Superradiance*

Nicholas E. Rehler† and Joseph H. Eberly
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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We introduce a straightforward quantum-electrodynamic approach to the problem of superradiant spontaneous emission from a system of two-level atoms, and also discuss two classical superradiant systems. The same physical concepts underlie our treatment of both the quantum and classical cases. Explicit expressions are found which describe the time evolution and directional character of the radiated power. Our approach applies to an arbitrary number of atoms which are confined to a region with linear dimensions which may be large compared with the mean wavelength of the emitted radiation. The radiative decay half-life of a large many-atom system is found to be much shorter than the natural lifetime of a single atom, but not as short as the half-life of a small atomic system with the same number of atoms. The far-field radiation pattern appropriate to the case of a circular cylinder of emitters which have been excited by a plane-wave pulse is considered in detail. Polar plots are presented of the radiation patterns produced when the atoms are excited by a plane-wave pulse. Exciting pulses traveling in directions both parallel and at an angle of 10° to the axis of the cylinder are considered. A discussion and generalization of Dicke's "coherence-brightening" criterion is given.

I. INTRODUCTION

In a classic paper[1] in 1954, R. H. Dicke calculated the rate at which radiation is emitted spontaneously by an assembly of two-level atoms. By considering the entire collection of atoms as a single quantum-mechanical system, he found that under certain conditions the individual atoms cooperate to emit radiation at a rate which is much greater than their incoherent emission rate. A system which exhibits cooperative effects of this nature is said to be "superradiant."

A new interest in superradiant phenomena has arisen in the past few years. This is due to the recently acquired experimental ability to manipulate coherently large collections of optically resonant atoms. Testimony to this ability is provided by experimental observations of such effects as photon echoes,[2] self-induced transparency,[3] optical nutation,[4] and optical adiabatic inversion.[5]

Many of these effects are familiar because their analogs have been studied at microwave and longer wavelengths. However, at the shorter wavelengths typical of the infrared and optical regions, spontaneous emission begins to be a serious competitor of such stimulated processes. This is all the more true for cooperative, or superradiant, spontaneous emission because it is most effective when the atomic system is characterized by a large induced electric-dipole moment. It is the presence of just such a dipole moment which is typical of all the effects mentioned above. It is also important to realize that in the optical region of the spectrum it is no longer practical to assume that all of the active atoms are confined to a region which has linear dimensions smaller than a wavelength. Because of this, propagation effects, notably self-induced transparency[3] and optical amplification,[6] become newly important. Nevertheless, there is no discussion[7] of any of these coherent effects in which cooperative spontaneous emission is taken account of consistently. Thus it seems desirable in the context of geometrically large collections of emitters and the optical region of the spectrum to reexamine the notion of superradiance.

A. Previous Work

In his treatment,[1] Dicke discussed separately the time dependence and the directional character of the emitted radiation. The three principal results were a treatment of the correlations of successively emitted photons; an approximate evaluation of the radiated intensity, or power, as a function of time for a geometrically small system; and the derivation of an expression for the initial radiation rate of an excited geometrically large system.

Subsequently, similar problems have been studied by a number of authors. These studies can be recognized to fall into one of three groups. The largest of these groups[8] is comprised of investigations in which the emitting atoms either are confined to a region smaller than the wavelength of emitted light or are able to couple with only one radiation mode. This situation permits important mathematical simplifications, and "exact" solutions for the time evolution of the system may be obtained. An important contribution[9] to this class of problems has recently been made by Bonifacio and Preparata.

Another group of studies has been made of spontaneous emission from small numbers of atoms. Early studies of this type were made by Jaynes and Cummings[10] and Stephen,[11] who considered one

and two atoms, respectively. Recently Dillard and Robl[12] have described results for as many as eight atoms.

The third group of investigations allows many atoms, spread over large regions, to interact with all the modes of the radiation field. Not surprisingly, the treatments which fall within this group have been limited in number. Both Dicke's derivation[1] of the initial radiation rate of a large excited system and one by Fain[13] leave open the important question of the duration of the emission process. In an impressive piece of work, Ernst and Stehle[14] have been able to give an implicit expression for the time evolution of the energy of a large system of spontaneously emitting atoms. However, because their method is a direct extension of the Weisskopf-Wigner method for treating the decay of a single excited atom, their conclusions are applicable only to the situation in which all of the atoms are simultaneously in their upper states at $t=0$. Three more recent treatments of spontaneous emission from large systems have been published by Agarwal, Lehmberg, and Dialetis.[15]

B. Present Paper

The investigations reported in this paper also fall within the third group. We are concerned with an arbitrary number of atoms. The atoms couple to all radiation modes. We assume them to be contained in a volume which may be large or small compared with the cube of the average emitted wavelength. The results given here serve to extend the conclusions of earlier work[16] already reported by us. Here we describe the directional as well as temporal dependence of the emitted radiation.

In addition, we describe superradiance as it occurs in two classical radiating systems. As in the quantum two-level atom case, the principle of radiation reaction provides the basic dynamical law. We find explicit formulas for the time dependence of the radiation rate and for the emitting-system energy. The rate constants which characterize the radiation process, as well as the angular distribution of the radiation, depend in a simple way on the geometry of the emitting system and on the initial phase coherence of the emitters.

In both the quantum and the classical situations the rate of energy loss from a geometrically large system of N radiators can be proportional to N^2 and thus many orders of magnitude larger than the purely incoherent rate. However, the rate is considerably *smaller* than that calculated by Dicke for a geometrically small N-atom system. To say this another way: The radiative decay half-life of a large many-atom system is much shorter than the natural lifetime of a single atom, but *longer* than the half-life of a small atomic system with the same number of atoms.

The relation between the radiation rate and energy for each type of emitter determines whether the radiated intensity can grow to a peak during the emission process, or merely decays monotonically from its initial value. In all the cases we have considered we have found that if a peak is present the higher it is the sooner it occurs in time, and following the peak the radiation rate falls off essentially exponentially in time.

The angular dependence of the emitted radiation itself may or may not be time dependent, but it is always strongly dependent on the geometry of the emitting system. A plane-wave-excited system always shows an emission maximum in the same direction as the exciting wave. In the latter case, the peak forward radiation rate is very nearly $\frac{1}{4}N^2$ time the single-emitter rate, showing the possibility of maximal superradiance even from a geometrically large system.

The remainder of the paper is organized in the following way. Section II is devoted to two classical radiating systems: a collection of N linear charged harmonic oscillators, and a collection of N magnetic dipoles precessing in a strong homogeneous magnetic field. Section III is concerned with a quantum-electrodynamic derivation of the power radiated from an extended system of N two-level atoms. We introduce in Sec. IV an approximate interaction picture N-atom state vector which is used to find expectation values of quantum-mechanical operators. In this approximation the temporal behavior of the atomic system is found to be governed by a simple nonlinear differential equation for the expectation value of the atomic energy. Similarities and differences between the quantum and classical cases are pointed out.

In Secs. V and VI we concentrate on the quantum case, and study the temporal development and angular distribution of the emitted radiation in detail. Explicit analytic solutions are given for the time dependence of the atomic energy and the radiated intensity. The spectral width of the emitted radiation is estimated and the superradiant lifetime of the emitting system is calculated. We consider in detail the radiation patterns appropriate to the case of a right circular cylinder of emitters which are plane-wave excited. A contour plot is given of the shape factor which governs both the rate of radiation and its angular distribution. The radiation patterns themselves are displayed for several sizes of emitting cylinders and several densities of active atoms, both when the exciting plane wave travels along the cylinder axis and at an angle of 10° to the axis. Some remarks are made about the "coherence-brightening" criterion of Dicke.[17] Finally, we conclude with Sec. VII in which our results are briefly summarized. There are two Appendices devoted to restrictions on our work im-

plied by the "area theorem" of McCall and Hahn,[3] and the coherent "ringing" discussed by Burnham and Chiao.[18]

II. TWO CLASSICAL MODELS

Since many of the salient features of superradiance are exhibited by a collection of classical radiators, it may be well to begin with a review of two classical problems. The two problems which we shall consider are (a) N identical charged harmonic oscillators and (b) N identical magnets precessing in a strong homogeneous magnetic field.

In each of these examples we have a collection of N point dipoles [electric dipoles in (a) and magnetic dipoles in (b)] which are fixed at positions $\vec{r}_1$, $\vec{r}_2, \ldots, \vec{r}_N$. The distances between the dipoles will be assumed to be sufficiently large that direct dipole-dipole forces can be ignored. [See Note (*a*) added in proof.] These N dipoles will initially be excited in some way, and then undergo free oscillation. Each dipole will thus radiate and gradually lose energy to the radiation field. In each case we will be able to find an equation of motion for the energy of the dipoles by equating the total radiated power to the dipole energy loss. All of the statements of this paragraph apply equally well to our treatment of two-level atoms in Secs. III and IV.

A. Harmonic Oscillators

The polarization density of the system may be written as

$$\vec{P}(\vec{r}, t) = \sum_{l=1}^{N} \vec{D}_l(t)\delta^3(\vec{r} - \vec{r}_l) , \tag{2.1}$$

where $\vec{D}_l(t)$ is the dipole moment associated with the lth oscillator. We shall take the form of $\vec{D}_l(t)$ to be given by

$$\vec{D}_l(t) = \vec{D}(t)\cos(\omega t - \phi_l) , \tag{2.2}$$

where $\vec{D}(t)$ is a slowly varying vector amplitude and ϕ_l is a constant phase. Notice that although we have assumed the oscillators have been excited in such a way that their amplitudes are identical, their phases may still be different.

