule:

$$\frac{m^{\mathrm{expt}}(\mathrm{Pd}) - m^{\mathrm{expt}}(\mathrm{Ag})}{m^{\mathrm{expt}}(\mathrm{Ag}) - m^{\mathrm{expt}}(\mathrm{Cd})} = 1.14\ ,$$

$$\frac{(I+30J) - (I+22J)}{(I+22J) - (I+15J)} = \frac{8}{7}\ .$$

Franzini and Radicati made more than a hundred such comparisons and many of them showed the sort of agreement obtained in the above example. For examples that more or less fail these comparisons, one may use the above hook-length procedure to write a three-body or four-body sum rule and then make further comparisons with observed mass values.

However, in order to demonstrate beyond reasonable doubt the existence or nonexistence of the higher-body effects, one needs to study a large number of isobaric families (each family must have at least four nuclei) while taking into account other important effects.[11] The results of this sort of program will be reported in a later paper.

In the meantime it has been noted[12] that three-body operator spectra will differ essentially from that of a two-body operator in odd-A nuclei only. This was seen when the hook-length formula was converted to an algebraic formula involving the isospin T_3 of the extreme isobar. These formulas for intervals between adjacent odd-A mass values are written between the lines in the level schematic shown in Fig. 6.

APPENDIX A

The dimension of a given IR of S_n is easily computed using the following formula involving hook lengths of the tableau for that IR[13]:

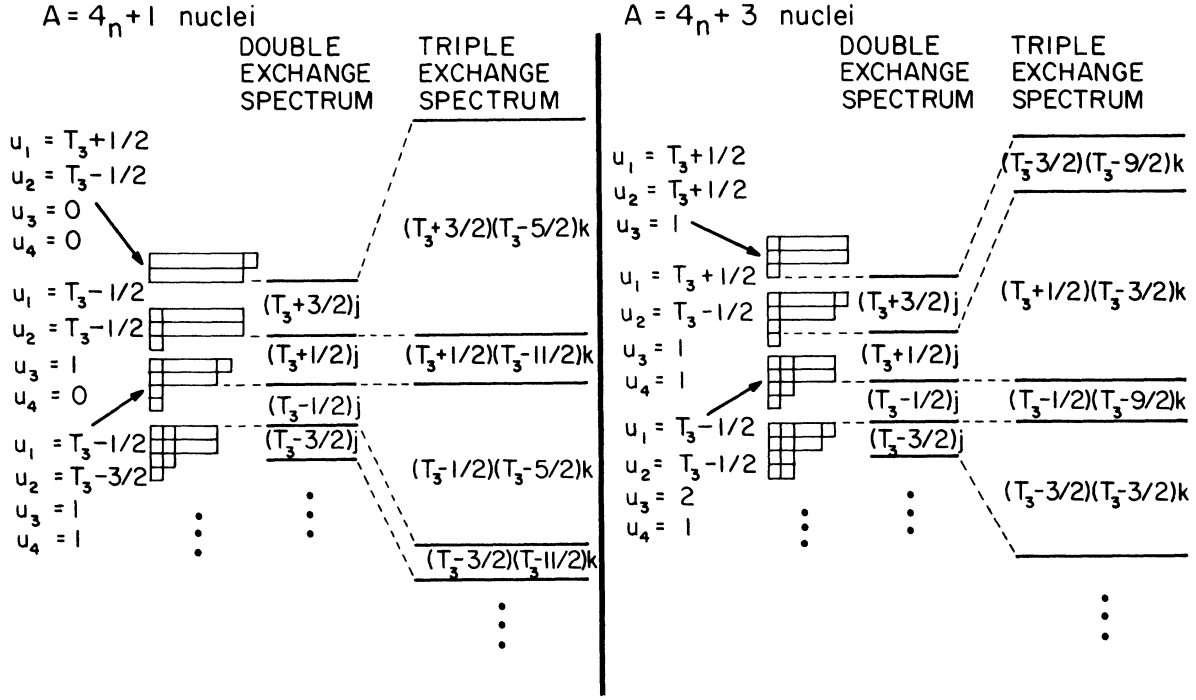


FIG. 6. Schematic representation of the effect of a two-body and then a three-body force in odd-A nuclear supermultiplet spectral intervals. T_3 is the greatest isospin found in the uppermost supermultiplet level. The constants j and k are exchange integrals.

