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Dynamics of a Q-Switched Laser near Threshold*

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The time evolution of the optical field produced by a Q-switched He:Ne laser has been investigated experimentally, in the neighborhood of the threshold of oscillation. The experiment was based on a measurement of the variation of the photoelectric counting probability with time, following the turn-on of the laser. Q switching was achieved with the help of an external mirror to extinguish the laser, and a Pockels cell to switch out the mirror. The working point of the laser could be held anywhere in the threshold region, by use of a feedback arrangement which controlled the cavity length, and therefore the atomic gain. Results are presented showing the evolution of the mean light intensity, the variance of the light intensity, and the counting probability with time, for several different values of the laser pump parameter. The results are found to be in very good agreement with the calculations of Risken and Vollmer, based on a rotating-wave van der Pol oscillator model of the laser.

I. INTRODUCTION

Although many investigations have been concerned with the operation of the laser under steady-state conditions, much less effort seems to have been devoted to the study of its transient characteristics. These characteristics become particularly important for a repetitively pulsed or Q-switched laser. There appears to have been only one experiment directed towards the measurement of the statistical properties of the emitted light as a function of time[1] under pulsed conditions, although a number of authors have examined the growth of the mean light intensity.[1–6] The corresponding theoretical problem has been tackled by several authors,[7–12] particularly Risken and Vollmer,[9] who have presented curves showing the evolution of the statistical features after the laser is turned on. But while most of the theoretical treatments have been applicable to the situation where the final steady state is not too far above the threshold of oscillation, very few dynamical measurements appear to have been carried out in this region. As a result, some of the most sensitive predictions of the theory have not so far been tested.

Since most lasers operate far above threshold in the saturation region, and tend towards this final state when Q switched, this situation is perhaps not altogether surprising. However, with a certain amount of elaboration it is quite possible to control a laser so that its steady state is in the threshold region, and to examine its characteristics.

By using the same optical feedback control system as described previously,[13–15] together with an optical shutter acting as Q switch, we have been able to study the transient characteristics of a Q-switched He:Ne laser near threshold. The investigation was based on photoelectric counting measurements, and the evolution of the counting probability with time after the laser is turned on. The counting probability, and the corresponding moments, are simply related to the probability distribution and the moments of the laser light intensity, whose time evolution was evaluated by Risken and Vollmer on the basis of a rotating-wave nonlinear oscillator model of the laser.[9] We are therefore able to make a fairly direct test of the theory. The results are found to be in very good agreement with predictions based on Risken and Vollmer's calculation.

We begin by outlining the theoretical background and the principle of the method. We then describe the apparatus, and the method used to determine certain key parameters. A number of corrections for dead time, background light, and finite counting time are required, and these are described in some detail. The results include the time evolution of the first two moments of the light intensity[16] and of the photon counting probability itself.

II. PRINCIPLE OF METHOD

Risken and Vollmer[9] have studied the transient behavior of a laser which is not too far from threshold, in terms of the rotating-wave van der Pol oscillator model. By solving the associated Fokker-Planck equation, they were able to calculate the probability distribution of the instantaneous light intensity at various times after the laser was turned on. While their treatment is semiclassical, rather similar equations are obtained from a fully quantum-mechanical treatment in terms of the phase-space representation of the optical field,[17] when the mean number of photons in the cavity near threshold is large.[18]

Let us denote the complex amplitude of the optical field (assumed polarized) at time t by $V(t)$, and write

$$V(t)=[(I)e^{-i\omega_0 t+i\phi}]^{1/2}, \tag{1}$$

where $I(t)$ is the instantaneous light intensity, ω_0 is the midfrequency of the light, and $\phi(t)$ is the instantaneous phase. If $W(I,\phi,t,a)$ is the joint probability density of I and ϕ at time t, the Fokker-Planck equation for W may be written[9]

$$\frac{\partial W}{\partial \tilde{t}}=-\frac{\partial}{\partial \tilde{I}}\{W[2\tilde{I}(a-\tilde{I})+4]\}+\frac{\partial^2}{\partial \tilde{I}^2}[4W\tilde{I}]+\frac{1}{\tilde{I}}\frac{\partial^2 W}{\partial\phi^2}, \tag{2}$$

while the associated Langevin equation is given by

$$\frac{\partial \tilde{I}}{\partial \tilde{t}}=2\tilde{I}(a-\tilde{I})+4+N(\tilde{t}), \tag{3}$$

in which $N(\tilde{t})$ is a δ-correlated noise current, which is more fully described below. Here a is a parameter, the so-called pump parameter, which is characteristic of the excitation of the laser in the steady state. It has the value zero at threshold. $\tilde{I}$ and $\tilde{t}$ are the light intensity and time expressed in dimensionless units, which can be defined via the relations

$$\tilde{I}(t,a)\equiv I(t,a)\langle\tilde{I}(t=\infty,\ a=0)\rangle/\langle I(t=\infty,\ a=0)\rangle \tag{4}$$

and

$$\tilde{t}\equiv t\tilde{T}_c(a=0)/T_c(a=0), \tag{5}$$

in which $T_c(a)$ is the intensity correlation time. The numerical values of $\tilde{I}(t=\infty,\ a=0)$ and $\tilde{T}_c(a=0)$ are given below.

The stationary $(t\to\infty)$, phase-independent solution of Eq. (2) is easily found to be

$$W(\tilde{I},\ t=\infty,a)=C\,e^{-\frac{1}{4}\tilde{I}^2+\frac{1}{2}a\tilde{I}}, \tag{6}$$

where C is a constant that ensures the normalization of W. From Eq. (6) the first two moments of the stationary distribution are

$$\langle\tilde{I}(t=\infty,a)\rangle=a+\frac{2e^{-a^2/4}}{(\sqrt{\pi})[1+\Phi(\frac{1}{2}a)]}, \tag{7}$$

$$\langle\tilde{I}^2(t=\infty,a)\rangle=2+a^2+\frac{2ae^{-a^2/4}}{(\sqrt{\pi})[1+\Phi(\frac{1}{2}a)]}, \tag{8}$$

where $\Phi(x)$ is the Gaussian error integral. When these expressions are evaluated at threshold, $a=0$, we obtain

$$\langle\tilde{I}(t=\infty,\ a=0)\rangle=2/\sqrt{\pi}\approx 1.128, \tag{9}$$

$$\langle\tilde{I}^2(t=\infty,a=0)\rangle/\langle\tilde{I}(t=\infty,a=0)\rangle^2=\tfrac{1}{2}\pi\approx 1.571. \tag{10}$$

Equation (10) shows that the threshold of the laser can be identified by the relative fluctuations of the light intensity.