Given Eqs. (2.1) and (2.2) we can easily calculate the real electric and magnetic fields in the radiation zone. From the fields, the Poynting flux $\vec{S}$ may be computed in the usual way to find $I(\hat{k}, t)d\Omega_{\hat{k}}$ the power radiated by the dipoles into an element of solid angle $d\Omega_{\hat{k}}$ about the direction of the unit vector $\hat{k}$. In particular,

$$I(\hat{k}, t)d\Omega_{\hat{k}} = \bar{\vec{S}} \cdot \hat{k}\, d(\text{area}) . \tag{2.3}$$

The bar in Eq. (2.3) indicates an average over a small time interval to eliminate the rapid oscillations at twice the natural frequency. The result for $I(\hat{k}, t)d\Omega_{\hat{k}}$ is

$$I(\hat{k}, t)d\Omega_{\hat{k}} = I_0(\hat{k})N^2 r^2(t)\left|\frac{1}{N}\sum_{l=1}^{N} e^{i(\phi_l - \vec{k}\cdot\vec{r}_l)}\right|^2 d\Omega_{\hat{k}} . \tag{2.4}$$

Here $\vec{k} = (\omega/c)\hat{k}$, $I_0(\hat{k})$ is the power radiated per unit solid angle by a single dipole oscillator of constant amplitude $D(0)$, and $r(t)$ is the ratio of the dipole amplitude $D(t) = |\vec{D}(t)|$ to its initial value $D(0)$:

$$r(t) = D(t)/D(0) . \tag{2.5}$$

Since c is the velocity of light in vacuum we have assumed (as we will throughout the rest of this paper) that the index of refraction of the underlying medium in which the dipoles are imbedded is unity.

The energy of the N charged oscillators (averaged over an oscillator period) may be written as

$$W(t) = NM\omega^2[D(0)/Q]^2 r^2(t) , \tag{2.6}$$

where M and Q are the mass and charge of each oscillator. In Eq. (2.6) we have neglected contributions to the energy which are smaller than the terms retained by the ratio $(\dot{D}/\omega D)^2$, invoking the slowly varying assumption about the amplitude $D(t)$.

By integrating Eq. (2.4) over all directions $\hat{k}$ and eliminating $r(t)$ by means of Eq. (2.6), we find the total power radiated per unit time $I(t) = \int I(\hat{k}, t)d\Omega_{\hat{k}}$ is related to the total oscillator energy by the equation

$$I(t) = N\mu W(t)/\tau_0 , \tag{2.7}$$

where

$$\mu = \frac{\tau_0}{M\omega^2}\left(\frac{Q}{D(0)}\right)^2 \int I_0(\hat{k})\left|\frac{1}{N}\sum_{l=1}^{N} e^{i(\phi_l - \vec{k}\cdot\vec{r}_l)}\right|^2 d\Omega_{\hat{k}} \tag{2.8}$$

and

$$\tau_0^{-1} = \frac{1}{M\omega^2}\left(\frac{Q}{D(0)}\right)^2 \int I_0(\hat{k})d\Omega_{\hat{k}} ; \tag{2.9}$$

τ_0 is the radiative lifetime of a single isolated oscillator.

Setting the total power radiated equal to the energy loss rate $-\dot{W}$ then yields a simple differential equation for $W(t)$. Its solution is

$$W(t) = W(0)e^{-t/\tau} , \tag{2.10}$$

where

$$\tau = \tau_0/N\mu . \tag{2.11}$$

Notice the two features of the above analysis which are basic to the concept of superradiance: (i) The radiated power is proportional to the square of the number of participating dipoles [cf. Eq. (2.7) at $t = 0$, and recall that $W(0)$ is proportional to N], and (ii) the effective lifetime of the system of N dipoles is shortened by a factor proportional to $1/N$ compared with the lifetime of a single isolated

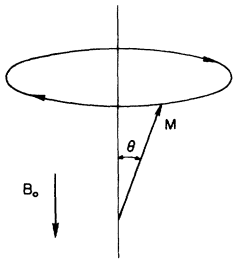

FIG. 1. The configuration of a classical magnetic dipole of moment $\vec{M}$ in a strong homogeneous magnetic field $\vec{B}_0$.

oscillator [cf. Eq. (2.11)].

B. Precessing Magnets

The configuration of a typical magnet of moment $\vec{M}$ in the assumed strong homogeneous magnetic field $\vec{B}_0$ is shown in Fig. 1. We assume that all N magnets are precessing with the same polar angle $\theta(t)$ (although perhaps with different azimuthal angles ϕ_l). This is strictly analogous to taking the electric dipole moments in Sec. II A to have a common amplitude $D(t)$ but arbitrary phases ϕ_l.

The power radiated from the system of magnets may be calculated by the same method used in Sec. II A to calculate the power radiated from the system of harmonic oscillators. The only difference is that the electric dipoles are replaced by magnetic dipoles with corresponding changes in the fields. The result for the power radiated into the solid angle $d\Omega_{\hat{k}}$ is

$$I(\hat{k},t)d\Omega_{\hat{k}} = I_0(\hat{k})N^2\sin^2\theta\left|\frac{1}{N}\sum_{l=1}^{N} e^{i(\phi_l-\vec{k}\cdot\vec{r}_l)}\right|^2 d\Omega_{\hat{k}} . \tag{2.12}$$

Here the ϕ_l's are the constant azimuthal plane angles of the precessing magnetic dipoles; $I_0(\hat{k})$ is the power radiated per unit solid angle by a single precessing magnetic dipole of moment $\vec{M}$ and polar angle $\theta = \frac{1}{2}\pi$; and $\vec{k} = (\omega/c)\hat{k}$, where ω is the Larmor precession frequency.

The energy (in units of $2MB_0$) of the N identical precessing magnets is

$$W(t) = -\tfrac{1}{2}N\cos\theta . \tag{2.13}$$

Upon integrating Eq. (2.12) over all directions $\hat{k}$, and eliminating the polar angle $\theta(t)$ by means of Eq. (2.13), we find

$$I(t) = (2MB_0\mu/\tau_0)(\tfrac{1}{2}N + W(t))(\tfrac{1}{2}N - W(t)) , \tag{2.14}$$

where

$$\tau_0^{-1} = (2/MB_0)\int I_0(\hat{k})\, d\Omega_{\hat{k}} , \tag{2.15}$$

$$\mu = \frac{2\tau_0}{MB_0}\int I_0(\hat{k})\left|\frac{1}{N}\sum_{l=1}^{N} e^{i(\phi_l-\vec{k}\cdot\vec{r}_l)}\right|^2 d\Omega_{\hat{k}} , \tag{2.16}$$

and $I(t)$ is defined as in Sec. II A.

Then, after equating the radiated power to the energy loss, a nonlinear differential equation is found of the form

$$-\frac{dW}{dt} = \frac{\mu}{\tau_0}(\tfrac{1}{2}N + W)(\tfrac{1}{2}N - W) . \tag{2.17}$$

This equation of motion may be easily integrated to yield the total dipole energy as an explicit function of time

$$W(t) = -\tfrac{1}{2}N\tanh[(t-t_0)/2\tau] \tag{2.18}$$

and the radiated power

$$I(t) = (N^2\mu MB_0/2\tau_0)\operatorname{sech}^2[(t-t_0)/\tau] . \tag{2.19}$$

In Eqs. (2.18) and (2.19), τ is defined by the relation

$$\tau = \tau_0/N\mu , \tag{2.20}$$

and t_0 is the time of peak radiation rate. It is obviously related to the initial conditions.

There are several remarks which must be made about these results. In the first place, the two marks of superradiance are evident in Eqs. (2.19) and (2.20): The radiated power is proportional to N^2 and the effective lifetime τ is proportional to $1/N$. Second, the results given here are far from new. They have been obtained before in a classical context, apparently first by Ginzburg,[19] and they have been discussed within the framework of nuclear magnetic resonance much more recently.[20] Finally, we see in Eq. (2.19) that the magnets provide an example of a classical emitting system with a nonmonotonic behavior for the radiated power. We will find that the radiated power from a system of two-level atoms is also nonmonotonic; in general it exhibits a delayed peak.

The precession law for magnets is the same as for a massive pendulum in a gravitational field. Except for modifications due to true quantum spontaneous emission, the equation of motion we have found above is also the same as the equation of motion which we shall find in Sec. IV for two-level atoms with electric dipole moments. In all three of these cases it is possible to make use of language appropriate to a pendulum. For example, the angle θ defined in Fig. 1 has an analog in the analysis of two-level atoms; in Sec. IV the total atomic energy W will again be given by Eq. (2.13) with θ having the significance that $\sin^2\frac{1}{2}\theta$ gives the probability that an atom is excited.

One notable distinction between the classical and quantum-electrodynamic treatments in pendulum language is that a classical pendulum exactly inverted (i.e., $\theta = \pi$) will never fall. A quantum pendulum exactly inverted does fall, and in a finite amount of time. (That is, an excited atom always decays.)

III. QUANTUM ELECTRODYNAMICS AND TWO-LEVEL ATOMS

In quantum electrodynamics the relation between

the power radiated and the electric and magnetic fields is a Heisenberg operator relation of the form

$$\mathbf{I}(\hat{k},t)d\Omega_{\hat{k}} = (c/8\pi) : (\vec{\mathbf{E}}\times\vec{\mathbf{B}} - \vec{\mathbf{B}}\times\vec{\mathbf{E}})\cdot\hat{k} : d(\text{area}) . \tag{3.1}$$

Here $d\Omega_{\hat{k}}$ is the solid angle about the direction of the unit vector $\hat{k}$ subtended by the differential surface element d(area). As usual, the colons indicate normal ordering.[21]

To perform the normal ordering in Eq. (3.1), it is necessary to separate the field operators into their positive and negative frequency parts. For a general operator $\mathbf{X}$, let $\mathbf{X}^{(+)}$ denote the positive frequency part and $\mathbf{X}^{(-)}$ the negative frequency part, that is,

$$\mathbf{X} = \mathbf{X}^{(+)} + \mathbf{X}^{(-)} . \tag{3.2}$$

Explicitly normally ordered, $\mathbf{I}(\hat{k},t)$ takes the form

$$\mathbf{I}(\hat{k},t)d\Omega_{\hat{k}} = (c/4\pi)(\vec{\mathbf{E}}^{(-)}\times\vec{\mathbf{B}}^{(+)} - \vec{\mathbf{B}}^{(-)}\times\vec{\mathbf{E}}^{(+)})\cdot\hat{k}\, d(\text{area}). \tag{3.3}$$

In writing $\mathbf{I}(\hat{k},t)$ as in Eq. (3.3), we imply having averaged over a small time interval just as in the classical case (cf. Sec. II).

