the value of ΔV.

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Master-Equation Approach to Spontaneous Emission.* II. Emission from a System of Harmonic Oscillators

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In a previous investigation, a general theory for spontaneous emission from a system of N identical atoms or molecules was developed. This theory was based on the master equation recently derived in another paper. In that paper, the master equation relating to spontaneous emission from a system of harmonic oscillators was also derived. In the present investigation, the normally ordered correlation functions for the oscillator system are calculated and these are then used to calculate the radiation-field correlation functions in the far zone. These correlation functions are compared for two different modes of excitation, viz., (i) when each of the oscillators is excited initially to a Fock state, and (ii) when each of the oscillators is excited to some coherent state. It is found that the even-order ($2n$) correlation functions for the second mode of excitation (superradiant excitation) are of order N^{2n} higher than those for the first mode of excitation. It is also shown that the photoelectron counting distribution for the superradiant excitation is Poissonian. Finally, the non-Markoffian effects in the spontaneous emission are studied in detail and their connection with exact results is described.

I. INTRODUCTION

In a previous paper,[1] a general theory for the spontaneous emission from a system of N identical atoms or molecules was developed which was based on the master equation that the author recently derived.[2] This treatment provided a quantum theory of the superradiant phenomenon,[3] which also has the important feature that it allows the development of a systematic procedure for making successive approximations. There, the master equation that described spontaneous emission from a system of harmonic oscillators was also derived. This master equation was solved and the superradiant state of the harmonic-oscillator system was discussed.

In Sec. II of the present paper, we first consider briefly spontaneous emission from a system of oscillators which are not confined to a region smaller than a wavelength, and the enhancement of the decay rate due to the presence of other oscillators is found. In Sec. III, we obtain the normally ordered correlation functions for the oscillator system. These are then used in Sec. IV to obtain the radiation-field correlation functions in the far zone. The correlation functions for two different modes of excitation, viz., (i) when each of the oscillators was excited to a Fock state, and (ii) when each of the oscillators was excited to some coherent state, are given explicitly. We call the second mode of excitation the "superradiant excitation." Various characteristics of this superradiant excitation are studied in detail. In particular, we find that the $2n$th ordered correlation functions for the superradiant excitation are of the order $(N)^{2n}$ higher than those for the first mode of excitation. We also find that the photoelectron counting distribution for superradiant emission is Poissonian, which is typical of a coherent field. In Sec. V, we study the non-Markoffian effects in spontaneous emission and establish the connection with more exact results. In particular, we show that is is possible to introduce a function $f(t)$ whose exact value is given by Eq. (5.13) which describes the entire dynamics of spontaneous emission. Finally, in Sec. VI, we study the spontaneous emission into a single mode. The radiation rate for the two above-mentioned modes of excitation are studied. In particular, we show that the superradiant emission into a single mode leads to a coherent field distribution with an amplitude which is an oscillatory function of time.

II. SPONTANEOUS EMISSION FROM LARGE SYSTEMS

We found that the emission from a system of harmonic oscillators is described by the following master equation:

$$\frac{\partial F_s^{(A)}}{\partial t}=\gamma_0\left(\sum_{ij}\frac{\partial}{\partial z_i}(z_j F_s^{(A)})+\text{c.c.}\right). \tag{2.1}$$

The notation is the same as that of I. The master equation (2.1) was derived for a system of oscillators which were confined to a region smaller than a wavelength. For large systems, one obtains the following master equation:

$$\partial F_s^{(A)}=\sum_{ij}\gamma_{ij}\left(\frac{\partial}{\partial z_i}(z_j F_s^{(A)})+\text{c.c.}\right). \tag{2.2}$$

The master equation (2.2) is equivalent to the following Langevin equation:

$$\dot z_i=-\textstyle\sum_j \gamma_{ij} z_j\ . \tag{2.3}$$

In matrix notation Eq. (2.3) becomes

$$\dot z(t)=-\gamma z(t)\ , \tag{2.4}$$

where $z(t)$ is the column matrix with element $z_i(t)$ and γ is the matrix with elements γ_{ij}. The solution to (2.4) is

$$z(t)=e^{-\gamma t}z(0)\ . \tag{2.5}$$

This solution can be used to calculate the time dependence of the normally ordered moments. In particular, the total energy associated with the oscillator system is

$$\begin{aligned}W(t)&=\omega_0\textstyle\sum_i\langle a_i^\dagger(t)a_i(t)\rangle\\ &=\omega_0\langle z^\dagger(t)z(t)\rangle\ ,\end{aligned} \tag{2.6}$$

where in the second line the average is taken with respect to the phase-space distribution function $F_S^{(A)}$. Using Eq. (2.5), Eq. (2.6) reduces to

$$\begin{aligned}W(t)&=\omega_0\langle z^\dagger(0)e^{-2\gamma t}z(0)\rangle\\ &=\omega_0\sum_{ij}(e^{-2\gamma t})_{ij}\langle a_i^\dagger(0)a_j(0)\rangle,\end{aligned} \tag{2.7}$$

and hence the radiation rate is given by

$$I(t)=-\frac{dW}{dt}=2\omega_0\sum_{ij}(\gamma e^{-2\gamma t})_{ij}\langle a_i^\dagger(0)a_j(0)\rangle\ . \tag{2.8}$$

We specialize (2.8) to the two following cases:

(i) If at $t=0$ the oscillators are excited such that

$$F_S^{(A)}(\{z_i\},\{z_i^*\};0)=\prod_{j=1}^{N}F_j^{(A)}(z_j,z_j^*;0),\quad \langle a_i(0)\rangle=0 \tag{2.9}$$

which would be the case, for example, if each of the oscillators was in some Fock state $|n_i\rangle$, then (2.8) reduces to

$$I_F(t)=2\omega_0\textstyle\sum_i(\gamma e^{-2\gamma t})_{ii}\langle a_i^\dagger(0)a_i(0)\rangle. \tag{2.10}$$

(ii) At $t=0$ each of the oscillators was in some coherent state $|z_0\rangle$, i.e.,

$$F_S^{(A)}(\{z_i\},\{z_i^*\};0)=\prod_{i=1}^{N}[\pi\delta^{(2)}(z_i-z_0)], \tag{2.11}$$

then (2.9) reduces to

$$I_C(t)=2\omega_0|z_0|^2\sum_{ij}(\gamma e^{-2\gamma t})_{ij}. \tag{2.12}$$

The radiation rate can thus be calculated if the matrix $e^{-2\gamma t}$ is known. This matrix can, in principle, be computed by using the method of contour integration. For the special case, when $\gamma_{ij}=\gamma_0$ for all i and j, it is easy to show that

$$(e^{-2\gamma t})_{ii}=1-(1/N)(1-e^{-2N\gamma_0 t}), \tag{2.13a}$$

$$(e^{-2\gamma t})_{ij}=-(1/N)(1-e^{-2N\gamma_0 t}). \tag{2.13b}$$

Equations (2.13a) and (2.13b) then lead to the results given in Sec. V of I. For the case of two oscillators, we find that

$$(e^{-2\gamma t})_{ii}=\cosh(2\gamma_{12}t)e^{-2\gamma_0 t}, \tag{2.14a}$$

$$(e^{-2\gamma t})_{ij}=-\sinh(2\gamma_{12}t)e^{-2\gamma_0 t}. \tag{2.14b}$$

On substituting (2.14) in (2.10) and (2.12), we obtain

$$\begin{aligned}I_F(t)&=2\omega_0[\gamma_0\cosh(2\gamma_{12}t)-\gamma_{12}\sinh(2\gamma_{12}t)]\\ &\quad\times\textstyle\sum_i\langle a_i^\dagger(0)a_i(0)\rangle\, e^{-2\gamma_0 t},\end{aligned} \tag{2.15}$$

$$I_C(t)=4\omega_0|z_0|^2(\gamma_0+\gamma_{12})\,e^{-2t(\gamma_0+\gamma_{12})}. \tag{2.16}$$

A result similar to (2.15) has been found recently by Lehmberg[4] who concluded from this that the the oscillators tend to trap the radiation. On the other hand, our result (2.16) shows that the oscillators also exhibit superradiance when each of them is excited initially to some coherent state. It also shows the *enhancement of the decay rate*.[5] In the case of small systems, the decay rate is exactly twice the decay rate of an isolated oscillator.