From the Green's function associated with the Fokker-Planck equation (2), it is also possible to derive the normalized intensity correlation function $\lambda(\tilde{\tau},a)$, defined by

$$\lambda(\tilde{\tau},a)\equiv\langle\Delta\tilde{I}(\tilde{t},a)\Delta\tilde{I}(\tilde{t}+\tilde{\tau},a)\rangle/\langle[\Delta\tilde{I}(\tilde{t},a)]^2\rangle, \tag{11}$$

for a laser described by Eq. (2) which is operating in its steady state. $\lambda(\tilde{\tau},a)$ can be expressed as a sum of exponential terms in the form[8,9]

$$\lambda(\tilde{\tau},a)=\sum_{r=1}^{\infty}M_r(a)\,e^{-\lambda_{0r}(a)\tilde{\tau}}, \tag{12}$$

and the results of numerical calculations of $\lambda_{0r}(a)$ and $M_r(a)$ for various r values and various pump parameters a have been published.[8,9] The intensity correlation time $\tilde{T}_c(a)$ associated with the foregoing correlation function $\lambda(\tilde{\tau},a)$ can be defined by

$$\tilde{T}_c(a)\equiv\int_0^{\infty}\lambda(\tilde{\tau},a)\,d\tilde{\tau},$$

and from Eq. (12)

$$\tilde{T}_c(a)=\sum_{r=1}^{\infty}M_r(a)/\lambda_{0r}(a). \tag{13}$$

From the published values for $\lambda_{0r}(a)$ and $M_r(a)$, $\tilde{T}_c(a)$ can be evaluated, and we find, at threshold,

$$\tilde{T}_c(a=0)\approx 0.171. \tag{14}$$

With the help of Eqs. (9) and (14), Eqs. (4) and (5) become

$$\tilde{I}(t,a)=1.128\,I(t,a)/\langle I(t=\infty,a=0)\rangle, \tag{15}$$

$$\tilde{t}=0.171t/T_c(a=0). \tag{16}$$

While the light intensity $I(t,a)$ is not measured directly, it is simply related to the output of a photoelectric detector which is illuminated by the laser beam. For a time interval t to $t+T$, which is sufficiently short that the instantaneous light intensity $I(t,a)$ does not change significantly, the probability $p(n,t,T,a)$ that $n(t,T,a)$ photoelectric counts are registered during the interval t to $t+T$, is related to the probability density $W(I,t,a)$ by the formula[19]

$$p(n,t,T,a)=\int_0^{\infty}[\alpha I(t,a)T]^n/ \times(n!)e^{-\alpha I(t,a)T}W(I,t,a)\,dI. \tag{17}$$

Here α is a constant characteristic of the detector. It follows immediately from this equation that the moments of $n(t,T,a)$ and $I(t,a)$ are related by

$$\langle n(t,T,a)\rangle = \alpha T\langle I(t,a)\rangle \ , \tag{18}$$

$$\langle n(t,T,a)[n(t,T,a)-1]\rangle = (\alpha T)^2\langle I^2(t,a)\rangle \ . \tag{19}$$

The normalized second factorial moment of the counting distribution is therefore independent of the intensity scaling, the counting time T, and the detector efficiency α, and we can write

$$\frac{\langle n(t,T,a)[n(t,T,a)-1]\rangle}{\langle n(t,T,a)\rangle^2}=\frac{\langle I^2(t,a)\rangle}{\langle I(t,a)\rangle^2} \ . \tag{20}$$

In particular, it follows from Eq. (10) that the threshold $(a=0)$ of the laser can be readily identified from the steady-state ratio of the moments[20]

$$\frac{\langle n(t=\infty,T,\ a=0)[n(t=\infty,T,\ a=0)-1]\rangle}{\langle n(t=\infty,T,a=0)\rangle^2} = \tfrac{1}{2}\pi\approx 1.571 \ . \tag{21}$$

We see, therefore, that the moments of the light intensity $I(t,a)$ are simply related to, and derivable from, the moments of the photoelectric counts registered by a photodetector. Since Risken and Vollmer[9] have presented curves showing the variation of $\langle(\tilde{I},\tilde{t},a)\rangle$, $\langle[\Delta\tilde{I}(\tilde{t},a)]^2\rangle$, and $W(\tilde{I},\tilde{t},a)$ with $\tilde{t}$ for several values of a, we can test the theory by carrying out photoelectric counting measurements. The scaling parameters required in Eqs. (4) and (5) are also readily derived. With the laser set to operate at threshold, the steady-state mean light intensity $\langle I(t=\infty,a=0)\rangle$ follows from the data with the help of Eq. (18), while the intensity correlation time $T_c(a=0)$ can be determined from a separate measurement, as described below.

The transient phase-independent solutions of Eq. (2) for $W(\tilde{I},\tilde{t},a)$, unlike the steady-state ones, are rather complicated, but can be expressed in terms of the eigensolution of a certain associated Schrödinger equation.[8,9] For short times $\tilde{t}$, the solution can be approximated by

$$W(\tilde{I},\tilde{t},a)=\frac{a}{2(e^{2a\tilde{t}}-1)}\ \exp\left(\frac{-a\tilde{I}}{2(e^{2a\tilde{t}}-1)}\right), \tag{22}$$

which implies

$$\langle\tilde{I}(\tilde{t},a)\rangle\approx 4\tilde{t} \tag{23}$$

for $a\tilde{t}\ll 1$. Risken and Vollmer[9] have presented computed forms of $W(\tilde{I},\tilde{t},a)$ for certain combinations of $a(=0,4,8)$ and $\tilde{t}$. We have taken advantage of these calculations and made measurements with the laser set to operate at the same values of the pump parameter a.

The results of these measurements give the probabilities $p(n,t,T,a)$ and the moments of $n(t,T,a)$ for various combinations of t and a. However, Eq. (17) is not easily inverted to yield $W(I,t,a)$ from $p(n,t,T,a)$. Although attempts have been made to invert the equation,[21] some features of $W(I,t,a)$ are extremely sensitive to small inaccuracies in the tail of $p(n,t,T,a)$. We have therefore chosen to derive only the first two moments of the light intensity from the measurements, via Eqs. (18) and (20), and to compare the measured forms of $p(n,t,T,a)$ with those derived from Eq. (17) with the help of Risken and Vollmer's calculations.