A. Free Fields and Dipole Sources

The fields can be divided further into their free-field parts and their source-dependent parts. Let us denote the free-field solutions to the homogeneous Maxwell equations by $\vec{\mathbf{E}}_F$ and $\vec{\mathbf{B}}_F$. Then one has

$$\begin{aligned}\vec{\mathbf{E}}^{(\pm)} &= \vec{\mathbf{E}}_F^{(\pm)} + \nabla\times\nabla\times\vec{\mathbf{Z}}^{(\pm)} ,\\ \vec{\mathbf{B}}^{(\pm)} &= \vec{\mathbf{B}}_F^{(\pm)} + (1/c)\nabla\times\dot{\vec{\mathbf{Z}}}^{(\pm)} .\end{aligned} \tag{3.4}$$

Here $\vec{\mathbf{Z}}$ is the Hertz dipole operator in the Heisenberg picture, which satisfies the same dynamical equation as its usual classical counterpart.[22]

After substituting Eqs. (3.4) into Eq. (3.3) we have

$$\begin{aligned}\mathbf{I}(\hat{k},t)d\Omega_{\hat{k}} = (1/4\pi)[&(\vec{\mathbf{E}}_F^{(-)}\times\vec{\mathbf{B}}_F^{(+)} - \vec{\mathbf{B}}_F^{(-)}\times\vec{\mathbf{E}}_F^{(+)})c\\ &+\vec{\mathbf{E}}_F^{(-)}\times(\nabla\times\dot{\vec{\mathbf{Z}}}^{(+)}) - \vec{\mathbf{B}}_F^{(-)}\times(\nabla\times\nabla\times\vec{\mathbf{Z}}^{(+)})\\ &+(\nabla\times\nabla\times\vec{\mathbf{Z}}^{(-)})\times\vec{\mathbf{B}}_F^{(+)} - (\nabla\times\dot{\vec{\mathbf{Z}}}^{(-)})\times\vec{\mathbf{E}}_F^{(+)}\\ &+(\nabla\times\nabla\times\vec{\mathbf{Z}}^{(-)})\times(\nabla\times\dot{\vec{\mathbf{Z}}}^{(+)})\\ &-(\nabla\times\dot{\vec{\mathbf{Z}}}^{(-)})\times(\nabla\times\nabla\times\vec{\mathbf{Z}}^{(+)})]\cdot\hat{k}\, d(\text{area}) .\end{aligned} \tag{3.5}$$

Next we compute the expectation value of Eq. (3.5) in the Heisenberg state $|\psi_H\rangle = |\{0\}\rangle\otimes|\psi\rangle$, a product of the free-field vacuum state and an arbitrary N-atom atomic state. Thus we consider the problem of an excited system of N atoms interacting with the quantized electromagnetic field.

To attack this problem we make a fundamental physical assumption: The reaction of the field back on the source is a small effect. More precisely, we suppose that the source operators change with time only in the Hilbert space of the source. The free-field operators change with time, but always have zero expectation value in the state $|\psi_H\rangle$. This means that only the last two terms inside the square brackets of Eq. (3.5) will give a nonzero contribution to the expectation value. In particular, we have

$$\begin{aligned}\langle\mathbf{I}(\hat{k},t)\rangle d\Omega_{\hat{k}} = (1/4\pi)\langle\psi_H|\,[&(\nabla\times\nabla\times\vec{\mathbf{Z}}^{(-)})\times(\nabla\times\dot{\vec{\mathbf{Z}}}^{(+)})\\ &-(\nabla\times\dot{\vec{\mathbf{Z}}}^{(-)})\times(\nabla\times\nabla\times\vec{\mathbf{Z}}^{(+)})]\cdot\hat{k}\,|\,\psi_H\rangle\, d(\text{area}) .\end{aligned} \tag{3.6}$$

As in the classical examples, we assume that the fields are generated by a collection of N elementary dipole sources. If $\vec{\mathbf{D}}_l(t)$ is the atomic dipole operator of the atom at position $\vec{r}_l (l = 1, 2, \ldots, N)$, then

$$\vec{\mathbf{Z}}^{(\pm)}(\vec{r},t) = \sum_{l=1}^{N}\frac{\vec{\mathbf{D}}_l^{(\pm)}(t - |\vec{r}-\vec{r}_l|/c)}{|\vec{r}-\vec{r}_l|} . \tag{3.7}$$

By taking the origin of the coordinate system to be near the center of the atomic system, we can rewrite Eq. (3.7) in the approximate form

$$\vec{\mathbf{Z}}^{(\pm)}(\vec{r},t) = \frac{1}{r}\sum_{l=1}^{N}\vec{\mathbf{D}}_l^{(\pm)}\left(T + \frac{\vec{k}\cdot\vec{r}_l}{\omega}\right) \tag{3.8}$$

for all $\vec{r}$ such that $r \equiv |\vec{r}| \gg |\vec{r}_l|$. Here $\vec{k} \equiv (\omega/c)\hat{k} = (\omega/c)\hat{r}$ and $T = t - r/c$. We shall suppose that our detector is in the far field of the source so that Eq. (3.8) applies.

While all of the above equations have been written in the Heisenberg picture, some of the equations which follow are in the Schrödinger picture and others are interpreted in the interaction picture. Throughout this section, our convention for a general operator $\mathbf{X}$ is demonstrated by the following relations:

$$\mathbf{X}(t) = U^{-1}(t,0)XU(t,0) , \tag{3.9a}$$

$$X^I(t) = U_0^{-1}(t,0)XU_0(t,0) , \tag{3.9b}$$

$$X = \mathbf{X}(0) = X^I(0) . \tag{3.9c}$$

As usual $U_0(t,0)$ and $U(t,0)$ are the unitary evolution operators which satisfy the differential equations

$$i\hbar\frac{dU_0}{dt} = H_0U_0 , \tag{3.10}$$

$$i\hbar\frac{dU}{dt} = HU , \tag{3.11}$$

respectively, H being the total Hamiltonian and H_0 being the sum of all the noninteracting parts of H.

Also, it is customary to define the *interaction* evolution operator $u(t,t')$ by the relation $U(t,t')$

$= U_0(t,t')u(t,t')$, so that Eqs. (3.9a) and (3.9b) can be combined in the form

$$\mathbf{X}(t) = u^{-1}(t,0)X^I(t)u(t,0) \ . \tag{3.12}$$

Consider the lth atomic dipole operator. Making use of Eq. (3.12) we have the relation

$$\vec{\mathbf{D}}_l^{(\pm)}(T+\vec{k}\cdot\vec{r}_l/\omega) = u^{-1}(T+\vec{k}\cdot\vec{r}_l/\omega,0) \times \vec{\mathbf{D}}_l^{I(\pm)}(T+\vec{k}\cdot\vec{r}_l/\omega)u(T+\vec{k}\cdot\vec{r}_l/\omega,0) \ . \tag{3.13}$$

The group property of the interaction evolution operator allows us to rewrite Eq. (3.13) in the form

$$\vec{\mathbf{D}}_l^{(\pm)}(T+\vec{k}\cdot\vec{r}_l/\omega) = u^{-1}(T,0)u^{-1}(T+\vec{k}\cdot\vec{r}_l/\omega,T) \times \vec{\mathbf{D}}_l^{I(\pm)}(T+\vec{k}\cdot\vec{r}_l/\omega)u(T+\vec{k}\cdot\vec{r}_l/\omega,T)u(T,0) \ . \tag{3.14}$$

Next we suppose that the linear dimensions of our atomic system are sufficiently small that a light signal can pass between any two atoms in a time short compared with the time required for any secular change in the atomic state. (This condition is examined in detail in Appendix B.) Then $u(T+\vec{k}\cdot\vec{r}_l/\omega, T) \cong 1$ and Eq. (3.14) becomes

$$\vec{\mathbf{D}}_l^{(\pm)}(T+\vec{k}\cdot\vec{r}_l/\omega) \cong u^{-1}(T,0)\vec{\mathbf{D}}_l^{I(\pm)}(T+\vec{k}\cdot\vec{r}_l/\omega)u(T,0) \ . \tag{3.15}$$

Finally, in this approximation, the Hertz dipole operator becomes simply

$$\vec{\mathbf{Z}}^{(\pm)}(\vec{r},t) = \frac{u^{-1}(T,0)}{r}\left[\sum_{l=1}^{N}\vec{\mathbf{D}}_l^{I(\pm)}\left(\frac{T+\vec{k}\cdot\vec{r}_l}{\omega}\right)\right]u(T,0) \ . \tag{3.16}$$

To go further requires an explicit atomic model. We will work with the same model introduced by Dicke.[1]

B. Two-Level Atomic Dipoles

In this model, the N dipole sources are simply N identical atoms which have only one transition of interest (the same one for every atom) between two nondegenerate states. [See Note (*b*) added in proof.] We denote the upper state of the lth atom in the Schrödinger picture by $|+\rangle_l$ and take its energy to be $+\frac{1}{2}\hbar\omega$, the lower state by $|-\rangle_l$ and take its energy to be $-\frac{1}{2}\hbar\omega$.