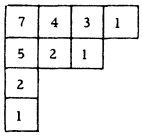


FIG. 7. An IR tableau with the hook length of each box given within the box.

$$l^{[\mu]}=\frac{n!}{\text{product of } [\mu] \text{ hook lengths}} .$$

A hook length of a given tableau box is the number of boxes to the right and below that box plus one. For example, the hook length of each box in the tableau in Fig. 7 is written within it. The dimension of IR of S_9 designated by this tableau is given below:

$$l^{[4,3,1,1]}=9!/(7\times4\times3\times5\times2\times2)=216 .$$

APPENDIX B

The character $\chi^{[\mu]}_{1^{\alpha}2^{\beta}3^{\gamma}\ldots}$ belonging to a given IR tableau $[\mu]=[\mu_1\mu_2\ldots]$ and class $1^{\alpha}2^{\beta}3^{\gamma}\ldots$ of S_n is easily found using the formula below and the definitions that follow[14]:

$$\chi^{[\mu_1,\ldots,\mu_p]}_{1^{\alpha}2^{\beta}3^{\gamma}\ldots}=\partial_1^{\alpha}\partial_2^{\beta}\partial_3^{\gamma}\ldots\begin{vmatrix}\mu_1+p-1\\ \vdots\\ \mu_{p-2}+2\\ \mu_{p-1}+1\\ \mu_p\end{vmatrix};$$

Def. 1:

$$\partial_m\begin{vmatrix}a\\ b\\ c\\ \vdots\end{vmatrix}=\begin{vmatrix}a-m\\ b\\ c\\ \vdots\end{vmatrix}+\begin{vmatrix}a\\ b-m\\ c\\ \vdots\end{vmatrix}+\begin{vmatrix}a\\ b\\ c-m\\ \vdots\end{vmatrix}+\cdots;$$

Def. 2:

$$\begin{vmatrix}p-1\\ \vdots\\ 2\\ 1\\ 0\end{vmatrix}=1 ;$$

Def. 3:

$$\begin{vmatrix}a\\ b\\ c\\ \vdots\end{vmatrix}=0$$

if any two numbers in the column are equal, or if any number is less than zero;

Def. 4:

$$\begin{vmatrix}a\\ b\\ c\\ \vdots\end{vmatrix}=-\begin{vmatrix}b\\ a\\ c\\ \vdots\end{vmatrix}$$

interchanging any two numbers gives a change of sign.

For example, here is the character of the [56, 13] IR of class 2 11 56 of S_{69}:

$$\chi^{[56,13]}_{2\,11\,56}=\partial_2\partial_{11}\partial_{56}\begin{vmatrix}57\\ 13\end{vmatrix}=\partial_2\partial_{11}\begin{vmatrix}1\\ 13\end{vmatrix}$$

$$=\partial_2\begin{vmatrix}1\\ 2\end{vmatrix}=\begin{vmatrix}1\\ 0\end{vmatrix}=1 .$$

Sometimes the hook-length formula can make the above formula even more convenient. For example, since the dimension is a character of the unicycle class, one has the following for any IR of S_n:

$$l^{[\mu]}=\chi^{[\mu]}_{1^n2^0\ldots} .$$

This may be used when calculating a character of a class that is nearly all unicycles:

$$\chi^{[6,6,1]}_{1^1 1^1_2}=\partial_1^{11}\partial_2\begin{vmatrix}8\\ 7\\ 1\end{vmatrix}=\partial_1^{11}\begin{vmatrix}6\\ 7\\ 1\end{vmatrix}+\partial_1^{11}\begin{vmatrix}8\\ 5\\ 1\end{vmatrix}$$

$$=-l^{[5,5,1]}+l^{[6,4,1]} .$$

Now the hook-length formula given in Appendix A may be used.

APPENDIX C

The problem of finding which spin-isospin multiplets $[SU(2)\times SU(2)]$ are contained in a given supermultiplet $[SU(4)]$ is solved once the Clebsch-Gordan series for IR or S_n are known.[14] For example, the Clebsch-Gordan series for all possible inner products of S_3 IR are shown in (C1):

$$\begin{aligned}&[3]\otimes[3]=[3],\quad [3]\otimes[2,1]=[2,1],\\ &[3]\otimes[1,1,1]=[1,1,1] ,\\ &[2,1]\otimes[2,1]=[3]\oplus[1,1,1]\oplus[2,1],\\ &[2,1]\otimes[1,1,1]=[2,1],\\ &[1,1,1]\otimes[1,1,1]=[3] .\end{aligned}\qquad (C1)$$