III. EXPERIMENTAL ARRANGEMENT AND PROCEDURE

A block diagram of the experimental arrangement is shown in Fig. 1. The light source for this experiment was a single-mode He : Ne laser (Spectra Physics model 119) operating at 6328 Å. The output intensity of this laser was stabilized with the help of a feedback arrangement, consisting of a monitor phototube, operational amplifier, and piezoelectric mount for mirror M_2, as described previously.[13-15] We gained considerable experience with this system during the course of a number of measurements of the stationary laser characteristics.[22] These measurements demonstrated that the working point of the laser could be stabilized and held anywhere from well below to well above threshold.

In order to measure the statistical properties of the laser output intensity, the beam was allowed to fall on a fast-counting phototube (cooled in order to reduce dark current). The output pulses of this tube were amplified, shaped by a discriminator, and counted by a scaler. The discriminator was disabled except for the duration of a gate pulse. At the end of the gate pulse, the number of counts n stored in the scaler was used to select the corresponding channel number n in a 100-channel analyzer, and the counting logic caused the analyzer to add one unit to that channel. The scaler was then reset and the counting system was prepared for another counting interval. After a large number N of counting intervals, the number accumulated in channel n of the analyzer provided a measure of $Np(n)$.

In order to examine the transient characteristics of the laser, a third, external mirror M_3 was introduced on the laser axis, as shown in Fig. 1, and was aligned so as to form a stable Fabry-Perot cavity with mirror M_2. M_3 was mounted on a magnetostrictive device which allowed its spacing rel-

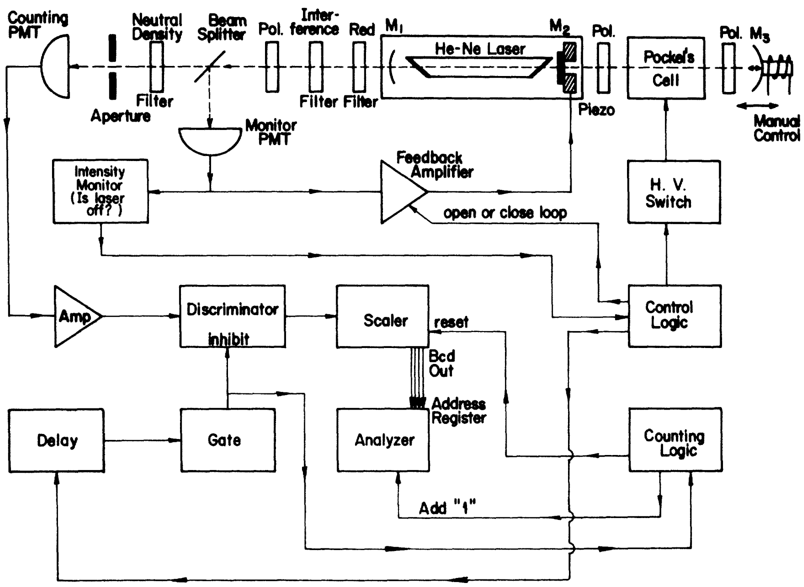

FIG. 1. Block diagram of the apparatus.

ative to M_2 to be varied. Movement of the external mirror led to a modulation of the effective optical cavity Q of the laser, and therefore of its light output. Such an arrangement has previously been used to modulate the output of a laser.[23] A Pockels cell and two polarizers were introduced into the external cavity, and these, together with the associated high-voltage switch, acted as a high-speed optical shutter, which allowed the external mirror to be switched in or out, and thereby permitted substantial and rapid changes of the laser cavity Q. The spacing between M_2 and M_3 was adjusted so as to minimize the light intensity with the shutter open, when the laser was almost (but not completely) extinguished, while the laser was stabilized in the threshold region with the shutter closed. Because the extinction was not quite complete, a small correction to the data was required, as described in Sec. IV.

The sequence of events for one measurement begins with the shutter closed, while the feedback amplifier stabilizes the laser at the desired operating level. The optical shutter is then opened and the laser is effectively extinguished. When the intensity monitor determines that the laser is extinguished, the shutter is suddenly closed, and the laser output rises from far below threshold to the steady-state operating level. The effective switching time (about 50 nsec) is negligible compared with the characteristic rise time (~ 100 μsec) of the laser. The laser is allowed to dwell in its steady state for about 20 msec following the turn-on, when it is again extinguished, and the switching cycle is repeated about 30 times per second. The feedback loop controlling the laser operates only during the 20-msec dwell time in the steady state.

A delay pulse of adjustable duration is initiated by the laser switch-on pulse. The end of this pulse triggers a standard-width gate pulse of 2-μsec duration (short compared with the characteristic 100-μsec rise time of the laser), which allows the phototube pulses to be counted and stored as described above. By switching the laser many times in this way, it is possible to explore the variation of the photon counting probability with time following the switch-on, and with the steady-state operating level.

The same experimental arrangement could be and was used to measure the steady-state photoelectric counting statistics of the laser. In that case the optical shutter was kept closed, the laser was stabilized continuously via the feedback loop, and the frequency and width of the gate pulse were varied as desired.

The first step in the experiment was to identify the threshold region of the laser, in order to determine $\langle n(t=\infty, T, a=0)\rangle$ and $T_c(a=0)$, and thereby make contact with the calculations of Risken and Vollmer.[9] This was done by adjusting the cavity length, with the laser in the steady state, until the first and second moments of the photoelectric counts satisfied condition (21). In practice, it was possible to set the laser at threshold with an uncertainty in the threshold intensity of about 1%.

With the laser set at threshold, the intensity correlation time $T_c(a=0)$, which is needed for the time-scaling formula (16), was determined by the method described previously.[15] This involved measuring the steady-state normalized second-order factorial moment of $n(t=\infty, T, a=0)$ for various values of T greater than $T_c(a=0)$, and making an extrapolation to $T=0$. For the purpose of this auxiliary experiment, the light intensity was reduced by insertion of a neutral density filter, which made dead-time correction (as described in Sec. IV) unnecessary. In practice, $T_c(a=0)$ could be determined with a statistical uncertainty of about 2%, which represents the limit of the accuracy of $\bar{t}$.