Also we define the Schrödinger picture operators R_{l+}, R_{l-}, and R_{l3} which operate only on the lth-atom state and have the properties[1]

$$\begin{aligned} R_{l\pm}|\pm\rangle_l &= 0 \ , \\ R_{l\pm}|\mp\rangle_l &= |\pm\rangle_l \ , \\ R_{l3}|\pm\rangle_l &= \pm\tfrac{1}{2}|\pm\rangle_l \ . \end{aligned} \tag{3.17}$$

It is straightforward to verify that these operators satisfy the commutation relations

$$[R_{l3}, R_{l\pm}] = \pm R_{l\pm} \ . \tag{3.18}$$

With our atomic model thus specified, we finish the evaluation of the radiated power for the allowed electric dipole transitions $\Delta m = 0$ and $\Delta m = \pm 1$.

First, suppose the transition is characterized by $\Delta m = 0$. The electric dipole moment operator for the lth atom $\vec{\mathbf{D}}_l$ and the unperturbed energy operator H_0 can then be written in the Schrödinger picture as

$$\vec{\mathbf{D}}_l = p\hat{z}_l(R_{l+}+R_{l-}) \ , \tag{3.19}$$

$$H_0 = \hbar\omega\sum_{l=1}^{N}R_{l3} + H_F \ . \tag{3.20}$$

Here $\hat{z}_l$ is a unit vector in the direction of the matrix element $(\vec{\mathbf{D}}_l)_{+-}$, $p = |(\vec{\mathbf{D}}_l)_{+-}|$, and H_F is the energy operator for the noninteracting part of the field. We have assumed that the phases of the atomic states have been chosen such that $(\hat{z}_l\cdot\vec{\mathbf{D}}_l)_{+-}$ is real.

Using Eq. (3.20) it is easily verified that the solution to Eq. (3.10) is given by

$$U_0(t,0) = \exp\left[i\left(\alpha - \sum_{l=1}^{N}(\omega t+\phi_l)R_{l3} - \frac{H_F t}{\hbar}\right)\right] \ , \tag{3.21}$$

where α and the $\phi_l (l = 1,2,\ldots,N)$ are $N+1$ arbitrary real constants. All physical results will, of course, be independent of α, but the ϕ_l are determined by the initial conditions (which have not yet been specified).

The positive and negative frequency parts of $\vec{\mathbf{D}}_l$ are identified most easily in the interaction picture. Using Eqs. (3.19), (3.21), and the commutation relations of Eq. (3.18) we find

$$\vec{\mathbf{D}}_l^I(t) = p\hat{z}_l(R_{l+}e^{i(\omega t+\phi_l)} + R_{l-}e^{-i(\omega t+\phi_l)}) \ ; \tag{3.22}$$

thus the positive and negative frequency parts of D_l^I are given by

$$\vec{\mathbf{D}}_l^{I(\pm)}(t) = p\hat{z}_l R_{l\mp}e^{\mp i(\omega t+\phi_l)} \ . \tag{3.23}$$

At this point we drop the subscripts on the unit vectors $\hat{z}_l$, thus specializing to the case where the N atomic dipoles have a common initial direction. This would be the case, for example, if the excitation mechanism were a linearly polarized plane-wave pulse.

By substituting Eq. (3.23) into Eq. (3.16), we find

$$\vec{\mathbf{Z}}^{(\pm)}(\vec{r},t) = u^{-1}(T,0)(p\hat{z}S_{\vec{k}\mp}e^{\mp i\omega T}/r)u(T,0) \ , \tag{3.24}$$

where we define

$$S_{\vec{k}\pm} = \sum_{l=1}^{N}R_{l\pm}e^{\pm i(\vec{k}\cdot\vec{r}_l+\phi_l)} \ . \tag{3.25}$$

To find the radiated power as given by Eq. (3.6), we must take space and time derivatives of $\vec{\mathbf{Z}}^{(\pm)}(\vec{r},t)$. Since it is assumed that the detector is

in the far field of the source, we ignore all derivatives of $1/r$. Thus, a first derivative of $\vec{\mathbf{Z}}^{(\pm)}(\vec{\mathbf{r}},t)$ will have two terms involving derivatives of u and u^{-1} and one term involving the derivative of $e^{\mp i\omega T}$. In the spirit of a slowly varying envelope approximation, we suppose that $|\dot{u}| \ll \omega|u|$ so that derivatives of u and u^{-1} can be ignored. This approximation is clearly consistent with other approximations we have made in this section and will be reexamined in Appendix B.

With these approximations, the relevant derivatives of $\vec{\mathbf{Z}}^{(\pm)}(\vec{\mathbf{r}},t)$ are found from Eq. (3.24) to be

$$\nabla\times\nabla\times\vec{\mathbf{Z}}^{(\pm)} = u^{-1}(T,0)\left[\hat{k}\times(\hat{z}\times\hat{k})\left(\frac{p\omega^2}{c^2 r}\right)S_{\vec{k}\mp}e^{\mp i\omega T}\right] \times u(T,0)\,, \tag{3.26}$$

$$\nabla\times\dot{\vec{\mathbf{Z}}}^{(\pm)} = u^{-1}(T,0)\left[(\hat{k}\times\hat{z})\left(\frac{p\omega^2}{cr}\right)S_{\vec{k}\mp}e^{\mp i\omega T}\right]u(T,0)\,. \tag{3.27}$$

Substituting Eqs. (3.26) and (3.27) into Eq. (3.6) and using the relation $d(\text{area}) = r^2 d\Omega_{\hat{k}}$, we find

$$\langle \mathbf{I}(\hat{k},t)\rangle = (p^2\omega^4/2\pi c^3)[1-(\hat{k}\cdot\hat{z})^2] \times\langle\psi_H|\,u^{-1}(T,0)S_{\vec{k}+}S_{\vec{k}-}u(T,0)\,|\psi_H\rangle\,. \tag{3.28}$$

For $\Delta m = \pm 1$ transitions, Eq. (3.19) is replaced by the equation

$$\vec{\mathbf{D}}_l = (p/\sqrt{2})[(\hat{x}\mp i\hat{y})R_{l+} + (\hat{x}\pm i\hat{y})R_{l-}]\,,$$

where $\hat{x}$ and $\hat{y}$ are orthogonal unit vectors in the same plane as the vector $(\vec{\mathbf{D}}_l)_{-+}$. Equations (3.20) and (3.21) remain the same. The same procedure which led to (3.28) then leads to a similar expression with $[1-(\hat{k}\cdot\hat{z})^2]$ replaced by $\frac{1}{2}[2-(\hat{k}\cdot\hat{x})^2 - (\hat{k}\cdot\hat{y})^2]$. Thus if we define

$$I_0(\hat{k}) = \begin{cases} (p^2\omega^4/2\pi c^3)[1-(\hat{k}\cdot\hat{z})^2] & \text{for } \Delta m = 0 \\ (p^2\omega^4/4\pi c^3)[2-(\hat{k}\cdot\hat{x})^2-(\hat{k}\cdot\hat{y})^2] & \text{for } \Delta m = \pm 1 \end{cases} \tag{3.29}$$

we may write, for the expectation of the intensity radiated by the N atoms, the following expression:

$$\langle \mathbf{I}(\hat{k},t)\rangle = I_0(\hat{k})\langle\psi_H|\,u^{-1}(T,0)S_{\vec{k}+}S_{\vec{k}-}u(T,0)\,|\psi_H\rangle\,. \tag{3.30}$$

The relation given in Eq. (3.30) is the principal result of this section. Its chief importance for our work is that it was derived without recourse to perturbation theory, and thus is free from restrictions to the short time scales associated with ordinary transition rates. Note also the fact that the time dependence of the radiated power in (3.30) is shown to be due solely to the interaction evolution operator $u(T,0)$ and its adjoint. Since we have required that u be slowly varying, in the sense that $|\dot{u}| \ll \omega|u|$, we obtain an added degree of confidence in our fundamental assumption introduced above Eq. (3.6) which implies that $\langle \mathbf{I}(\hat{k},t)\rangle$ is slowly varying in the same sense. Finally, we must point out again that although (3.30) was derived within a quantum-electrodynamic framework, the physical ideas are very ordinary ones. We have rederived (as we show below) and generalized Dicke's central result for a large superradiant system, using only the common picture of a dipole and its radiation field. Nowhere is it necessary to resort to "cooperation operators," or to be concerned with their commutation properties or their eigenvalues.

Before proceeding further, it is appropriate to show the connection between (3.30) and the corresponding result of Dicke.[1] In order to do this, it suffices to rewrite (3.30) in the Schrödinger picture, which yields the formula

$$\langle \mathbf{I}(\hat{k},t)\rangle = I_0(\hat{k})\langle\psi_S(T)|\,R_{\vec{k}+}R_{\vec{k}-}\,|\psi_S(T)\rangle\,. \tag{3.31}$$

Here $|\psi_s(T)\rangle$ is the Schrödinger picture state given by $|\psi_s(T)\rangle = U(T,0)\,|\psi_H\rangle$ and $R_{\vec{k}\pm}$ are the Schrödinger picture operators denoted by the same symbols by Dicke, and are obtained from our expressions for $S_{\vec{k}\pm}$ simply by setting $\phi_l = 0$ for all l [see Eq. (3.25)].

In the absence of retardation (3.31) is identical to Dicke's perturbation-theory result. Note that in this form the expression for the intensity is considerably less useful to us, and may be misleading, due to the appearance of the Schrödinger state $|\psi_S(T)\rangle$. It is a rapidly varying quantity and introduces terms oscillating at the transition frequency ω, in general. Equation (3.30) shows explicitly that such terms actually do not appear in the expectation value.