III. NORMALLY ORDERED CORRELATION FUNCTIONS FOR OSCILLATOR SYSTEM

In this section, we will calculate the normally ordered correlation functions for the oscillator system. It is well known that the normally ordered correlation function can be computed in terms of the Green's function $K^{(A)}(z, z^*, t|z_0, z_0^*, 0)$ associated with the master equation for $F_S^{(A)}$, and the phase-space distribution function $F_S^{(A)}$, e.g., $\langle a^\dagger(t_1)\cdots, a^\dagger(t_1)\, a(t_n)\cdots, a(t_1)\rangle$ is given by

$$\langle a^\dagger(t_1)\cdots a^\dagger(t_n)a(t_n)\cdots a(t_1)\rangle$$

$$=\frac{1}{\pi}\int\cdots\int d^2\{z_i\}\, d^2z_0 F_S^{(A)}(z_0, z_0^*; 0)$$

$$\times\prod_{j=1}^{N}[K^{(A)}(z_j, z_j^*, t_j|\, z_{j-1}, z_{j-1}^*, t_{j-1})\,|\,z_j|^2]. \quad (3.1)$$

We now introduce the Green's function $K^{(A)}(\{z_i\}, \{z_i^*\}, t|\{z_i^0\}, \{z_i^{0*}\}, 0)$ which satisfies Eq. (2.1), namely,

$$\frac{\partial K^{(A)}}{\partial t}=\gamma_0\Big(\sum_{ij}\frac{\partial}{\partial z_i}(z_j K^{(A)})+\text{c. c.}\Big), \quad (3.2)$$

and the initial condition

$$K^{(A)}(\{z_i\}, \{z_i^*\}, 0|\{z_i^0\}, \{z_i^{0*}\}, 0)=\prod_{i=1}^{N}\delta^{(2)}(z_i-z_i^0). \quad (3.3)$$

It is easily shown that the solution of (3.2) subject to the initial condition (3.3) is

$$K^{(A)}(\{z_i\}, \{z_i^*\}, t|\{z_i^0\}, \{z_i^{0*}\}, 0)=\prod_{i=1}^{N}\delta^{(2)}(\bar{z}_i-z_i^0)\, e^{2N\gamma_0 t}, \quad (3.4)$$

where $\bar{z}_i$ is related to z_i by the relation

$$\bar{z}_i=z_i+(1/N)(e^{\gamma_0 tN}-1)\sum_{j=1}^{N}z_j. \quad (3.5a)$$

The relation inverse to (3.5a) is given by

$$z_i=\bar{z}_i-(1/N)(1-e^{-\gamma_0 tN})\sum_{j=1}^{N}\bar{z}_j. \quad (3.5b)$$

We now use the solution (3.4) to compute various correlation functions. We have, for example,

$$\Gamma_{i;j}(t;0)\equiv\langle a_i^\dagger(t)a_j(0)\rangle$$

$$=\frac{1}{\pi}\int\cdots\int d^2\{z_i^{(1)}\}d^2\{z_i^{(2)}\}\, z_i^{(1)*}z_j^{(2)}K^{(A)}[\{z_i^{(1)}\}, \{z_i^{(1)*}\}, t|\{z_i^{(2)}\}, \{z_i^{(2)*}\}, 0]\quad F_S^{(A)}(\{z_i^{(2)}\}, \{z_i^{(2)*}\}; 0)$$

$$=\frac{1}{\pi}\int\cdots\int d^2\{\bar{z}_i^{(1)}\}d^2\{z_i^{(2)}\}z_j^{(2)}\left[\bar{z}_i^{(1)}-\frac{1}{N}(1-e^{-\gamma_0 tN})\sum_j\bar{z}_j^{(1)}\right]^*\prod_{j=1}^{N}\delta^{(2)}(\bar{z}_j^{(1)}-z_j^{(2)})\,F_S^{(A)}(\{z_i^{(2)}\}, \{z_i^{(2)*}\}; 0)$$

$$=\langle a_i^\dagger(0)\, a_j(0)\rangle-(1/N)(1-e^{-\gamma_0 tN})\sum_{l=1}^{N}\langle a_l^\dagger(0)\, a_j(0)\rangle$$

or

$$\Gamma_{i;j}(t;0)=\Gamma_{i;j}(0;0)-(1/N)(1-e^{-\gamma_0 tN})\sum_{l=1}^{N}\Gamma_{l;j}(0;0)\,. \quad (3.6)$$

Similarly, the correlation function $\langle a_i^\dagger(t)\, a_j^\dagger(t)\, a_k(0)\, a_l(0)\rangle$ is given by

$$\Gamma_{ij;kl}(t, t; 0, 0)\equiv\langle a_i^\dagger(t)\, a_j^\dagger(t)\, a_k(0)\, a_l(0)\rangle$$

$$=\frac{1}{\pi}\int\cdots\int d^2\{z_j\}\Big(z_i^*-(1/N)(1-e^{-N\gamma_0 t})\sum_{m=1}^{N}z_m^*\Big)\Big(z_j^*-(1/N)(1-e^{-N\gamma_0 t})\sum_{n=1}^{N}z_n^*\Big)z_k z_l\, F_S^{(A)}(\{z_j\}, \{z_j^*\}; 0)$$

$$=\Gamma_{ij;kl}(0, 0; 0, 0)-(1/N)(1-e^{-N\gamma_0 t})\sum_{n=1}^{N}\Gamma_{in;kl}(0, 0; 0, 0)$$

$$-(1/N)(1-e^{-N\gamma_0 t})\sum_{m=1}^{N}\Gamma_{mj;kl}(0, 0; 0, 0)+(1/N^2)(1-e^{-N\gamma_0 t})^2\sum_m\sum_n\Gamma_{mn;kl}(0, 0; 0, 0)\,. \quad (3.7)$$

Equations (3.6) and (3.7) give the time dependence of the correlation functions $\Gamma_{i;j}(t;0)$ and $\Gamma_{ij;kl}(t, t; 0, 0)$. We shall now exhibit explicitly this time dependence for two different modes of excitation.