In order to adjust the laser for other values of the pump parameter a, it was necessary only to change the cavity length while the laser was operating in the steady state, until the mean counting rate changed in the ratio required by Eqs. (7) and (18).

IV. CORRECTIONS TO EXPERIMENTAL DATA

Three corrections to the data were necessary in practice, before information about the light intensity could be derived from the measured photoelectric counts. These corrections allowed for the effects of dead time in the counting circuits (~ 10 nsec), for background light from the laser gas discharge, and for the finite counting time interval. The measured moments were corrected for these various effects in the order mentioned.

The need for a finite counting time correction arises because the measurements were made with a counting interval $T=2$ μsec which is not completely negligible compared with the correlation time T_c of the laser, whereas Eqs. (17)–(19) are strictly valid only for $T \ll T_c$. The more general formula for $p(n, t, T, a)$ is [19,24]

$$p(n, t, T, a) = \int_0^\infty \frac{1}{n!} \left[\alpha U(t, T, a)\right]^n \times e^{-\alpha U(t,T,a)} \mathcal{P}(U, t, T, a)\, dU , \tag{24}$$

in which $U(t, T, a)$ is the time-integrated light intensity defined by

$$U(t, T, a) \equiv \int_t^{t+T} I(t', a)\, dt' , \tag{25}$$

and $\mathcal{P}(U, t, T, a)$ is its probability density. From Eq. (24) we may readily show that the rth moments of $n(t, T, a)$ and $U(t, T, a)$ are related by

$$\langle n(t, T, a)[n(t,T,a)-1]\cdots[n(t, T, a)-r+1]\rangle = \alpha^r \langle U^r(t, T, a)\rangle . \tag{26}$$

Equations (18) and (19) can be seen to be special cases of this equation when T is sufficiently short.

The effect of finite counting time has previously been discussed for stationary fields, whose intensity correlation function is known.[25] In particular, if the steady-state normalized intensity correlation function is of the multiple exponential form of Eq. (12), the second moment of the counting distribution varies with the counting time T according to the relation[15]

$$g(T)/g(0) = 2\sum_{r=0}^{\infty} (M_r/\lambda_{0r}^2 T^2)\left[\lambda_{0r} T + e^{-\lambda_{0r} T} - 1\right] = 1 - \sum_{r=0}^{\infty} \left(\tfrac{1}{3} M_r \lambda_{0r} T\right)\left[1 - \tfrac{1}{4}\lambda_{0r} T + \cdots\right] , \tag{27}$$

where

$$g(T) \equiv [\langle n^2(t, T, a)\rangle - \langle n(t, T, a)\rangle^2 - \langle n(t, T, a)\rangle]/\langle n(t, T, a)\rangle^2 . \tag{28}$$

In order to examine the effects of finite counting time on a measurement of the nonstationary system, as for a Q-switched laser, it is convenient to go back to the basic Langevin equation (3). We may then show (see Appendix) that the second moments of $n(t, T, a)$, $U(t, T, a)$, and $I(t, a)$ are related via the formula

$$\frac{\langle n(\tilde{t}_0, \tilde{T}, a)[n(\tilde{t}_0, \tilde{T}, a)-1]\rangle}{\langle n(\tilde{t}_0, \tilde{T}, a)\rangle^2} = \frac{\langle U^2(\tilde{t}_0, \tilde{T}, a)\rangle}{\langle U(\tilde{t}_0, \tilde{T}, a)\rangle^2} = \frac{\langle \tilde{I}^2(\tilde{t}_0 + \frac{1}{2}\tilde{T})\rangle}{\langle \tilde{I}(\tilde{t}_0 + \frac{1}{2}\tilde{T})\rangle^2} - \frac{4\tilde{T}}{3\langle \tilde{I}(\tilde{t}_0 + \frac{1}{2}\tilde{T})\rangle} + O[\tilde{T}^2] . \tag{29}$$

In this experiment the counting time $\tilde{T}$ in normalized units was less than 10^{-2}, and the $O[\tilde{T}^2]$ term in Eq. (29) could be neglected. In practice, the correction term $4\tilde{T}/3\langle \tilde{I}(\tilde{t}_0 + \frac{1}{2}\tilde{T})\rangle$ was calculated from the experimentally derived value of $\tilde{I}(\tilde{t}_0 + \frac{1}{2}\tilde{T})\rangle$. It was added to the normalized second factorial moment of n (which had previously been corrected for dead time and background light), to yield $\langle \tilde{I}^2(\tilde{t}_0 + \frac{1}{2}\tilde{T})\rangle/\langle \tilde{I}(\tilde{t}_0 + \frac{1}{2}\tilde{T})\rangle^2$.

It is not obvious how to correct $p(n, t, T, a)$ for the finite counting time T, and we have not done so. Since the finite counting time correction is effectively zero for the first moment, and was found to be always less than the statistical uncertainty for the second moment, the correction for $p(n, t, T, a)$ should be small also, except possibly in the tail of the distribution, where $p(n, t, T, a)$ is very small.

The background light from the discharge in the plasma tube is statistically independent of the laser light of interest. Also, the bandwidth of the background intensity fluctuations is very large compared with the reciprocal of the counting time, so that the background light alone leads to stationary, Poissonian counting statistics.[24] For these reasons, if we denote by $U_L(t, T, a)$ and $U_B(T)$ the integrated light intensities due to the laser and the background light, respectively, we can write[15]

$$U(t, T, a) = U_L(t, T, a) + U_B(T) , \tag{30}$$

in which the fluctuations of U_B are negligible compared with those of U_L. It therefore follows that

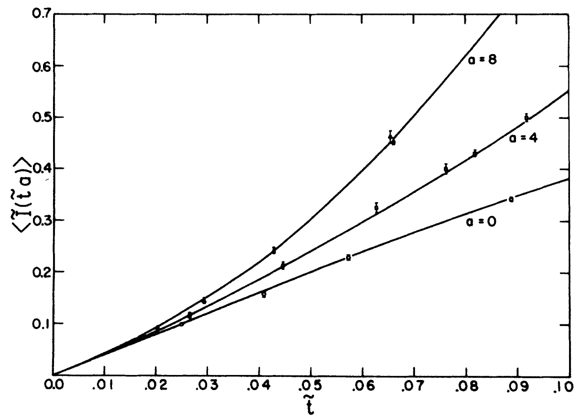

FIG. 2. Variation of the mean light intensity with time, in normalized units, for three different values of the pump parameter a. Error bars represent statistical uncertainties of the experimentally determined $\langle \tilde{I}(\tilde{t}, a)\rangle$ values. The full curves were computed by Risken and Vollmer (Ref. 9).