IV. ATOMIC EQUATION OF MOTION

In this section we apply the results of Sec. III to the situation of greatest interest: a collection of two-level atoms with nonoverlapping wave functions excited by an intense plane-wave pulse characterized by frequency ω and wave vector $\vec{k}_1$.

We assume that this exciting pulse may be described classically, and that it acts on each atom in the same way except for the time delays due to the finite separations of the atoms from each other. It is well known[1,3,23] that such a pulse leaves each atom in a coherent superposition of its upper and lower states. For the lth atom, after the passage of the pulse, we can write

$$|\psi_l\rangle = e^{i\chi_l/2}\sin\tfrac{1}{2}\theta\,|+\rangle_l + e^{-i\chi_l/2}\cos\tfrac{1}{2}\theta\,|-\rangle_l \equiv |\theta\rangle_l\,. \tag{4.1}$$

Here the parameter θ has both the significance ascribed to it by Dicke (that is, $\sin^2\frac{1}{2}\theta$ is the probability that the atom has been left in its upper

state), and the familiar[3,23] geometrical interpretation as well. The phase χ_l will be determined shortly.

Our interest is in the evolution of the atomic system after the plane-wave pulse has passed. In order to study this evolution we make the natural assumption that the only important phenomenon occurring is the radiation of energy into the field. Since the atoms are not directly coupled to each other, and since this radiation process is in any event a rather slow process [recall the discussion of Eq. (3.30)], we take the *form* of each atom's state to be unchanged and simply allow the parameter θ to become a slowly varying function of time with an initial value $\theta(\bar{t}) = \theta_0$. We define $\bar{t}$ to be the time at which the last two-level atom is excited. In other words, we make the assumption that the slowly varying state vector $u(T,0)|\psi\rangle$ which governs the time evolution of $\langle|(\hat{k},t)\rangle$ in (3.30) is given by

$$u(T,0)|\psi\rangle = \prod_{l=1}^{N} |\theta(T)\rangle_l \equiv |\theta(T)\rangle . \tag{4.2}$$

The atomic states $|\theta(T)\rangle$ might reasonably be called "radiation reaction" states since their dynamical evolution will be determined by the requirements of radiation reaction. The physical conditions implicit in Eq. (4.2) are discussed in Appendix A.

The different time origins for the action of the pulse at the different atomic locations can now be used to fix the unspecified phases ϕ_l and χ_l in Eqs. (3.21) and (4.1). In order to demonstrate this, let us evaluate the expectation of the lth dipole operator $\vec{D}_l$ defined in Eq. (3.19). We find

$$\langle\vec{D}_l(t)\rangle = \langle\theta(t)| U_0^{-1}\vec{D}_l U_0 |\theta(t)\rangle = \hat{z}p\sin\theta\cos(\omega t + \phi_l - \chi_l) . \tag{4.3}$$

Note that if $\sin\theta \neq 0$ and the relative time phases of the dipoles are to be determined *only* by the arrival time of the plane-wave pulse at the dipole sites $\vec{r}_l$, then for all l we must have

$$\phi_l - \chi_l = \phi_0 - \vec{k}_1 \cdot \vec{r}_l , \tag{4.4}$$

where ϕ_0 is an irrelevant common phase. Treatments of superradiance given to date have ignored possible time-dependent phase effects. We remain within this restriction by regarding ϕ_l and χ_l as constant. All subsequent results depend upon the validity of (4.4). We thus focus our attention upon cases where the time scales of interest are short compared with, for example, the times associated with collisions or other effects which disrupt the dipole phase coherence. [See S. Bloom (Ref. 20) for a classical treatment which includes a phenomenological collision time T_2.]

The singular cases in which θ is initially an even multiple of π are trivial; they correspond to the situation in which each of the two-level atoms is left in its ground state and does not radiate. The nontrival cases in which $\sin\theta = 0$ correspond to complete initial inversion. We will not consider this possibility here; it has been treated in detail by Ernst and Stehle.[14]

The basic expression given in (3.30) is now readily evaluated. One finds for the radiated intensity, as an implicit function of time, the expression

$$I(\hat{k},t) = I_0(\hat{k})\tfrac{1}{2}N\{1 - \cos\theta(t) + \tfrac{1}{2}N\sin^2\theta(t)\,[\Gamma(\hat{k},\hat{k}_1) - 1/N]\} , \tag{4.5}$$

where we have adopted for convenience the notation $I(\hat{k},t) = \langle|(\hat{k},t+r/c)\rangle$, and have defined

$$\Gamma(\hat{k},\hat{k}_1) = \left|\left(e^{i(\vec{k}-\vec{k}_1)\cdot\vec{r}}\right)_{\rm av}\right|^2 , \tag{4.6}$$

in which the average is to be taken over the positions $\vec{r}_l$ of the atoms. Again, in the absence of retardation and at $t = \bar{t}$, this is a result of Dicke, although derived quite differently.

In order to complete our program for this section and obtain the dynamical law implied by radiation reaction, we need expressions for the time-dependent intensity integrated over all directions $\hat{k}$, and for the time-dependent energy of the atomic system. In analogy to the classical cases [recall Eqs. (2.8) and (2.16)], let us define the quantities

$$I(t) = \int I(\hat{k},t)\,d\Omega_{\hat{k}} , \tag{4.7}$$

$$I_0 = \int I_0(\hat{k})\,d\Omega_{\hat{k}} . \tag{4.8}$$

Also we define μ by the relation

$$(1/N + \mu)I_0 = \int I_0(\hat{k})\Gamma(\hat{k},\hat{k}_1)\,d\Omega_{\hat{k}} . \tag{4.9}$$

In addition, the total atomic energy (in units of $\hbar\omega$) is given by

$$W(t) = \left\langle\theta(t)\left|\sum_{l=1}^{N} R_{l3}\right|\theta(t)\right\rangle = -\frac{N}{2}\cos\theta(t) . \tag{4.10}$$

By integrating Eq. (4.5) over all directions $\hat{k}$ and eliminating θ by means of Eq. (4.10), one then finds that $I(t)$ has the compact form

$$I(t) = (\mu\hbar\omega/\tau_0)[\tfrac{1}{2}N + W(t)][\tfrac{1}{2}N - W(t) + 1/\mu] , \tag{4.11}$$

where $\tau_0 = \hbar\omega/I_0$ is the single-atom lifetime.

Finally, we may write the radiation-reaction dynamical law of evolution for the emitting system as a nonlinear first-order differential equation:

$$-\frac{dW}{dt}=\frac{\mu}{\tau_0}\left(\frac{N}{2}+W\right)\left(\frac{N}{2}-W+\frac{1}{\mu}\right) . \qquad (4.12)$$

(N.B. There is no singularity at $\mu=0$.)

In concluding this section we must point out the close similarity between (4.12) and the corresponding classical expression (2.17). In many treatments of the radiative damping problem (cf. Bloom,[20] Ponte Gonçalves *et al.*,[9] Bloembergen and Pound[20]), the classical-quantum correspondence is taken to be complete by neglecting the $1/\mu$ term on the right-hand side of (4.12). However, as Bonifacio has emphasized[9] in treating the small-system case (many atoms, but all within a wavelength of each other), where $\mu\cong 1$, this term is essential for spontaneous emission to occur at all in the usual sense. As we will show in Sec. V, for a given number of atoms N, $1/\mu$ is an even more important term in the large-system case because μ is usually a very small number.

V. EVOLUTION IN TIME

We have seen in Sec. IV that a very few assumptions can lead to an approximate dynamical law for cooperative spontaneous emission by a system of N two-level atoms. It should be clear that, just as the Weisskopf-Wigner single-atom-approximation method can be regarded as a scheme for ensuring in a self-consistent way that the initial-state probability decreases as the probability associated with all other states increases, our rate equation (4.12) can be thought of as ensuring the same self-consistent balance for the N-atom energy. We expect from our approximate dynamics the same kind of advantages as accrue to the Weisskopf-Wigner method in the one-atom problem. In particular, the solutions will be valid over times much longer than those for which first-order perturbation theory is appropriate.

It is also worth pointing out that the nontrivial difference between the one-atom and N-atom situations is embodied here in the single constant μ. It is the presence of μ that makes the basic equation nonlinear. It is easy to establish that $\mu=0$ when $N=1$; and that Eq. (4.12) then has the ordinary exponentially decaying solution. The abstraction of all of the complications due to N cooperating emitters into a single constant is, needless to say, one of the principal attractions of the method we are following. It is significant that the much more elaborate analysis of Ernst and Stehle[14] leads in a very complicated way to a similar constant. The relation between the two results will be mentioned later in this section.

A. Time Dependences

The basic rate equation (4.12) is nonlinear but easily integrable. The solution, subject to the initial condition $W(0)=\frac{1}{2}N$, is

$$W(t)=-\tfrac{1}{2}N[e^{t/\tau}-(N\mu+2)]/(e^{t/\tau}+N\mu) , \qquad (5.1)$$

where

$$\tau=\tau_0/(N\mu+1) . \qquad (5.2)$$

The total radiated intensity $I(t)$ is therefore found from Eq. (4.11) to be

$$I(t)=NI_0(N\mu+1)^2e^{t/\tau}/[N\mu+e^{t/\tau}]^2 . \qquad (5.3)$$

Although τ is undefined when $N\mu=-1$, the solution to Eq. (4.12) in this case is the same as that found by simply taking the limit $N\mu\to-1$ in Eq. (5.1). When $\mu>0$, which is certainly the usual situation, the relations given in (5.1) and (5.3) can also be written in the form

$$W(t)=-\tfrac{1}{2}N\{(1+1/N\mu)\tanh[(t-t_0)/2\tau]-1/N\mu\} \qquad (5.4)$$

and

$$I(t)=(1/4\mu)I_0(N\mu+1)^2\,\mathrm{sech}^2[(t-t_0)/2\tau] , \qquad (5.5)$$

where t_0 is given by

$$t_0=\tau_0\ln(N\mu)/(N\mu+1)=\tau\ln(N\mu) . \qquad (5.6)$$

The functional forms given in Eqs. (5.4) and (5.5) are superficially similar to those found in a number of other investigations. There are important differences which should be pointed out.