A. Excitation Given by (2.9)

For the excitation described by Eq. (2.9), with $F_j^{(A)}$ representing the phase-space distribution function corresponding to Fock state $|n_j\rangle$, we have

$$\Gamma_{i;j}(0;0)=n_i\delta_{ij}\,;\quad n_i\equiv\Gamma_{i;i}(0;0)\,,$$
$$\Gamma_{ij;kl}(0,0;0,0)=n_i(n_i-1)\,\delta_{ij}\delta_{jk}\delta_{kl} \tag{3.8}$$
$$+n_in_j\delta_{ik}\delta_{jl}(1-\delta_{jk})+n_in_j\delta_{il}\delta_{jk}(1-\delta_{jl})\,. \tag{3.9}$$

On substituting (3.8) and (3.9) in (3.6) and (3.7), we find that

$$\Gamma_{i;j}(t;0)=n_i\delta_{ij}-(1/N)\,(1-e^{-N\gamma_0 t})\,n_j\,, \tag{3.10}$$

$$\Gamma_{ij;kl}(t,t;0,0)=n_i(n_i-1)\,\delta_{ij}\delta_{jk}\delta_{kl}+n_in_j\delta_{ik}\delta_{jl}(1-\delta_{kj})+n_in_j\delta_{il}\delta_{jk}(1-\delta_{ij})-(1/N)\,(1-e^{-\gamma_0 tN})\,[n_i(n_i-1)\,\delta_{ik}\delta_{kl}$$
$$+n_in_l\delta_{ik}(1-\delta_{kl})+n_in_k\delta_{il}(1-\delta_{kl})+\text{terms obtained by the replacement } i\to j]$$
$$+(1/N^2)\,(1-e^{-\gamma_0 tN})^2\,[n_k(n_k-1)\,\delta_{kl}+2n_kn_l(1-\delta_{kl})]\,. \tag{3.11}$$

Equations (3.10) and (3.11), in particular, lead to the following interesting results:

$$\langle a_i^\dagger(t)\,a_j(0)\rangle-\langle a_i^\dagger(t)\rangle\langle a_j(0)\rangle$$
$$=-(1/N)\,(1-e^{-\gamma_0 tN})\,n_j\quad(i\neq j) \tag{3.12}$$

and

$$\langle a_i^{\dagger 2}(t)\,a_j^2(0)\rangle-\langle a_i^{\dagger 2}(t)\rangle\langle a_j^2(0)\rangle$$
$$=(1/N^2)\,(1-e^{-\gamma_0 tN})^2\,n_j(n_j-1)\,. \tag{3.13}$$

These results show explicitly that because of spontaneous emission a finite correlation is produced between the ith and jth oscillator. It is also found that

$$\langle a_i^\dagger(t)\,a_j(t)\rangle-\langle a_i^\dagger(t)\rangle\langle a_j(t)\rangle$$
$$=-(1/N)\,(1-e^{-\gamma_0 tN})\,(n_i+n_j)$$
$$+(1/N^2)\,(1-e^{-\gamma_0 tN})^2\left(\sum_{i=1}^{N}n_i\right)\,. \tag{3.14}$$

In the special case when $n_i=n$, (3.14) reduces to

$$\langle a_i^\dagger(t)\,a_j(t)\rangle-\langle a_i^\dagger(t)\rangle\,\langle a_j(t)\rangle$$
$$=-(2n/N)\sinh(N\gamma_0 t)\,e^{-N\gamma_0 t}\,. \tag{3.15}$$

The fact that the correlation (3.15) is negative should be noted.

We now introduce the operator $D(t)$ defined by

$$D(t)=\sum_{i=1}^{N}a_i(t)\,. \tag{3.16}$$

In terms of $D(t)$, the radiation rate may be shown to be

$$I(t)=2\gamma_0\omega_0\langle D^\dagger(t)\,D(t)\rangle\,. \tag{3.17}$$

In Sec. IV, we will also show that the correlation function for the electric field associated with the radiated field is related to the correlation functions for $D(t)$. Using (3.10), (3.11), and (3.16) it can be shown that the normally ordered correlation functions involving the operator $D(t)$ are given by

$$\langle D^\dagger(t)\,D(0)\rangle=\sum_{ij}\Gamma_{i;j}(t;0)=\left(\sum_i n_i\right)e^{-N\gamma_0 t}\,, \tag{3.18a}$$

$$\langle D^\dagger(t)\,D^\dagger(t)\,D(0)\,D(0)\rangle=\sum_{ijkl}\Gamma_{ij;kl}(t,t;0,0)$$
$$=\left(\sum_i n_i(n_i-1)+2\sum_{\substack{ij\\i\neq j}}n_in_j\right)-(2/N)\,(1-e^{-\gamma_0 tN})\left(N\sum_i n_i(n_i-1)+2N\sum_{\substack{il\\i\neq l}}n_in_l\right)$$
$$+(1/N^2)\,(1-e^{-\gamma_0 tN})^2\left(N^2\sum_k n_k(n_k-1)+2N^2\sum_{k\neq l}\sum n_kn_l\right)$$
$$=\left(\sum_i n_i(n_i-1)+2\sum_{\substack{ij\\i\neq j}}n_in_j\right)e^{-2N\gamma_0 t}\,, \tag{3.18b}$$

and

$$\langle D^\dagger(t)\,D^\dagger(t)\,D(t)\,D(t)\rangle=\left(\sum_i n_i(n_i-1)+2\sum_{\substack{ij\\i\neq j}}n_in_j\right)e^{-4N\gamma_0 t}\,. \tag{3.18c}$$

The important point to notice is that the correlation functions decay exponentially with time. The decay rates are N times larger than the ones which would be obtained from a single oscillator.

B. Coherent State Excitation

For the excitation described by Eq. (2.11), we have

$$\Gamma_{i;j}(0;0) = |z_0|^2 , \tag{3.19a}$$

$$\Gamma_{ij;kl}(0,0;0,0) = |z_0|^4 . \tag{3.19b}$$

On substituting (3.19) in (3.6) and (3.7), it is found that

$$\Gamma_{i;j}(t;0) = |z_0|^2 e^{-N\gamma_0 t} , \tag{3.20a}$$

$$\begin{aligned}\Gamma_{ij;kl}(t,t;0,0) &= |z_0|^4 - (2/N)(1-e^{-N\gamma_0 t})N|z_0|^4 \\ &\quad + (1/N^2)(1-e^{-N\gamma_0 t})^2 N^2 |z_0|^4 \\ &= |z_0|^4 e^{-2N\gamma_0 t} .\end{aligned} \tag{3.20b}$$

It may be also readily shown that

$$\langle a_i^\dagger(t)\, a_j(t)\rangle - \langle a_i^\dagger(t)\rangle \langle a_j(t)\rangle = 0 . \tag{3.21}$$

Equation (3.21) shows that no correlation is induced between the ith and jth oscillator. In fact, for this case, we found the following solution for the phase-space distribution function:

$$F_S^{(A)} = \prod_{j=1}^{N} [\pi\delta^{(2)}(\bar{z}_j - z_0)]\, e^{2N\gamma_0 t} ,$$

which by change of variables reduces to

$$F_S^{(A)} = \prod_{j=1}^{N} [\pi\delta^{(2)}(z_j - z_0 e^{-N\gamma_0 t})] . \tag{3.22}$$

The result (3.22) shows that the density operator for a system of N oscillators at time t simply factorizes or, in other words, ith and jth oscillator are uncorrelated. The lowest-order correlation functions for the operator $D(t)$ are given by

$$\langle D^\dagger(t)\, D(0)\rangle = N^2 |z_0|^2 e^{-N\gamma_0 t} , \tag{3.23a}$$

$$\langle D^\dagger(t)\, D^\dagger(t)\, D(0)\, D(0)\rangle = N^4 |z_0|^2 e^{-2N\gamma_0 t} , \tag{3.23b}$$

$$\langle D^\dagger(t)\, D^\dagger(t)\, D(t)\, D(t)\rangle = N^4 |z_0|^2 e^{-4N\gamma_0 t} . \tag{3.23c}$$

On comparison of (3.23) with (3.18), it is found that

$$\frac{\langle D^\dagger(t)\, D(0)\rangle_{\text{coh}}}{\langle D^\dagger(t)\, D(0)\rangle_F} = N , \tag{3.24a}$$

$$\frac{\langle D^\dagger(t)\, D^\dagger(t)\, D(0)\, D(0)\rangle_{\text{coh}}}{\langle D^\dagger(t)\, D^\dagger(t)\, D(0)\, D(0)\rangle_F} = O(N^2) . \tag{3.24b}$$

Equations (3.24) are obviously characteristic of the superradiant phenomenon.