$$\mathcal{P}(U,t,T,a)=\mathcal{P}_L[U(t,T,a)-U_B(T)]\ , \tag{31}$$

$$\langle n_B(T)\rangle=\alpha U_B(T)\ , \tag{32}$$

$$\langle n_L(t,T,a)\rangle=\alpha\langle U_L(t,T,a)\rangle\ , \tag{33}$$

with an obvious extension of the notation. From these relations and Eq. (24), we have

$$p(n,t,T,a)=\int_0^\infty \frac{1}{n!}\,[\alpha U_L+\langle n_B(T)\rangle]^n \times e^{-\alpha U_L-\langle n_B(T)\rangle}\,\mathcal{P}_L(U_L,t,T,a)\,dU_L\ , \tag{34}$$

and

$$\langle n(t,T,a)\rangle=\langle n_L(t,T,a)\rangle+\langle n_B(T)\rangle\ , \tag{35}$$

$$\langle[\Delta n(t,T,a)]^2\rangle=\langle[\Delta n_L(t,T,a)]^2\rangle+\langle n_B(T)\rangle\ , \tag{36}$$

from which corrections for the effect of background light can be made. $\langle n_B(T)\rangle$, the mean number of photoelectric counts due to the background, is determined from a separate measurement, with the laser cavity detuned as far as possible. In practice, the parameter α is adjusted to make the mean of the measured distribution $p(n,t,T,a)$ coincide with that given by Eq. (34).

The problem of correction for the dead time of the counting circuits has been treated by several authors.[25–27] If δ is the ratio of dead-time to the counting time T, then, up to terms of the first order in δ, Eq. (34) has to be modified to read

$$p(n,t,T,a)=\int_0^\infty \frac{1}{n!}\,[\alpha U_L+\langle n_B(T)\rangle]^n e^{-\alpha U_L-\langle n_B(T)\rangle} \times[1+n\delta(\alpha U_L+\langle n_B(T)\rangle+1-n)] \times\mathcal{P}_L(U_L,t,T,a)\,dU_L\ . \tag{37}$$

It follows from this that the numbers of counts n' and n before and after dead-time correction, respectively, are related by

$$\langle n\rangle=\langle n'\rangle+\langle n'(n'-1)\rangle\delta\ , \tag{38}$$

$$\langle n(n-1)\rangle=\langle n'(n'-1)\rangle(1+2\delta) +\langle n'(n'-1)(n'-2)\rangle 2\delta\ . \tag{39}$$

As was mentioned earlier, the extinction produced by the external mirror and the Pockels cell was not quite complete, so that, at the beginning of the turn-on, $\langle \tilde{I}(\tilde{t}=0,a)\rangle$ was not strictly zero (although it was always less than 0.1 in normalized units). We confirmed from the counting statistics that the probability density of $\tilde{I}(\tilde{t}=0,a)$ was exponential, as predicted by the theory for short times following the turn-on. In other words, the light intensity behaved as if it had started to rise from zero at some short time preceding the turn-on. Since $\langle \tilde{I}(\tilde{t},a)\rangle$ should equal $4\tilde{t}$ for $\tilde{t}\ll 1$, according to Eq. (23), we determined the mean normalized light intensity before the laser was turned on, and divided this by 4 to arrive at a small correction time $|\tilde{t}_0|$ (~ 0.02) to be added to the actual delay time. This ensured that $\langle \tilde{I}(\tilde{t},a)\rangle=0$ at $\tilde{t}=0$.

V. RESULTS

The time development of $\langle \tilde{I}(\tilde{t},a)\rangle$ for a Q-switched laser derived from the experiment (for pump parameters $a=0$, 4, 8), together with the curves calculated by Risken and Vollmer,[9] is shown in Figs. 2 and 3. Figure 2 describes the first 25 μsec of growth of the light intensity on an expanded time scale, while Fig. 3 describes the entire develop-

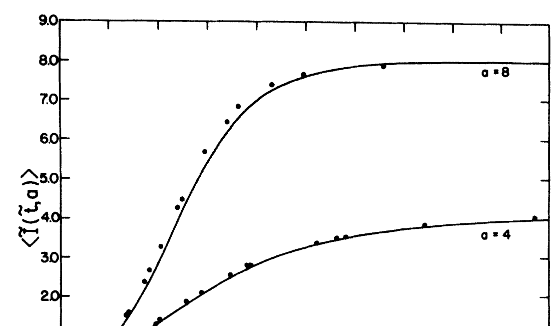

FIG. 3. Variation of the mean light intensity with time, in normalized units, for three different values of the pump parameter a. Statistical uncertainties of the experimentally determined $\langle \tilde{I}(\tilde{t}, a)\rangle$ values are too small to show. The full curves were computed by Risken and Vollmer (Ref. 9).

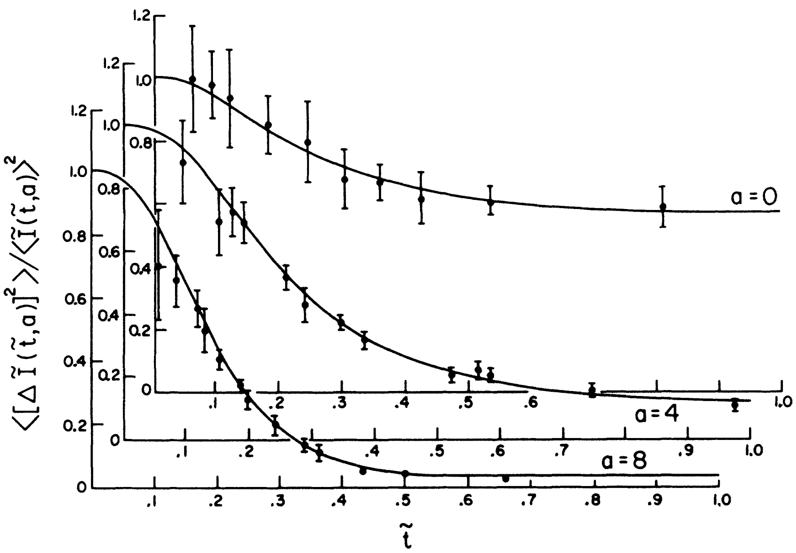

FIG. 4. Variation of the normalized variance of the light intensity with time, in normalized units, for three different values of the pump parameter a. Error bars represent statistical uncertainties of the experimentally determined values. The full curves were computed by Risken and Vollmer (Ref. 9).

ment towards the steady state. It will be seen that there is very good agreement between theory and experiment, and, in particular, that the convergence of all the curves for $\tilde{t} \ll 1$, as predicted by Eq. (23), is confirmed.