The appearance of a field amplitude $\epsilon\sim I^{1/2}$ which has a hyperbolic-secant time dependence does not imply any relation with the radiation fields of self-induced transparency.[3] Obviously, here the atoms are not returned to their initial states after the emission process. It is interesting, however, that the solution given in (5.4) for $W(t)$ in the limit $N\mu\gg 1$ is also the solution to the McCall-Hahn "area-theorem" equation[3] $d\theta/dz=-\frac{1}{2}\alpha\sin\theta$ if $t\to z$, $\tau\to 1/\alpha$, and $W=-\frac{1}{2}N\cos\theta$.

The dependence on time of $I(t)$ changes qualita-

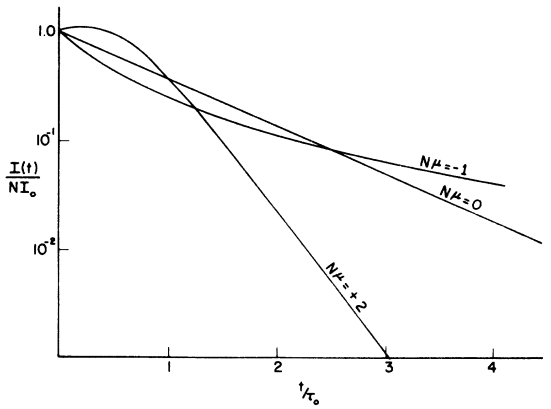

FIG. 2. The radiation rate $I(t)$ for small values of $N\mu$. The decay time constant is on the order of τ_0, the lifetime of a single excited atom.

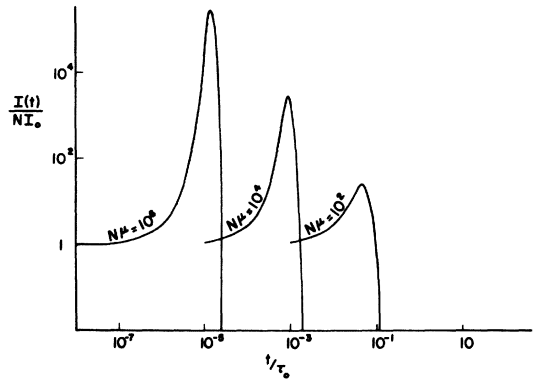

FIG. 3. The total radiation rate $I(t)$ for large values of $N\mu$.

tively as well as quantitatively when $N\mu$ is varied. In Figs. 2 and 3 we have plotted the ratio of the radiation rate $I(t)$ to the incoherent emission rate from N excited atoms as a function of time for maximum initial inversion and six values of $N\mu$. Notice that at $t=0$ the ratio is always unity, i.e., the N atoms emit incoherently in the limit in which each atom is excited into its upper state by the incident pulse. When $N\mu$ is large, the ratio quickly attains a value much greater than unity and the system is said to be "superradiant." The dependence on time shown here is strikingly similar to that displayed by Ernst and Stehle.[14]

In general, if the initial excitation energy is $W(\bar{t})=\bar{W}$ (where $\bar{W}$ is the initial energy), Eqs. (5.1) and (5.3) apply only for times $t\geq\bar{t}$, where

$$\bar{t}=\tau\ln\left(\frac{(\frac{1}{2}N-\bar{W})\mu+1}{\bar{W}/N+\frac{1}{2}}\right)\geq 0\ . \tag{5.7}$$

Thus Figs. 1 and 2 give $I(t)$ correctly in the case of partial initial inversion with $\bar{W}<\frac{1}{2}N$ only if the $t<\bar{t}$ parts of the curves are ignored.

We define the radiative-decay half-life T_1 to be the time during which the energy decreases from its initial value $\bar{W}$ to $\frac{1}{2}(\bar{W}-\frac{1}{2}N)$. Inverting Eq. (5.1) and using Eq. (5.7), we find

$$T_1=\tau\ln\left(\frac{2+(\frac{3}{2}N-\bar{W})\mu}{1+(\frac{1}{2}N-\bar{W})\mu}\right)\ . \tag{5.8}$$

In the limit that $\bar{W}\to\frac{1}{2}N$ and $N\mu\gg 1$,

$$T_1\cong(\tau_0/N\mu)\ln(N\mu)\cong t_0\ . \tag{5.9}$$

As in Ernst and Stehle's treatment[14] the time at which one-half of the atoms are in their ground states closely coincides with the time at which the radiation rate is maximum.[24]

One can associate an amplitude $\epsilon(t)$ with the intensity $I(t)$ by defining

$$\epsilon(t)=\begin{cases}[I(t)]^{1/2}e^{i\omega t}, & t\geq\bar{t}\\ 0\ , & t<\bar{t}\ .\end{cases}$$

The spectral width $\Delta\omega$ can then be defined in the usual way:

$$(\Delta\omega)^2\equiv\int(\omega'-\bar{\omega})^2\left|\tilde{\epsilon}(\omega')\right|^2d\omega'/\int\left|\tilde{\epsilon}(\omega')\right|^2d\omega'\ , \tag{5.10}$$

where $\tilde{\epsilon}(\omega')$ is the Fourier transform of $\epsilon(t)$, and

$$\bar{\omega}=\int\omega'\left|\tilde{\epsilon}(\omega')\right|^2d\omega'/\int\left|\tilde{\epsilon}(\omega')\right|^2d\omega'\ .$$

If this is done one finds that, for an initial energy $\bar{W}=0$ and $N\mu\gg 1$,

$$\Delta\omega\geq 1/2.34\tau=N\mu/2.34\tau_0\ , \tag{5.11}$$

using the well-known relation between the mean spectral extent of a Fourier integral and the mean extent of its Fourier transform.[25]

B. Shape Factor μ

It is clear that the radiated intensity can be orders of magnitude greater than the incoherent intensity. Let f be the enhancement factor defined by the ratio of the peak intensity $I(t_0)$ to the incoherent intensity radiated by the number of atoms which are excited at $t=t_0$. We find from Eqs. (5.4) and (5.5) for the energy and radiation rate

$$f\cong N\mu/2 \tag{5.12}$$

when $N\mu\gg 1$. This is generally much smaller than the enhancement factor for a system of N atoms confined to a region with dimensions much smaller than λ, where $\lambda=2\pi c/\omega$. However, when N is large and the system does have dimensions which are small compared with λ, then $\mu\cong 1$ and Eqs. (5.11) and (5.12) agree with results already found by Dicke[1] while Eq. (5.9) reduces to a result found by Fain.[13]

For these reasons the product $N\mu$ may be considered to be the parameter of large-system dynamics which plays the same role that the total number N plays in small-system dynamics. It is interesting that Ernst and Stehle[14] have also found such a parameter which they call s, and that $N\mu$ is identical to s if $|\vec{k}_1|=0$ [i.e., simultaneous excitation of all the atoms, recall Eq. (4.4)].

μ is a complicated function of the size and shape of the volume in which the two-level atoms are contained. However, it is often possible to estimate μ for the case of practical interest in which the number of atoms per λ^3 is large and they are distributed at random throughout the volume.

Consider, for example, the case in which the confining volume is a circular cylinder. $\Gamma(\hat{k},\hat{k}_1)$, originally defined by Eq. (2.14), can be written

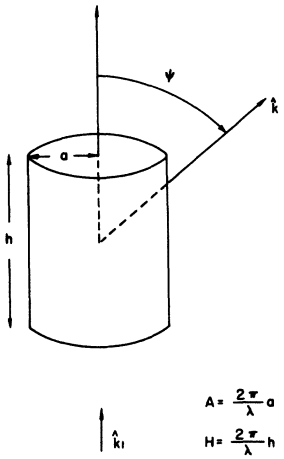

FIG. 4. Definition of A, H, and ψ. $\hat{k}_1$, the direction of the excitation pulse, is parallel to the cylinder axis, as is assumed in Figs 5–9.

explicitly as

$$\Gamma(\hat{k},\hat{k}_1)=\left|\frac{1}{N}\sum_{j=1}^{N}\exp[i(\vec{k}-\vec{k}_1)\cdot\vec{r}_j]\right|^2$$

$$=\frac{1}{N^2}\sum_{j=1}^{N}\sum_{l=1}^{N}\exp[i(\vec{k}-\vec{k}_1)\cdot(\vec{r}_j-\vec{r}_l)]\ . \quad (5.13)$$

Taking the ensemble average of both sides of Eq. (5.13) and, for convenience, denoting the ensemble average of $\Gamma(\hat{k},\hat{k}_1)$ again by $\Gamma(\hat{k},\hat{k}_1)$ leads to the result

$$\Gamma(\hat{k},\hat{k}_1)=1/N+[(N-1)/NV^2]$$

$$\times\int d^3x\int d^3x'\exp[i(\vec{k}-\vec{k}_1)\cdot(\vec{x}-\vec{x}')]\ , \quad (5.14)$$

where V is the cylinder volume. The $1/N$ arises from the diagonal terms in the double sum of Eq. (5.13) which must be treated separately from the off-diagonal terms when taking the ensemble average.[26]

Performing the integrations in Eq. (5.14), and taking $|\vec{k}|=|\vec{k}_1|=2\pi/\lambda$, yields

$$\Gamma(\hat{k},\hat{k}_1)=\frac{1}{N}+\frac{4(N-1)}{N}\left(\frac{\sin[\frac{1}{2}H(1-\cos\psi)]}{\frac{1}{2}H(1-\cos\psi)}\right)^2$$

$$\times\left(\frac{J_1(A\sin\psi)}{A\sin\psi}\right)^2 \quad (5.15)$$

and

$$\mu=\frac{6(N-1)}{NA^2H^2}\int_{-1}^{+1}\frac{dx(1+x^2)}{(1-x)^2(1-x^2)}$$

$$\times\sin^2[\tfrac{1}{2}H(1-x)]J_1^2[A(1-x^2)^{1/2}]. \quad (5.16)$$

Here J_1 is the Bessel function of the first kind, order one, and A, H, and ψ are defined in Fig. 4. We have assumed the atoms to undergo $\Delta m=\pm1$ transitions so that the emitted wave is circularly polarized. The exciting-pulse propagation vector $\vec{k}_1$ is along the cylinder axis.