We have so far considered the correlation functions only for small systems. For large systems, the correlation functions can be calculated by the use of (2.5) and the quantum regression theorem. For example, $\langle a_i^\dagger(t)\, a_j(0)\rangle$ would be given by

$$\langle a_i^\dagger(t)\, a_j(0)\rangle = \textstyle\sum_k (e^{-\gamma t})_{ik} \langle a_k^\dagger(0)\, a_j(0)\rangle . \tag{3.25}$$

In particular, for the case of two oscillators, we find that

$$\begin{aligned}\langle a_i^\dagger(t)\, a_i(0)\rangle &= \cosh(\gamma_{12} t)\, e^{-\gamma_0 t} \langle a_i^\dagger(0)\, a_i(0)\rangle \\ &\quad - \sinh(\gamma_{12} t)\, e^{-\gamma_0 t} \langle a_k^\dagger(0)\, a_i(0)\rangle \quad (i \neq k) ,\end{aligned} \tag{3.26a}$$

$$\begin{aligned}\langle a_i^\dagger(t)\, a_j(0)\rangle &= \cosh(\gamma_{12} t)\, e^{-\gamma_0 t} \langle a_i^\dagger(0)\, a_j(0)\rangle \\ &\quad - \sinh(\gamma_{12} t)\, e^{-\gamma_0 t} \langle a_k^\dagger(0)\, a_j(0)\rangle \quad (k \neq i) .\end{aligned} \tag{3.26b}$$

For the case of excitation described by (2.9), (3.26) reduces to

$$\langle a_i^\dagger(t)\, a_i(0)\rangle = \cosh(\gamma_{12} t)\, e^{-\gamma_0 t} n_i , \tag{3.27a}$$

$$\langle a_1^\dagger(t)\, a_2(0)\rangle = -\sinh(\gamma_{12} t)\, e^{-\gamma_0 t} n_2 , \tag{3.27b}$$

$$\langle a_2^\dagger(t)\, a_1(0)\rangle = -\sinh(\gamma_{12} t)\, e^{-\gamma_0 t} n_1 . \tag{3.27c}$$

For the case of excitation described by (2.11), (3.26) reduces to

$$\langle a_i^\dagger(t)\, a_j(0)\rangle = |z_0|^2 e^{-\gamma_0 t - \gamma_{12} t} . \tag{3.28}$$

Equation (3.28) shows that the correlation function decays exponentially with the enhancement of decay rate, which is again a characteristic of superradiance.

IV. NORMALLY ORDERED CORRELATION FUNCTIONS FOR RADIATION FIELD OPERATORS

So far we have been calculating the correlation functions for the oscillator system. In this section, we will show how the knowledge of the correlation functions for the oscillator system may be used to obtain the correlation functions for the spontaneously emitted radiation field. The Hamiltonian for this problem is

$$H = \omega_0 \sum_{i=1}^{N} a_i^\dagger a_i + \sum_{ks} \omega_{ks}\, a_{ks}^\dagger a_{ks} + \sum_{ksj} (g_{ks}^* a_{ks}^\dagger a_j + \text{H.c.}) . \tag{4.1}$$

The Heisenberg equations of motion for the operators a_i and a_{ks} are given by [5a]

$$i\frac{da_{ks}}{dt} = \omega_{ks} a_{ks} + \sum_i g_{ks}^* a_i , \tag{4.2}$$

$$i\frac{da_i}{dt} = \omega_0 a_i + \sum_{ks} g_{ks}\, a_{ks} . \tag{4.3}$$

From (4.2) and (4.3), it follows that

$$-\frac{d^2 a_{ks}}{dt^2}=\omega_{ks}\left(\omega_{ks}a_{ks}+\sum_i g^*_{ks}a_i\right)+\sum_i g^*_{ks}\left(\omega_0 a_i+\sum_{k's'}g_{k's'}a_{k's'}\right). \qquad (4.4)$$

The field operators are given by[5a]

$$\vec{A}^{(+)}(\vec{r},t)=\left(\frac{2\pi c}{L^3}\right)^{1/2}\sum_{ks}\frac{1}{\sqrt{k}}e^{i\vec{k}\cdot\vec{r}}\vec{\epsilon}_{ks}a_{ks}(t), \qquad (4.5a)$$

$$\vec{E}^{(+)}(\vec{r},t)=i\left(\frac{2\pi c}{L^3}\right)^{1/2}\sum_{ks}(\sqrt{k})e^{i\vec{k}\cdot\vec{r}}\vec{\epsilon}_{ks}a_{ks}(t). \qquad (4.5b)$$

From (4.5a) and (4.4), one obtains the following equation of motion for the vector potential operator $\vec{A}^{(+)}(\vec{r},t)$:

$$\left(\nabla^2\vec{A}^{(+)}-\frac{1}{c^2}\frac{d^2\vec{A}^{(+)}}{dt^2}\right)=\left(\frac{2\pi c}{L^3}\right)^{1/2}\sum_{ks}\frac{1}{\sqrt{k}}e^{i\vec{k}\cdot\vec{r}}\vec{\epsilon}_{ks}\left(-k^2-\frac{1}{c^2}\frac{d^2a_{ks}}{dt^2}\right)$$
$$=\frac{1}{c^2}\left(\frac{2\pi c}{L^3}\right)^{1/2}\sum_{ks}\frac{1}{\sqrt{k}}e^{i\vec{k}\cdot\vec{r}}\vec{\epsilon}_{ks}\times\left(2\omega_0\sum_i g^*_{ks}a_i+\sum_{ik's'}g^*_{ks}g_{k's'}a_{k's'}\right). \qquad (4.6)$$

We rewrite (4.6) in the form

$$\Box^2\vec{A}^{(+)}=-(4\pi/c)\vec{j}^{(+)}, \qquad (4.7)$$

where $\vec{j}^{(+)}$ is the positive-frequency part of the current operator. It is obvious from (4.6) that in the lowest order of the coupling constant $\vec{j}^{(+)}$ is given by

$$\vec{j}^{(+)}(\vec{r},t)=-\frac{1}{4\pi c}\left(\frac{2\pi c}{L^3}\right)^{1/2}\sum_{ksi}\frac{1}{\sqrt{k}}\vec{\epsilon}_{ks}g^*_{ks}e^{i\vec{k}\cdot\vec{r}}(2\omega_0)a_i(t). \qquad (4.8)$$