Now the $SU(2)\times SU(2)$ multiplets found in, for example, the $SU(4)$ supermultiplet [2, 1] are represented by those combinations of tableaux ([3], [2, 1]), ([2, 1], [3]), ([2, 1], [2, 1]), ([1, 1, 1], [2, 1]), and ([2, 1], [1, 1, 1]) that are capable of giving [2, 1] in an S_3 inner product. Now, following the procedure leading to Fig. 3, we identify each of the tableaux in these pairs with $SU(2)$ multiplets $[2S+1]$ or $[2T+1]$, and this yields the desired reduction shown in (C2):

[2, 1] of $SU(4)$

$$= ([4]\times[2]+[2]\times[4]+[2]\times[2]) \text{ of } SU(2)\times SU(2) \ . \tag{C2}$$

Procedures for deducing Clebsch-Gordan series of S_n for large n exist,[13] but for our purposes we needed only to realize that the single horizontal row of n boxes is the tableau of the scalar representation of S_n. And the inner product of this with an IR $[\mu]$ of S_n is just $[\mu]$ itself.

APPENDIX D

One may use the hook-length procedure to derive a formula for the coefficient (D1) of the two-body exchange integrals in terms of tableau row lengths μ_j:

$$\tfrac{1}{2}n(n-1)\ (\chi_{1^{n-2}2^2}^{[\mu_1\mu_2\mu_3\mu_4]}/l^{[\mu_1\mu_2\mu_3\mu_4]})$$
$$= \tfrac{1}{2}\left[\mu_1(\mu_1-1)+\mu_2(\mu_2-3)+\mu_3(\mu_3-5)+\mu_4(\mu_4-7)\right] \ . \tag{D1}$$

The coefficient (D2) of the three-body exchange integrals is also computed this way[15]:

$$\frac{n(n-1)(n-2)}{3!}\ \frac{\chi_{1^{n-3}3}^{[\mu_1\mu_2\mu_3\mu_4]}}{l^{[\mu_1\mu_2\mu_3\mu_4]}} = (1/3!)\left[\mu_1(\mu_1-1)(\mu_1-2)+\mu_2(\mu_2-2)(\mu_2-4)+\mu_3(\mu_3-4)(\mu_3-5)\right.$$
$$\left.+\mu_4(\mu_4^2-12\mu_4+38)-3(\mu_1\mu_2+\mu_1\mu_3+\mu_1\mu_4+\mu_2\mu_3+\mu_2\mu_4+\mu_3\mu_4)\right] \ . \tag{D2}$$

Equation (D1) can be compared with the Franzini-Radicati formulas[8,16] by defining the quantum numbers λ_1, λ_2, and λ_3 which are the differences (D3) of the tableau row lengths:

$$\lambda_1 = \mu_1 - \mu_2 \ ,$$
$$\lambda_2 = \mu_2 - \mu_3 \ , \tag{D3}$$
$$\lambda_3 = \mu_3 - \mu_4 \ ,$$

$$\frac{n(n-1)}{2}\ \frac{\chi_{1^{n-2}2^2}^{[\mu_1\mu_2\mu_3\mu_4]}}{l^{[\mu_1\mu_2\mu_3\mu_4]}} = \frac{n^2-16n}{8} + \frac{1}{2}\langle C\rangle \ . \tag{D4}$$

The quantity $\langle C\rangle$ given below in (D5) is the eigenvalue of a Casimir operator (D6):

$$\langle C\rangle = 3\lambda_1+4\lambda_2+2\lambda_3+\left(\frac{\lambda_1+\lambda_3}{2}\right)^2 + \left(\frac{\lambda_1-\lambda_3}{2}\right)^2+\left(\frac{\lambda_1+2\lambda_2+\lambda_3}{2}\right)^2$$
$$= [S(S+4)+T(T+2)+Y^2]. \tag{D5}$$

In the last line of (D5), Wigner's[17] quantum numbers

$$S = \tfrac{1}{2}(\lambda_1+2\lambda_2+\lambda_3)\,,$$
$$T = \tfrac{1}{2}(\lambda_1+\lambda_3)\,,$$
$$Y = \tfrac{1}{2}(\lambda_1-\lambda_3)$$

are used. The Casimir operator C in question is shown below [(D6)] and can be related to the Majorana exchange operator M given in (D7):

$$C = \sum_{ij}^{n}\{\vec{s}(i)\cdot\vec{s}(j)+\vec{t}(i)\cdot\vec{t}(j)+4[\vec{s}(i)\cdot\vec{s}(j)][\vec{t}(i)\cdot\vec{t}(j)]\}\,, \tag{D6}$$

$$M = \sum_{ij}[\tfrac{1}{2}+2\vec{s}(i)\cdot\vec{s}(j)][\tfrac{1}{2}+2\vec{t}(i)\cdot\vec{t}(j)]. \tag{D7}$$

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