From computer evaluations of the integral in Eq. (5.16) we have made a contour map showing constant values of μ. This is displayed in Fig. 5. Clearly, for macroscopic collections of atoms, μ is very small. The enhancement factor $f\cong\frac{1}{2}N\mu$ is usually orders of magnitude smaller than N.

The same integral can be estimated asymptotically for the two limits of a large disk and a long needle. The results for these two limits are

$$\mu=\begin{cases}\dfrac{3}{2A^2}\left(1+\dfrac{\sin^2H}{H^2}\right), & A\gg1 \text{ and } H\ll(\frac{1}{10}A)^2 \text{(disk)}\\[2ex] \dfrac{3\pi}{4H}\ , & H\gg1 \text{ and } A\ll\frac{1}{10}\sqrt{H} \text{ (needle)}\end{cases} \quad (5.17)$$

All integrations may be performed analytically in the case of a sphere. For both the cylinder and the sphere μ approaches $1-1/N$ when the linear dimensions are small compared with λ. This must be true for any shape.

In concluding this section we must emphasize that we have found what we might call a serious impedance mismatch between free space and a geometrically large system of spontaneously emitting atoms. That is, we have found the shape factor μ to be so small that the effectiveness of many-atom cooperation is reduced much below that found in geometrically small systems. $I(t)$ is so much smaller than N^2I_0 that at this point it is doubtful whether the term superradiance sensibly applies. In Sec. VI an investigation of the angular dependence of the emission will resolve these doubts.

VI. ANGULAR DISTRIBUTION OF SPONTANEOUS RADIATION

The matter of the time dependence of the radiated intensity, which was studied in Sec. III, is not fully settled without an examination of angular de-

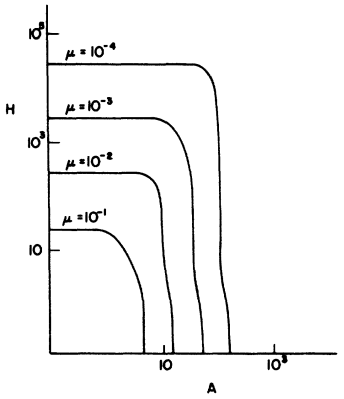

FIG. 5. A contour map showing constant values of μ for right circular cylinders of various heights and radii.

pendence as well. That this is not an entirely trivial question is suggested by Eq. (4.5). It is clear that $I(\hat{k}, t)$ cannot be broken into separate temporal and directional factors. The angular variation of the emitted radiation changes in the course of the emission process.

The discussions by Dicke of "coherence brightening,"[17] and by Ernst and Stehle of "ray" formation,[14] already make it clear that pronounced peaks in the angular distribution of emitted intensity should be expected. However, at the beginning of the emission process (if $\overline{W}$ is near $\frac{1}{2}N$) radiation is given off almost isotropically. In this section, we study the time development of the angular distribution.

The combination of Eqs. (4.5) and (4.10) gives

$$I(\hat{k}, t) = I_0(\hat{k})\{\tfrac{1}{2}N + W(t) + [\tfrac{1}{2}N + W(t)][\tfrac{1}{2}N - W(t)] \times[\Gamma(\hat{k}, \hat{k}_1) - 1/N]\} . \quad (6.1)$$

By substituting for $W(t)$ from Eq. (5.1) we obtain an explicit solution for the temporal and directional dependence of the emitted intensity which has the form

$$I(\hat{k}, t) = NI_0(\hat{k})[(N\mu + 1)/(N\mu + e^{t/\tau})] \times\{1 + [N(e^{t/\tau} - 1)/(N\mu + e^{t/\tau})][\Gamma(\hat{k}, \hat{k}_1) - 1/N]\} . \quad (6.2)$$

Equation (6.2) shows that in the limit that $t \to 0$, the angular distribution of the emitted intensity is given by

$$I(\hat{k}, 0) = NI_0(\hat{k}) . \quad (6.3)$$

Thus the far-field radiation pattern of a nearly inverted system of two-level atoms is initially the same as the radiation pattern of a single excited atom.

However, consider the time $t = t_0$ when the total radiation rate $I(t)$ is maximum. Using the definitions of t_0 and τ, Eq. (6.2) yields

$$I(\hat{k}, t_0) = I_0(\hat{k})\left(\frac{N\mu + 1}{2\mu}\right)\left[1 + \left(\frac{N\mu - 1}{2\mu}\right)\left(\Gamma(\hat{k}, \hat{k}_1 - \frac{1}{N}\right)\right] \quad (6.4)$$

for the radiation pattern at that time. For the case that $N\mu \gg 1$, Eq. (6.4) reduces to

$$I(\hat{k}, t_0) = \tfrac{1}{4}NI_0(\hat{k})[1 + N\Gamma(\hat{k}, \hat{k}_1)] . \quad (6.5)$$

In other words the system is emitting, at time $t = t_0$, at a rate approximately N^2 times the single-atom rate in any direction $\hat{k}$ for which the function $\Gamma(\hat{k}, \hat{k}_1)$ is of order unity.

Thus we see that the principal effect of the typically very small values of μ encountered with large systems is to delay the time of maximum emission. However, when that time is reached, the radiation is sensibly described as superradiance, at least in certain directions. Let us now look in some detail at the function $\Gamma(\hat{k}, \hat{k}_1)$ which defines these directions.

First we note from Eq. (4.6) that $\Gamma(\hat{k}, \hat{k}_1)$ is always maximum and equal to unity when $\hat{k} = \hat{k}_1$. Moreover, we see from Eq. (5.15) that for typical macroscopic systems of atoms this function falls off rapidly from unity as the direction $\hat{k}$ moves away from the direction $\hat{k}_1$, i.e., as ψ increases from zero. The larger the dimensions of the confining volume, the more rapidly it falls off. It is these considerations which lead to the most important feature of the angular distribution of the emitted intensity: For typical macroscopic systems of two-level atoms which have been excited by a pulse in the direction $\hat{k}_1$, most of the emitted radiation is emitted into a small cone about the direction $\hat{k}_1$. The larger the dimensions of the confining volume, the smaller the solid angle subtended by the cone.

To illustrate this feature consider again the problem of a right circular cylinder densely doped with two-level atoms. For this case $I_0(\hat{k})$ is given by

$$I_0(\hat{k}) = I_0(3/16\pi)(1 + \cos^2\psi) , \quad (6.6)$$

where $I_0 = \hbar\omega/\tau_0$ (providing, as before, the two-level atoms couple with circularly polarized radiation) and $\Gamma(\hat{k}, \hat{k}_1)$ is given by Eq. (5.15). After substituting these results into Eqs. (6.3) and (6.4) we find

$$i(\psi, 0) = \tfrac{1}{2}(1 + \cos^2\psi) \quad (6.7)$$

and

$$i(\psi, t_0) = i(\psi, 0)\left\{\frac{1}{2} + \frac{1}{2N\mu} + (N - 1)\left[1 - \left(\frac{1}{N\mu}\right)^2\right] \times\left(\frac{\sin(\frac{1}{2}H(1 - \cos\psi))}{\frac{1}{2}H(1 - \cos\psi)}\right)^2\left(\frac{J_1(A\sin\psi)}{A\sin\psi}\right)^2\right\} , \quad (6.8)$$

where we have defined

$$i(\cos^{-1}(\hat{k}\cdot\hat{k}_1), t) = \tfrac{8}{3}\pi I(\hat{k}, t)/NI_0 . \quad (6.9)$$

We shall refer to A^2/H as the "Fresnel number" of a particular right circular cylinder. When the Fresnel number is large, the cylinder is disk shaped; when the Fresnel number is small, the cylinder is needle shaped.

Equations (6.7) and (6.8) have been plotted in Figs. 6–9 for eight right circular cylinders. Each figure has plots for one Fresnel number (either 0.1 or 1000) and one density of active atoms (either[27] $61.2/\lambda^3$ or $61.2\times10^4/\lambda^3$) but *two* volumes. The continuous line in each figure represents the radiation emitted from a right circular cylinder with a volume such that $A^2H = 10^5$, while the broken line represents the radiation emitted from one with $A^2H = 10$. In each case the distance of a particular line from the origin in the direction ψ is proportional to $\log_{10}[10^5 i(\psi, t_0)]$, the direction $\psi = 0$ being

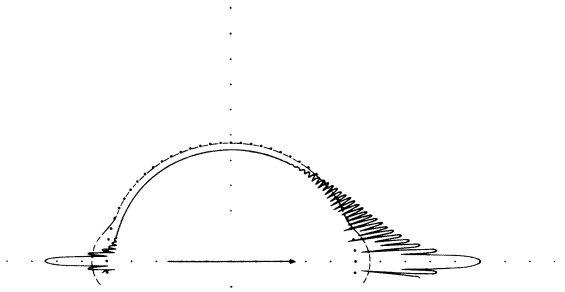

FIG. 6. The radiation patterns from two circular cylinders with Fresnel number = 1000, and density of active atoms = $61.2/\lambda^3$. As in Figs. 7–10 as well, the solid line shows $i(\psi, t_0)$, the normalized intensity at the time of maximum radiation rate $I(t)$ for a cylinder with $A^2H = 10^5$; the broken line shows $i(\psi, t_0)$ for a smaller cylinder with $A^2H = 10$. The dotted line is the incoherent radiation pattern from either cylinder when all of the atoms are excited. The arrow indicates the direction of the excitation pulse, assumed to be parallel to the cylinder axis. Note that the scale of the figures prevents accurate reproduction of the relative minima [cf. Eq. (6.8)].