The solution to Eq. (4.7) is

$$\vec{A}^{(+)}(\vec{r},t)=\frac{1}{c}\int d^3r'\,|\vec{r}-\vec{r}'|^{-1}\vec{j}^{(+)}(\vec{r}',t-|\vec{r}-\vec{r}'|/c). \qquad (4.9)$$

On substituting (4.8) in (4.9), we obtain

$$\vec{A}^{(+)}(\vec{r},t)=-\frac{1}{4\pi c^2}\left(\frac{2\pi c}{L^3}\right)^{1/2}\sum_{ks}\frac{1}{\sqrt{k}}\vec{\epsilon}_{ks}g^*_{ks}(2\omega_0)\times\sum_i\int d^3r'\,|\vec{r}-\vec{r}'|^{-1}a_i(t-|\vec{r}-\vec{r}'|/c)e^{i\vec{k}\cdot\vec{r}'}, \qquad (4.10)$$

which in the limit $L\to\infty$ becomes

$$\vec{A}^{(+)}(\vec{r},t)=-\left(\frac{i\omega_0^2}{c^2}\right)\frac{1}{(2\pi)^3}\int\frac{d^3k}{k}[\vec{d}-\hat{k}(\vec{d}\cdot\hat{k})]\times\sum_i\int\frac{d^3r'}{|\vec{r}-\vec{r}'|}a_i(t-|\vec{r}-\vec{r}'|/c)e^{i\vec{k}\cdot\vec{r}'}. \qquad (4.11)$$

We now make the approximation

$$a_i(t-|\vec{r}-\vec{r}'|/c)\approx a_i(t-|\vec{r}|/c)e^{-i\omega_0(t-|\vec{r}-\vec{r}'|/c)}, \qquad (4.12)$$

where the operator a_i on the right-hand side refers to the Heisenberg operator from which the time dependence due to free Hamiltonian has been separated out. This approximation is valid because we have assumed that the oscillator system is confined to a region smaller than a wavelength. We calculate $\vec{A}^{(+)}(\vec{r},t)$ only in the radiation zone, for which we have

$$|\vec{r}-\vec{r}'|\approx|\vec{r}|-\left(\frac{\vec{r}\cdot\vec{r}'}{|\vec{r}|}\right). \qquad (4.13)$$

On substituting (4.13) and (4.12) in (4.11), we obtain the following result for the asymptotic form of $\vec{A}^{(+)}(\vec{r},t)$:

$$\vec{A}^{(+)}(\vec{r},t)\sim-(i\omega_0^2/c^2)\sum_i a_i(t-|\vec{r}|/c)\,(e^{ik_0r-i\omega_0t}/r)\times\left(\frac{1}{2\pi}\right)^3\int\frac{d^3k}{k}[\vec{d}-\hat{k}(\vec{d}\cdot\hat{k})]\times\int d^3r'\,e^{i\vec{k}\cdot\vec{r}'}\left(\exp -ik_0\frac{\vec{r}\cdot\vec{r}'}{|\vec{r}|}\right)$$
$$=-\frac{i\omega_0^2}{c^2}\sum_i a_i(t-|\vec{r}|/c)(e^{ik_0r-i\omega_0t}/r)\times\int\frac{d^3k}{k}[\vec{d}-\hat{k}(\vec{d}\cdot\hat{k})]\delta^{(3)}(\vec{k}-k_0\hat{n}), \qquad (4.14)$$

where $\hat{n}$ is the unit vector in the direction of the vector $\vec{r}$ and $k_0=\omega_0/c$. On simplification, we obtain[6]

$$\vec{A}^{(+)}(\vec{r},t)\sim(-i\omega_0/c)[\vec{d}-\hat{n}(\vec{d}\cdot\hat{n})](e^{ik_0r-i\omega_0t}/r)\times\sum_i a_i(t-|\vec{r}|/c). \qquad (4.15)$$

Similarly, one can show that the asymptotic form of the electric field operator is given by

$$\vec{E}^{(+)}(\vec{r},t)\sim+(\omega_0^2/c^2)[\vec{d}-\hat{n}(\vec{d}\cdot\hat{n})](e^{ik_0r-i\omega_0t}/r)\times\sum_i a_i(t-|\vec{r}|/c). \qquad (4.16)$$

On rewriting (4.16) in terms of the operator $D(t)$ defined by Eq. (3.16), we have

$$\vec{E}^{(+)}(\vec{r}, t) \sim (+\omega_0^2/c^2)[\vec{d} - \hat{n}(\vec{d}\cdot\hat{n})](e^{ik_0 r - i\omega_0 t}/r) \times D(t - |\vec{r}|/c) . \quad (4.17)$$

We have thus shown that the positive-frequency part of the electric field operator in the radiation zone is, apart from other factors, equal to the operator $D(t - |\vec{r}|/c)$. Hence the correlation functions for the electric field are given by the correlation functions for the oscillator system. For example, one has

$$\langle \vec{E}^{(-)}(\vec{r}, t)\cdot\vec{E}^{(+)}(\vec{r}, t')\rangle = (\omega_0^4/c^4)(|\vec{d}\times\hat{n}|^2/r^2)e^{i\omega_0(t-t')} \times\langle D^{+}(t - |\vec{r}|/c)D(t' - |\vec{r}|/c)\rangle . \quad (4.18)$$

For the case of coherent field excitation, (4.18) reduces to

$$\langle \vec{E}^{(-)}(\vec{r}, t)\cdot\vec{E}^{(+)}(\vec{r}, t')\rangle = (\omega_0^4/c^4)(|\vec{d}\times\hat{n}|^2/r^2)e^{i\omega_0(t-t')} \times N^2|z_0|^2\exp[2N\gamma_0|r|/c - N\gamma_0(t+t')] . \quad (4.19)$$

In particular, we find from (4.19) that

$$\langle \vec{E}^{(-)}(\vec{r}, t+|\vec{r}|/c)\cdot\vec{E}^{(+)}(\vec{r}, |\vec{r}|/c)\rangle = (\omega_0^4/c^4)(|\vec{d}\times\hat{n}|^2/r^2)N^2|z_0|^2 e^{i\omega_0 t - N\gamma_0 t} . \quad (4.20)$$

It is worth noting that the correlation function (4.20) is of the same form as for a damped Hertzian dipole.[7] Again the N^2 dependence of the correlation function should be noted. Similarly, the fourth-order correlation function is given by

$$\langle \vec{E}^{(-)}(\vec{r}, t)\vec{E}^{(-)}(\vec{r}, t):\vec{E}^{(+)}(\vec{r}, t')\vec{E}^{(+)}(\vec{r}, t')\rangle = (\omega_0^8/c^8)(|\vec{d}\times\hat{n}|^4/r^4)e^{2i\omega_0(t-t')} \times e^{-2N\gamma_0(t+t')}e^{4N\gamma_0|r|/c}N^4|z_0|^4 . \quad (4.21)$$

From (4.20) and (4.21), we find the following results, if we integrate over all directions n:

$$(c/2\pi)\int d\Omega(r^2)\langle \vec{E}^{(-)}(\vec{r}, t+|r|/c)\cdot\vec{E}^{(+)}(\vec{r}, |r|/c+t')\rangle = N^2|z_0|^2(2\omega_0\gamma_0)e^{i\omega_0(t-t')-N\gamma_0(t+t')} , \quad (4.22)$$

$$(c^2/\pi)\int d\Omega(r^4)\langle \vec{E}^{(-)}(\vec{r}, t+|\vec{r}|/c)\vec{E}^{(-)}(\vec{r}, t+|r|/c):\vec{E}^{(+)}(\vec{r}, t'+|\vec{r}|/c)\vec{E}^{(+)}(\vec{r}, t'+|\vec{r}|/c)\rangle = \tfrac{24}{5}\omega_0^2\gamma_0^2N^4|z_0|^2e^{2i\omega_0(t-t')-2N\gamma_0(t+t')} . \quad (4.23)$$