The laser was switched between 10 000 and 25 000 times for each experimental point. In addition to the indicated statistical uncertainties of the derived $\langle \tilde{I}(\tilde{t}, a)\rangle$ values, there is a statistical uncertainty of about 2% in the $\tilde{t}$ values, due to the uncertainty of the measured scaling parameter $T_c(a=0)$ [cf. Eq. (5)]. In fact, $T_c(a=0)$ was redetermined for each run, corresponding to different pump parameter settings a, since we noticed a certain amount of slow drift of the laser parameters over a period of days, when the settings were changed. Thus, the time scaling parameter for the $a=8$ run may have been in error by a few percent.

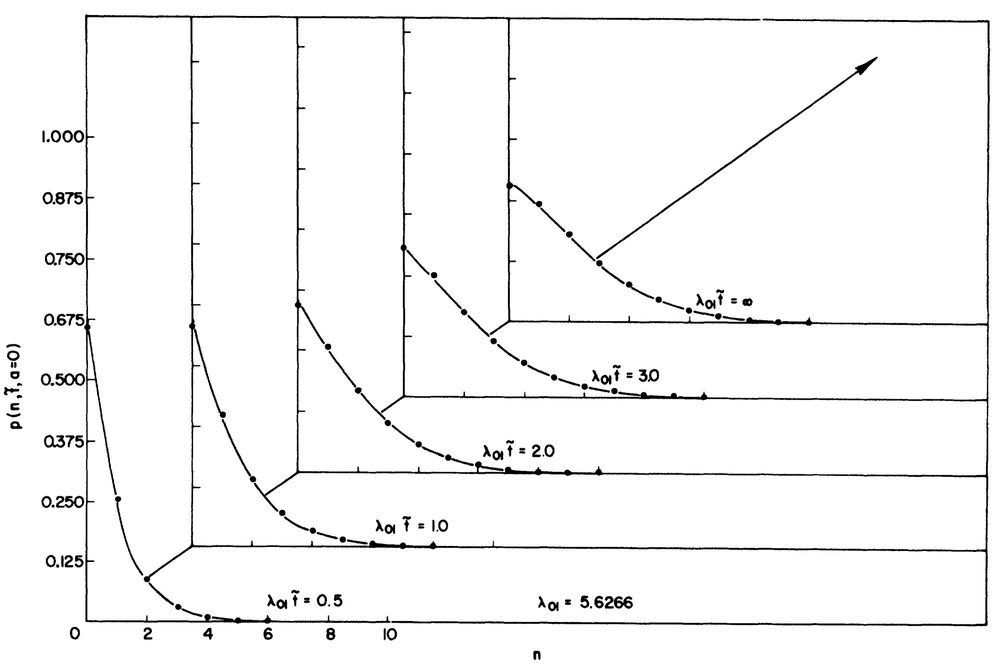

FIG. 5. Evolution of the photoelectric counting probability with time in normalized units, at threshold (with pump parameter $a=0$). Statistical uncertainties of the experimental points are too small to show. The full curves are based on the calculations of Risken and Vollmer (Ref. 9). λ_{01} is the first decay constant in the expansion (12).

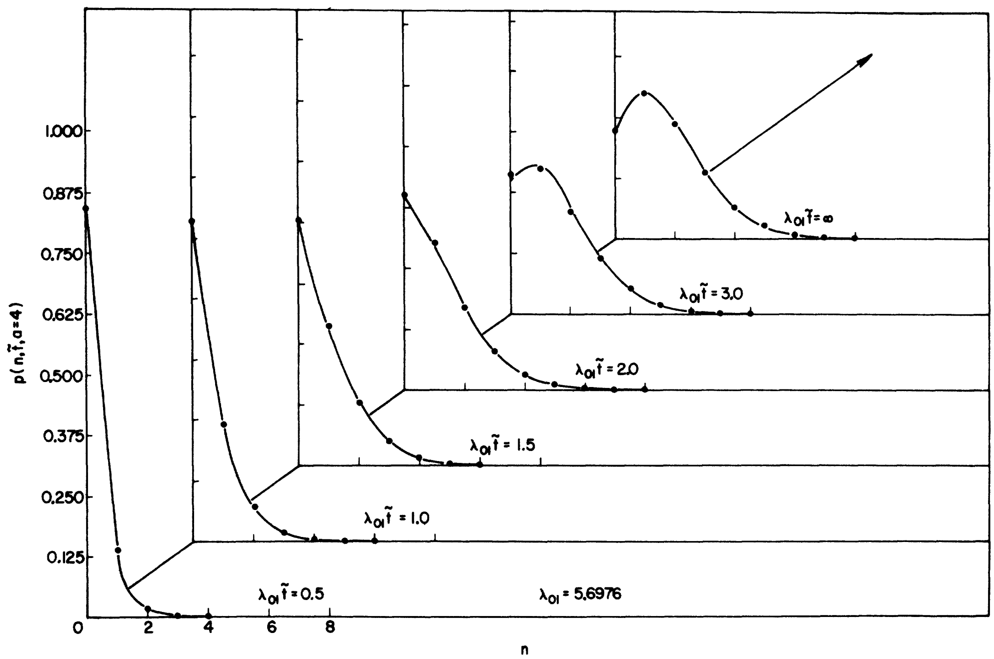

FIG. 6. Evolution of the photoelectric counting probability with time in normalized units, with pump parameter $a = 4$. Statistical uncertainties of the experimental points are too small to show. The full curves are based on the calculations of Risken and Vollmer (Ref. 9). λ_{01} is the first decay constant in the expansion (12).

The values of $\langle[\Delta \tilde{I}(\tilde{t}, a)]^2\rangle/\langle \tilde{I}(\tilde{t}, a)\rangle^2$ derived from the measurements, for the same three values of the pump parameter, together with the curves computed by Risken and Vollmer,[9] are shown in Fig. 4. Once again there is good agreement between theory and experiment within the statistical uncer-

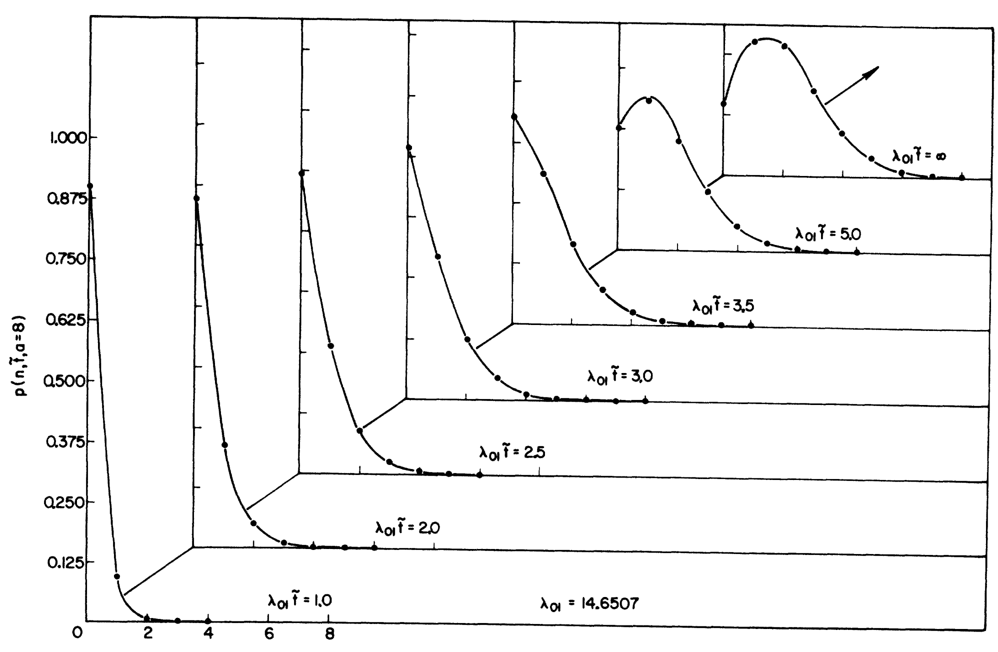

FIG. 7. Evolution of the photoelectric counting probability with time in normalized units, with pump parameter $a = 8$. Statistical uncertainties of the experimental points are too small to show. The full curves are based on the calculations of Risken and Vollmer (Ref. 9). λ_{01} is the first decay constant in the expansion (12).

tainties, which are inevitably rather greater for the second moments than for the first.

In order to test the validity of the curves for $W(\tilde{I}, \tilde{t}, a)$ computed by Risken and Vollmer,[9] we did not attempt an inversion of Eq. (17) [or Eq. (37) after various corrections are incorporated]. Instead we compared the measured distributions $p(n, \tilde{t}, T, a)$ with those derived from Eq. (37), when the Risken and Vollmer curves for $W(\tilde{I}, \tilde{t}, a)$ are used in place of $\mathcal{P}_L(U_L, \tilde{t}, T, a)$. Then $\alpha U_L(\tilde{t}, T, a)$ has, of course, to be replaced by $\alpha T I_L(\tilde{t}, a) = \tilde{I}\langle n_L(t=\infty, a=0)\rangle/1.128$ under the integral in Eq. (37). The test for $W(\tilde{I}, \tilde{t}, a)$ is therefore somewhat indirect. Nevertheless, we felt that a comparison of experiment with theory, for a large number of combinations of a and $\tilde{t}$, would tend to reveal discrepancies if they exist.

The results are indicated in Figs. 5–7. The experimentally determined probabilities are shown as dots, while the theoretically derived values are displayed as continuous curves for clarity, although, obviously, they are meaningful only for integral n. The transition of the optical field from the thermal state at $\tilde{t}=0$, for all a values, towards the almost coherent state for large $\tilde{t}$ and large a values, is clearly indicated. Once again there is good agreement between the experimental and the theoretically derived values.

It should be emphasized that no scaling parameters were arbitrarily adjusted to produce agreement in any of the figures, but that the scaling parameters were themselves determined by experiment. The predictions of dynamical behavior in the threshold region by the nonlinear oscillator theory of the laser are therefore very well confirmed.

Note added in proof. We note from a recent preprint by F. T. Arecchi and V. Degiorgio [Phys. Rev. (to be published)] that these authors have also investigated the evolution of the photoelectric counting probability, but for a laser operating much further above threshold.

APPENDIX: FINITE COUNTING TIME CORRECTION FOR SECOND MOMENT

It is convenient to go back to the basic Langevin equation (3) for the laser, in which we denote the right-hand side by $F(\tilde{I}, \tilde{t}) + N(\tilde{t})$, where $N(\tilde{t})$ is a δ-correlated noise current such that

$$\langle N(\tilde{t})\rangle = 0 \quad , \tag{A1}$$

$$\langle N(\tilde{t}_1)N(\tilde{t}_2)\rangle = \delta(\tilde{t}_1 - \tilde{t}_2)8\langle \tilde{I}(\tilde{t}_1)\rangle \ , \tag{A2}$$

$$\langle N(\tilde{t})F(\tilde{I}(\tilde{t}_0))\rangle = 0, \quad \tilde{t}_0 < \tilde{t} \ . \tag{A3}$$

The last equation is a reflection of the fact that the noise $N(\tilde{t})$ is statistically independent of the light intensity at earlier times $\tilde{t}_0 < \tilde{t}$, but not at later times $\tilde{t}_0 > \tilde{t}$.