The double dot in (4.21) denotes the dot product between the 1st and 3rd member and the 2nd and 4th member. Equation (4.22) also leads to the expression (I5.14) for the power radiated. It is also clear that $2n$th-order correlation function of the form (4.20) and (4.21) will depend on $2n$th power of the number of oscillators which is again characteristic of the superradiant phenomenon (c.f., also the results of Sec. VI). In fact, it can be shown that the probability $p(n, T; |\vec{r}|/c)$ that a detector placed at a distance $|\vec{r}|$ from the oscillator system will detect n photoelectrons in the time interval between $|\vec{r}|/c$ and $|\vec{r}|/c+T$ is given by

$$p(n, T; |\vec{r}|/c) = [(\alpha I_0)^n/n!]e^{-\alpha I_0} , \quad (4.24)$$

where

$$I_0 = N^2|z_0|^2(2\omega_0\gamma_0)\int_0^T e^{-2N\gamma_0 t}dt = N^2|z_0|^2(2\omega_0\gamma_0)e^{-N\gamma_0 T}[\sinh(N\gamma_0 T)/N\gamma_0] . \quad (4.25)$$

Thus we obtain the Poisson distribution for the photoelectron counting distribution with the parameter I_0 given by (4.25), which is characteristic of a coherent field.[8]

V. SOME EXACT RESULTS AND NON-MARKOFFIAN BEHAVIOR

The master equation (2.1) was derived under two assumptions, namely, a Born approximation was made and the memory effects were ignored. In this section, we will first obtain the radiation rate by solving the Heisenberg equations of motion and later establish the connection of the results so derived with those obtained from non-Markoffian master equation. The Heisenberg equations of motion for the operators a_{ks} and a_j were obtained in Sec. IV. These are

$$i\frac{da_{ks}}{dt} = \omega_{ks}a_{ks} + \sum_j g_{ks}^* a_j , \quad (5.1)$$

$$i\frac{da_j}{dt} = \omega_0 a_j + \sum_{ks} g_{ks}a_{ks} . \quad (5.2)$$

On taking the Laplace transform (indicated by tilde), we obtain the equations

$$[p\tilde{a}_{ks}(p) - a_{ks}(0)] = -i\omega_{ks}\tilde{a}_{ks}(p) - i\sum_j g_{ks}^*\tilde{a}_j(p) , \quad (5.3)$$

$$[p\tilde{a}_j(p) - a_j(0)] = -i\omega_0\tilde{a}_j(p) - i\sum_{ks} g_{ks}\tilde{a}_{ks}(p) . \quad (5.4)$$

From (5.3) we find that $\tilde{a}_{ks}(p)$ is given by

$$\tilde{a}_{ks}(p)=(i\omega_{ks}+p)^{-1}a_{ks}(0)-i(p+i\omega_{ks})^{-1}\sum_j g^*_{ks}\tilde{a}_j(p)\ . \quad (5.5)$$

On eliminating $\tilde{a}_{ks}(p)$ from (5.4), we find that $\tilde{a}_j(p)$ is given by

$$(p+i\omega_0)\tilde{a}_j(p)=a_j(0)-i\sum_{ks}g_{ks}(p+i\omega_{ks})^{-1}a_{ks}(0) -\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\sum_j\tilde{a}_j(p)\ . \quad (5.6)$$

On summing both sides of (5.6) over all values of j, we find that

$$(p+i\omega_0)\tilde{D}(p)=D(0)-iN\sum_{ks}g_{ks}(p+i\omega_{ks})^{-1}a_{ks}(0) -N\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\tilde{D}(p)\ , \quad (5.7)$$

where $\tilde{D}(p)$ is the Laplace transform of the operator $D(t)$, defined by (3.16). Equation (5.7) is easily solved for $\tilde{D}(p)$. We find that

$$\tilde{D}(p)=\Big((p+i\omega_0)+N\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\Big)^{-1}D(0) -iN\Big((p+i\omega_0)+N\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\Big)^{-1} \times\sum_{ks}g_{ks}(p+i\omega_{ks})^{-1}a_{ks}(0)\ . \quad (5.8)$$

It is clear from (5.8) that

$$(p+i\omega_0)^{-1}\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\tilde{D}(p) =-\tilde{D}(p)/N+(1/N)(p+i\omega_0)^{-1}D(0) -i\sum_{ks}g_{ks}(p+i\omega_{ks})^{-1}(p+i\omega_0)^{-1}a_{ks}(0)\ . \quad (5.9)$$

On combining (5.6) and (5.9), we find that

$$\tilde{a}_j(p)=a_j(0)(p+i\omega_0)^{-1}+\tilde{D}(p)/N -(1/N)(p+i\omega_0)^{-1}D(0)\ . \quad (5.10)$$

We now consider spontaneous emission. The initial state of the radiation field is the vacuum state and hence $\langle a_{ks}(0)\rangle=0$. We obtain from (5.8), on taking the average values,

$$\langle\tilde{D}(p)\rangle=\Big((p+i\omega_0)+N\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\Big)^{-1}\langle D(0)\rangle\ . \quad (5.11)$$

On inverting the Laplace transform we find that

$$\langle D(t)\rangle=\langle D(0)\rangle f(t)e^{-i\omega_0 t}\ , \quad (5.12)$$

where

$$f(t)=\frac{1}{2\pi i}\oint dp\, e^{(p+i\omega_0)t} \times\Big((p+i\omega_0)+N\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\Big)^{-1}\ . \quad (5.13)$$

The results derived so far are exact, i.e., we have made no approximation on the strength of the coupling constants. If we now make the usual approximation,[9] i.e., replace

$$\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\ \text{ by }\ \sum_{ks}|g_{ks}|^2(-i\omega_0+i\omega_{ks})^{-1}$$

and ignore the frequency shifts, then (5.12) reduces to

$$\langle D(t)\rangle=\langle D(0)\rangle e^{-i\omega_0 t-N\gamma_0 t}\ . \quad (5.14)$$

This shows the simple decay behavior for the average value of the operator $D(t)$. It is worth noting that this result is the analog of the result (I2.23) of Paper I which was found for two-level atoms. From (5.10), we also deduce that

$$a_j(t)=a_j(0)e^{-i\omega_0 t}+(1/N)D(t)-(1/N)D(0)e^{-i\omega_0 t}\ , \quad (5.15)$$

and in particular

$$\langle a_j(t)\rangle=e^{-i\omega_0 t}\langle a_j(0)\rangle-(1/N)e^{-i\omega_0 t} \times(1-e^{-N\gamma_0 t})\langle D(0)\rangle\ . \quad (5.16)$$

It should be noted that the approximate results (5.14) and (5.16) are identical to the results found from the master equation (2.1). An alternate expresssion for $a_j(t)$ can also be obtained by noting that

$$(p+i\omega_0)^{-1}\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\tilde{D}(p) =\frac{1}{N}\left((p+i\omega_0)^{-1}-\frac{\langle\tilde{D}(p)\rangle}{\langle D(0)\rangle}\right) -i\sum_{ks}g_{ks}(p+i\omega_{ks})^{-1}a_{ks}(0)\left((p+i\omega_0)^{-1}-\frac{\langle\tilde{D}(p)\rangle}{\langle D(0)\rangle}\right)\ . \quad (5.17)$$

Then $\tilde{a}_j(p)$ is given by

$$\tilde{a}_j(p)=a_j(0)(p+i\omega_0)^{-1}-\frac{D(0)}{N}\left((p+i\omega_0)^{-1}-\frac{\langle\tilde{D}(p)\rangle}{\langle D(0)\rangle}\right) -i\sum_{ks}g_{ks}(p+i\omega_{ks})^{-1}a_{ks}(0)\frac{\langle\tilde{D}(p)\rangle}{\langle D(0)\rangle}\ . \quad (5.18)$$

We rewrite (5.18) as

$$\tilde{a}_j(p)=a_j(0)(p+i\omega_0)^{-1}-\frac{D(0)}{N}\left((p+i\omega_0)^{-1}-\frac{\langle\tilde{D}(p)\rangle}{\langle D(0)\rangle}\right) +\sum_{ks}a_{ks}(0)\tilde{\beta}_{ks}(p) \quad (5.19)$$

or

$$a_j(t)=a_j(0)e^{-i\omega_0 t}-(1/N)e^{-i\omega_0 t}D(0)+\frac{1}{N}\frac{\langle D(t)\rangle}{\langle D(0)\rangle}D(0)+\sum_{ks}a_{ks}(0)\beta_{ks}(t)\ . \quad (5.20)$$

Since the radiation field is initially in the vacuum state, the operator $a_j(t)$ is *effectively equivalent* to the following operator:

$$a_j(t)=a_j(0)e^{-i\omega_0 t}-\frac{1}{N}D(0)\left(e^{-i\omega_0 t}-\frac{\langle D(t)\rangle}{\langle D(0)\rangle}\right), \quad (5.21)$$

as far as the calculation of the normally ordered correlations is concerned. The result (5.21) is also an exact result with $\langle D(t)\rangle$ given by (5.12). The radiation rate, in situations when each of the oscillators was excited initially to a coherent state $|z_0\rangle$, is given by

$$I(t)=-N|z_0|^2\omega_0\frac{d}{dt}|f(t)|^2, \quad (5.22)$$

where $f(t)$ is given by (5.13). This result then leads to the result (I 5.14) of Paper I, if the approximate result given by (5.14) is used. We again stress the fact that all the approximations have to be made in the calculation of the function $f(t)$.

We now derive these results from the master-equation approach. We derive a non-Markoffian equation describing spontaneous emission and we do make the Born approximation. The master equation so derived leads to the result (5.21). Let $\tilde{F}_s^{(A)}(\{z_i\}, \{z_i^*\}; p)$ be the Laplace transform of $F_s^{(A)}(\{z_i\}, \{z_i^*\}; t)$. It can be shown that $\tilde{F}_s^{(A)}(\{z_i\},\{z_i^*\}; p)$ satisfies the following equation:

$$[p\tilde{F}_s^{(A)}(\{z_i\},\{z_i^*\};p)-F_s^{(A)}(\{z_i\},\{z_i^*\};0)]=\left(\sum_{ij}\frac{\partial}{\partial z_i}[z_j\tilde{F}_s^{(A)}(\{z_i\},\{z_i^*\};p)]\times\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}+\text{c.c.}\right)\ . \quad (5.23)$$

It should be noted $F_S^{(A)}$ refers to the phase-space distribution function in the interaction picture. The master equation (5.23) has been obtained in Born approximation. In principle, we must retain all the higher-order terms. We conjecture that when this is done the equivalent Langevin equations of motion will be given by

$$[p\tilde{z}_i(p)-z_i(0)]=-\sum_j\tilde{z}_j(p)\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\ . \quad (5.24)$$

It is clear from Eq. (5.24) that

$$p\sum_i\tilde{z}_i(p)=\sum_i z_i(0)-N\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\sum_j\tilde{z}_j(p)\ . \quad (5.25)$$

This equation may be rewritten in the form

$$\langle\tilde{D}(p)\rangle=\langle D(0)\rangle\left(p+N\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}\right)^{-1}\ . \quad (5.26)$$

On using (5.25) and (5.26), it may be shown that

$$\sum_j\tilde{z}_j(p)p^{-1}\sum_{ks}|g_{ks}|^2(p+i\omega_{ks})^{-1}=\frac{1}{N}\sum_j z_j(0)\left(p^{-1}-\frac{\langle\tilde{D}(p)\rangle}{\langle D(0)\rangle}\right). \quad (5.27)$$

On combining (5.24) and (5.27), we finally find that

$$\tilde{z}_i(p)=p^{-1}z_i(0)-\frac{1}{N}\left(p^{-1}-\frac{\langle\tilde{D}(p)\rangle}{\langle D(0)\rangle}\right)\sum_j z_j(0)\ . \quad (5.28)$$

On taking the Laplace transform of (5.28), we obtain the formula

$$z_i(t)=z_i(0)-\frac{1}{N}\left(1-\frac{\langle D(t)\rangle}{\langle D(0)\rangle}\right)\sum_j z_j(0)\ . \quad (5.29)$$

The results (5.26) and (5.29) are identical with (5.12) and (5.21), respectively.[10]

VI. SPONTANEOUS EMISSION INTO SINGLE MODE

In this section, we will consider the problem of spontaneous emission by a system of oscillators into a single mode. The corresponding problem for the case of two-level atoms has been investigated in great detail[11,12] recently. The interaction Hamiltonian in the interaction picture is given by

$$H_I=\left(g\sum_{\delta=1}^{N}a_j^\dagger a+\text{H.c.}\right), \quad (6.1)$$

where g is the coupling constant. a and $a^\dagger$ are the annihilation and the creation operators for the radiation-field mode. The antinormally ordered equivalent of the density operator satisfies the following master equation[13]:

$$i\frac{\partial F^{(A)}}{\partial t}=\left(g\sum_{j=1}^{N}z_j^*z+\text{c.c.}\right)\times\left[\exp\left(-\frac{\overleftarrow{\partial}}{\partial z^*}\frac{\overrightarrow{\partial}}{\partial z}-\sum_{j=1}^{N}\frac{\overleftarrow{\partial}}{\partial z_j^*}\frac{\overrightarrow{\partial}}{\partial z_j}\right)\right]F^{(A)}-\text{c.c.}\ ,$$

which on simplification reduces to

$$\frac{\partial F^{(A)}}{\partial t}=\sum_j\left(ig^*z_j\frac{\partial}{\partial z}+igz\frac{\partial}{\partial z_j}+\text{c.c.}\right)F^{(A)}\ . \quad (6.2)$$

The corresponding Langevin equations are

$$\dot{z}=-ig^*\sum_{j=1}^{N}z_j\ , \quad (6.3)$$

$$\dot{z}_j=-igz\ . \quad (6.4)$$

From (6.3) and (6.4), it is easily shown that

$$z(t)=z(0)\cos[|g|(N)^{1/2}t] - \frac{ig^*}{(N)^{1/2}|g|}\sin[|g|(N)^{1/2}t]\sum_{j=1}^{N} z_j(0) \ , \quad (6.5)$$

$$z_i(t)=z_i(0)-\frac{ig\sin[|g|(N)^{1/2}t]}{|g|(N)^{1/2}}z(0) - (1/N)[1-\cos|g|(N)^{1/2}t]\sum_{j=1}^{N} z_j(0) \ . \quad (6.6)$$