If $\tilde{I}(\tilde{t}_0)$ is the intensity at time $\tilde{t}_0$, we may approximate the solution of Eq. (3) for some slightly later time $\tilde{t}$ by treating

$$F(\tilde{I}, \tilde{t}) \equiv 2\tilde{I}(a - \tilde{I}) + 4 \tag{A4}$$

as nearly constant, and writing

$$\tilde{I}(\tilde{t}) \approx \tilde{I}(\tilde{t}_0) + (\tilde{t} - \tilde{t}_0)F(\tilde{I}(\tilde{t}_0)) + \int_{\tilde{t}_0}^{\tilde{t}} N(\tilde{t}')\, d\tilde{t}' \ . \tag{A5}$$

On taking the average of this equation and using Eq. (A1) we obtain

$$\langle \tilde{I}(\tilde{t})\rangle = \langle \tilde{I}(t_0)\rangle + (\tilde{t} - \tilde{t}_0)\langle F(\tilde{I}(\tilde{t}_0))\rangle \ . \tag{A6}$$

Similarly, we can also find the second moment of the intensity at time $\tilde{t}$ from Eq. (A5),

$$\begin{aligned}\langle \tilde{I}^2(\tilde{t})\rangle = {} & \langle \tilde{I}^2(\tilde{t}_0)\rangle + 2(\tilde{t} - \tilde{t}_0)\langle \tilde{I}(\tilde{t}_0)F(\tilde{I}(\tilde{t}_0))\rangle \\ & + (\tilde{t} - \tilde{t}_0)^2\langle F^2(\tilde{I}(\tilde{t}_0))\rangle \\ & + 2\langle [\tilde{I}(\tilde{t}_0) + (\tilde{t} - \tilde{t}_0)F(\tilde{I}(\tilde{t}_0))] \int_{\tilde{t}_0}^{\tilde{t}} N(\tilde{t}')\, d\tilde{t}'\rangle \\ & + \int_{\tilde{t}_0}^{\tilde{t}} d\tilde{t}' \int_{\tilde{t}_0}^{\tilde{t}} d\tilde{t}''\langle N(\tilde{t}')N(\tilde{t}'')\rangle \ ,\end{aligned} \tag{A7}$$

which, with the help of Eqs. (A1)–(A3), simplifies to

$$\begin{aligned}\langle \tilde{I}^2(\tilde{t})\rangle = {} & \langle \tilde{I}^2(\tilde{t}_0)\rangle + 2(\tilde{t} - \tilde{t}_0)\langle \tilde{I}(\tilde{t}_0)F(\tilde{I}(\tilde{t}_0))\rangle \\ & + (\tilde{t} - \tilde{t}_0)^2\langle F^2(\tilde{I}(\tilde{t}_0))\rangle \\ & + 8(\tilde{t} - \tilde{t}_0)\langle \tilde{I}(\tilde{t}_0)\rangle + 4(\tilde{t} - \tilde{t}_0)^2\langle F(\tilde{I}(\tilde{t}_0))\rangle \ .\end{aligned} \tag{A8}$$

Let us now evaluate the moments of the time-integrated light intensity $\tilde{U}(\tilde{t}, \tilde{T}, a)$, which are proportional to the factorial moments of the photocount distribution according to Eq. (26). From

$$\begin{aligned}\tilde{U} \equiv \int_{\tilde{t}_0}^{\tilde{t}_0+\tilde{T}} \tilde{I}(\tilde{t})\, d\tilde{t} = {} & \tilde{T}\tilde{I}(\tilde{t}_0) + (\tfrac{1}{2}\tilde{T}^2)F(\tilde{I}(\tilde{t}_0)) \\ & + \int_{\tilde{t}_0}^{\tilde{t}_0+\tilde{T}} d\tilde{t} \int_{\tilde{t}_0}^{\tilde{t}} d\tilde{t}' N(\tilde{t}')\end{aligned} \tag{A9}$$

and Eqs. (A1) and (A6), we obtain

$$\begin{aligned}\langle \tilde{U}\rangle & = \tilde{T}[\langle \tilde{I}(\tilde{t}_0)\rangle + (\tfrac{1}{2}\tilde{T})\langle F(\tilde{I}(\tilde{t}_0))\rangle] \\ & = \tilde{T}\langle \tilde{I}(t_0 + \tfrac{1}{2}T)\rangle .\end{aligned} \tag{A10}$$

This confirms the intuitive notion that the mean number of observed counts is proportional to the mean light intensity at the middle of the counting interval. The situation for the second moment is, however, not quite so simple. From Eq. (A9) we obtain

$$\begin{aligned}\tilde{U}^2 = {} & \tilde{T}^2\tilde{I}^2(\tilde{t}_0) + (\tfrac{1}{4}\tilde{T}^4)F^2(\tilde{I}(\tilde{t}_0)) + \tilde{T}^3\tilde{I}(\tilde{t}_0)F(\tilde{I}(\tilde{t}_0)) \\ & + \int_{\tilde{t}_0}^{\tilde{t}_0+\tilde{T}} d\tilde{t}_1 \int_{\tilde{t}_0}^{\tilde{t}_0+\tilde{T}} d\tilde{t}_2 \int_{\tilde{t}_0}^{\tilde{t}_1} d\tilde{t}' \int_{\tilde{t}_0}^{\tilde{t}_2} d\tilde{t}'' N(\tilde{t}')N(\tilde{t}'')\end{aligned}$$

$$+ 2[\tilde{T}\tilde{I}(\tilde{t}_0) + (\tfrac{1}{2}\tilde{T}^2)F(\tilde{I}(\tilde{t}_0))] \int_{\tilde{t}_0}^{\tilde{t}_0+\tilde{T}} d\tilde{t} \int_{\tilde{t}_0}^{\tilde{t}} d\tilde{t}' N(\tilde{t}') \ , \quad \text{(A11)}$$

and when we average this, the last term vanishes as before. The fourfold integral can be rewritten in terms of $\langle \tilde{I}(\tilde{t}')\rangle$ and a δ function with the help of Eq. (A2), and can then be evaluated. We obtain

$$\begin{aligned}\langle \tilde{U}^2\rangle = {} & \tilde{T}^2\langle \tilde{I}^2(\tilde{t}_0)\rangle + (\tfrac{1}{4}\tilde{T}^4)\langle F^2(\tilde{I}(\tilde{t}_0))\rangle \\ & + \tilde{T}^3\langle \tilde{I}(\tilde{t}_0)F(\tilde{I}(\tilde{t}_0))\rangle \\ & + (\tfrac{8}{3}\tilde{T}^3)\langle \tilde{I}(\tilde{t}_0)\rangle + (\tfrac{2}{3}\tilde{T}^4)\langle F(\tilde{I}(\tilde{t}_0))\rangle \ , \end{aligned} \quad \text{(A12)}$$

which reduces to Eq. (29) when we make use of Eq. (A8) with $\tilde{t} = \tilde{t}_0 + \frac{1}{2}\tilde{T}$.

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