The solutions (6.5) and (6.6) may be used to calculate the normally ordered correlation functions. We first assume that the initial state is given by

$$F^{(A)}(\{z_i\},\{z_i^*\};\ z,z^*;0) = \prod_{j=1}^{N}[\pi\delta^{(2)}(z_j-z_0)][\pi\delta^{(2)}(z)] \ . \quad (6.7)$$

It is then easily shown on using (6.5)–(6.7), that

$$\langle a^\dagger(t)a(t)\rangle = N|z_0|^2\sin^2[|g|(N)^{1/2}t], \quad (6.8)$$

$$\textstyle\sum_j \langle a_j^\dagger(t)a_j(t)\rangle = N|z_0|^2\cos^2[|g|(N)^{1/2}t]. \quad (6.9)$$

From (6.8) and (6.9), it is evident that the energy oscillates between the oscillators and the field. The radiation rate in this case is given by

$$I_C(t) = -\omega_0\frac{d}{dt}\textstyle\sum_j\langle a_j^\dagger(t)a_j(t)\rangle = |z_0|^2\omega_0|g|N^{3/2}\sin[2|g|(N)^{1/2}t]. \quad (6.10)$$

One may similarly compute the radiation rate when each of the oscillators was initially excited to a state described by (2.9) and it is found that

$$I_F(t)=\omega_0|g|\sin[2|g|(N)^{1/2}t]\frac{1}{(N)^{1/2}}\sum_j\langle a_j^\dagger(0)a_j(0)\rangle. \quad (6.11)$$

On comparing (6.10) and (6.11), we find that

$$I_C(t)=NI_F(t) \ . \quad (6.12)$$

We thus conclude that the radiation rate from Fock-state excitation is N times smaller than that from a coherent state excitation. This conclusion is again reminiscent of the phenomenon of superradiance.

From (6.5), we also see that the normally ordered moments are given by

$$\langle a^{\dagger m}(t)a^n(t)\rangle = \left(\frac{ig}{(N)^{1/2}|g|}\right)^m\left(\frac{-ig^*}{(N)^{1/2}|g|}\right)^n \times\sin^{m+n}[|g|(N)^{1/2}t]\left\langle\left(\textstyle\sum_i a_i^\dagger(0)\right)^m\left(\textstyle\sum_j a_j(0)\right)^n\right\rangle,$$

which for the case of coherent state excitation reduces to

$$\langle a^{\dagger m}(t)a^n(t)\rangle = \left(\frac{ig}{(N)^{1/2}|g|}\right)^m\left(\frac{-ig^*}{(N)^{1/2}|g|}\right)^n \times\sin^{m+n}[|g|(N)^{1/2}t](z_0^*)^m(z_0)^nN^{m+n}. \quad (6.13)$$

In particular, we have from (6.13) the following result for even-ordered moments:

$$\langle a^{\dagger m}(t)a^m(t)\rangle = N^m|z_0|^{2m}\sin^{2m}[|g|(N)^{1/2}t] = [\langle a^\dagger(t)a(t)\rangle]^m. \quad (6.14)$$

A result of the form (6.14) was also exhibited recently by Birula[11] for the case of superradiant excitation of two-level atoms, the odd-ordered moments vanished for this excitation, whereas for an oscillator system they are finite. In fact, it is obvious from (6.13) that the reduced phase-space distribution function for the field mode, denoted by $F_0^{(A)}(z,z^*;t)$, is given by

$$F_0^{(A)}(z,z^*;t)=\pi\delta^{(2)}[z-z'], \quad (6.15a)$$

where

$$z'=\frac{-ig^*}{(N)^{1/2}|g|}\sin[|g|(N)^{1/2}t]Nz_0. \quad (6.15b)$$

We thus conclude that the field is found in a coherent state $|z'\rangle$ when all the oscillators were excited initially to a coherent state $|z_0\rangle$. Similarly, it can be shown that the reduced phase-space distribution function for the oscillator system is given by

$$F_S^{(A)}(\{z_i\},\{z_i^*\};t)=\prod_{i=1}^{N}\{\pi\delta^{(2)}[z-z_0\cos|g|(N)^{1/2}t]\}. \quad (6.16)$$

The striking similarity between the result (6.16) and (3.22) is worth noting.

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[5]Compare Y. C. Lee and D. L. Lin, Phys. Rev. 183, 147 (1969).

[5a]This decomposition in positive- and negative-frequency parts is justified because the annihilation operator $a_{ks}(t)$ for the present problem is of the form

$$a_{ks}(t)=\sum_{k_1s_1}u_{ks,k_1s_1}(t)\,a_{k_1s_1}(0)+\sum_j v_{rs,j}(t)a_j(0).$$

[6]A relation similar to (4.15) has been derived by Lehmberg (Ref. 4), who used the solution to Heisenberg equations of motion. A similar approach has also been employed by J. H. Eberly and N. E. Rehler, Phys. Rev. A 2, 1607 (1970).

[7]See, for example, M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1970), 4th ed., p. 83.

[8]L. Mandel and E. Wolf, Rev. Mod. Phys. 37, 231 (1965).

[9]See, for example, W. Heitler, *Quantum Theory of Radiation* (Oxford U. P., Cambridge, England, 1954).

[10]The non-Markoffian effects for the case of one oscillator interacting with a heat bath have been investigated recently by F. Haake, Z. Physik 223, 353 (1969).

[11]Z. Bialynicka-Birula, Phys. Rev. D 1, 400 (1970).

[12]R. Bonifacio and G. Preparata, Phys. Rev. A 2, 336 (1970).

[13]G. S. Agarwal and E. Wolf, Phys. Rev. Letters 21, 180 (1968); Phys. Rev. D 2, 2187 (1970).

PHYSICAL REVIEW A VOLUME 3, NUMBER 5 MAY 1971

COMMENTS AND ADDENDA

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Validity of the Potentials of Hg†

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The scattering of 300-eV electrons from atomic Hg was used to study the validity of some recently published analytical potentials. Comparison with experimental data showed rather poor agreement with a potential derived on a nonrelativistic basis and good agreement with a potential derived on a relativistic basis. The best agreement has been achieved by adding to the scattering potential an exchange contribution due to exchange between the incoming electron and the target electrons. Experiment and theory are found to deviate significantly at the minima in the differential cross section.

There are several ways to study the validity of a potential of an atom. In general, agreement between the experimentally determined bound states of the system and the calculated ones is considered as a very good indication of how close the calculated potential comes to the true one. Recently, Green *et al.*[1] have reversed this procedure. They have chosen an analytic function which contains two free parameters for the potential and which has a hydrogenlike behavior at large distances. Similar approximations to achieve analytic potentials are known in nuclear physics as optical potentials. The parameters were determined so that the energy levels of the bound states of the atoms were reproduced. The disadvantage of the parameters published in Ref. 1 is that they are based on a nonrelativistic approach which cannot be considered reliable for an atom like Hg. Using a relativistic treatment, Darewych *et al.*[2] have repeated the adjustment of the same analytic potential to the energy levels measured by electron spectroscopy for chemical analysis,[3] again by fitting two parameters.