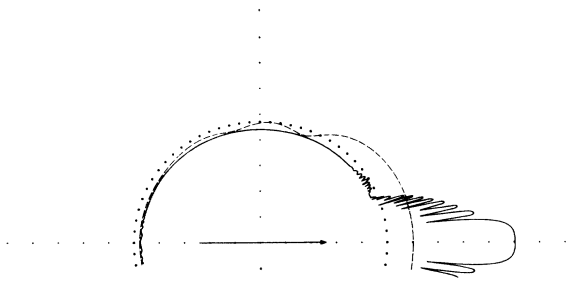

FIG. 7. The radiation patterns from two circular cylinders with Fresnel number = 0.1 and a density of 61.2 active atoms per λ^3. For other details see Fig. 6.

indicated by the arrow.

The dotted line (which is almost a semicircle) in each figure represents the incoherent intensity at $t = 0$; for this line the distance from the origin is proportional to $\log_{10}[10^5 i(\psi, 0)]$. Notice from Eq. (6.7) that $i(\psi, 0)$ is independent of the density of active atoms and the size and shape of the confining cylinder. Thus in each figure we need only one dotted line to represent the incoherent radiation from both of the two possible volumes. In actual fact, of course, the incoherent-radiation intensity is proportional to N [cf. Eq. (6.3), but for convenience we have divided out this dependence in defining $i(\psi, t)$ via Eq. (6.9)].

Although we have not completed the bottom halves of the patterns, in each case it can be obtained by reflection of the top half about the horizontal axis.

In each of these figures it is possible to determine orders of magnitude by counting + signs along the horizontal and vertical axes. Each + sign moving in a direction away from the origin indicates an increase of a factor of 10 in the intensity. Thus from Fig. 9, for example, one finds that for the cylinder with $A^2H = 10^5$ the coherent-radiation intensity in the forward direction ($\psi = 0$) at $t = t_0$ is greater by a factor of 10^9 than the incoherent intensity at $t = 0$.

To compare our result for the angular distribution of the emitted radiation with Ernst and Stehle's[14] "ray," it is convenient to integrate Eq. (6.2) over time. Let $J(\hat{k})$ be the total energy per unit solid angle in the direction $\hat{k}$ which is emitted from an inverted system of two-level atoms. That is, we define

$$J(\hat{k}) = \int_0^\infty I(\hat{k}, t)\, dt \ .$$

The integration may be performed explicitly, and leads to the relation

$$J(\hat{k}) = [\tau_0 I_0(\hat{k})/\mu^2]\{\mu \ln(N\mu + 1) + [N\mu - \ln(N\mu + 1)] \times [\Gamma(\hat{k}, \hat{k}_1) - 1/N]\} \ . \qquad (6.10)$$

It is not difficult to verify that this expression satisfies the requirement $\int J(\hat{k})\, d\Omega_{\hat{k}} = N\hbar\omega$, as it must.

Now if $N\mu \gg 1$, and $\hat{k}$ is sufficiently close to the forward direction $\hat{k}_1$, such that

$$\Gamma(\hat{k}, \hat{k}_1) \gg (1/N)[1 + \ln(N\mu + 1)] \ ,$$

then we find

$$J(\hat{k}) = (\tau_0 N/\mu) I_0(\hat{k}) \Gamma(\hat{k}, \hat{k}_1) \ . \qquad (6.11)$$

This is the same as the most probable angular distribution of emitted photons found by Ernst and

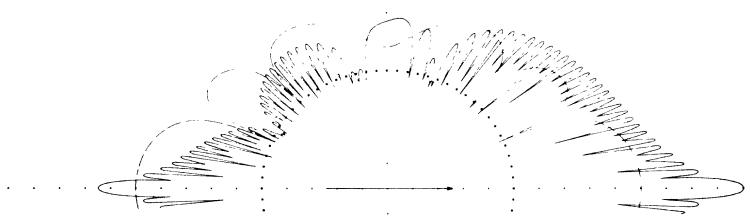

FIG. 8. The radiation patterns from two circular cylinders with Fresnel number = 1000 and a density of 61.2×10^4 active atoms per λ^3. For other details see Fig. 6.

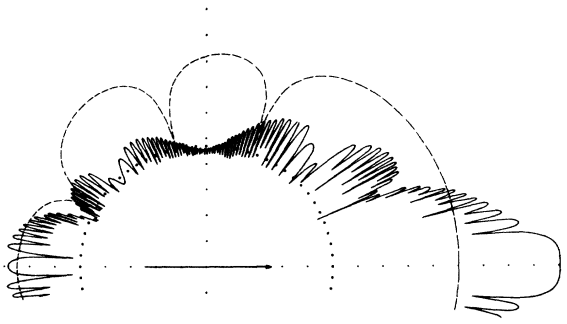

FIG. 9. The radiation patterns from two circular cylinders with Fresnel number = 0.1 and a density of 61.2×10^4 active atoms per λ^3. For other details see Fig. 6.

Stehle if we identify $\hat{k}_1$ with the direction of their "ray."

A further comparison is interesting to make. One can see that Dicke's criterion[17,28] for "coherence brightening" is reflected in the shape of our radiation patterns. Loosely speaking, coherence brightening can be expected whenever the number of emitters and the shape and size of the system are such as to produce pronounced lobes in the angular distribution of spontaneously emitted radiation.

Quantitatively, expressed in our notation, Dicke's criterion is $N \gg A^2$. Although it is not clearly stated in Dicke's paper, this criterion applies only when the Fresnel number is of the order of unity or larger. From our Figs. 6–9 one can verify that in every case of large Fresnel number when Dicke's criterion is met, the radiation pattern has highly pronounced lobes along the axis of the cylinder. This is the effect of "coherence brightening." If the system is described by a very small Fresnel number there is a corresponding criterion which is $N \gg H$. Again, when this criterion is satisfied, the radiation patterns are seen to have a forward lobe in which the vast majority of the emitted photons are concentrated. Note that unless $A^2 \gg 1$ (for a disk) or $H \gg 1$ (for a needle), the opening angle of the forward lobe need not be small (see, e.g., the small volume curve in Fig. 9).

As one would expect, the direction of the incident wave and the symmetry of the cylindrical volume of emitters are both important in determining the radiation rate. When the incident wave is not directed along the axis of the cylinder the radiation patterns are considerably changed. In Fig. 10 we show the radiation pattern from a cylinder excited by a wave traveling at an angle of 10° to the axis. The two patterns shown are in the plane through the cylinder axis which contains $\hat{k}_1$ and the plane perpendicular to that one. Except for the mode of excitation the situation is the same as for the large volume shown in Fig. 7.

VII. SUMMARY

Let us summarize the results which we have obtained for typical macroscopic collections of two-level atoms which have been excited by a short plane-wave pulse.

First, we find that follo .g an excitation pulse which transfers a maximum amount of energy to the atomic system, the radiation rate is incoherent (i.e., proportional to the total number of atoms excited). However, as the atomic energy decays, the radiation rate rises quickly to a peak value which can be many orders of magnitude greater than the incoherent rate. The atomic-system energy decays monotonically to zero with a half-life which closely coincides with the time interval during which the radiation intensity rises to its peak. The half-life of a geometrically large many-atom system is much shorter than the natural lifetime of a single atom, but not as short as the half-life of a small atomic system with the same number of atoms.

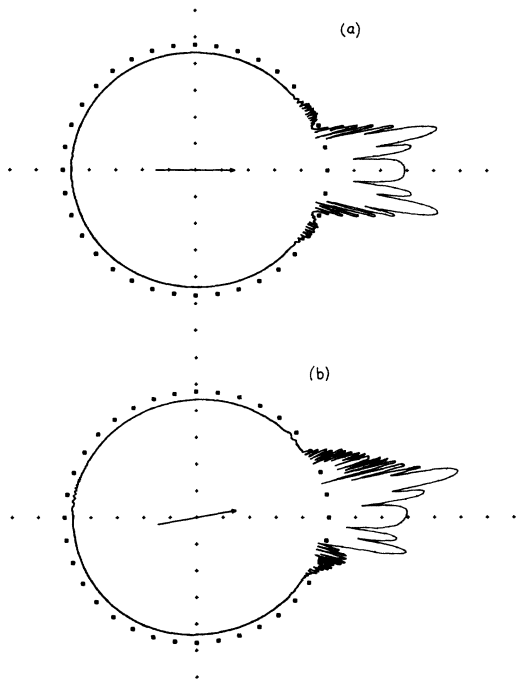

FIG. 10. The radiation pattern from the larger of the circular cylinders in Fig. 7 when the direction of the excitation pulse is at an angle of 10° to the cylinder axis. The top pattern (a) is $i(\psi, t_0)$ in a plane which intersects the cylinder axis and contains the vector $\hat{k}_1\times\hat{z}$, $\hat{z}$ being along the cylinder axis. The arrow is the projection of $\hat{k}_1$ on this plane. The lower pattern (b) is $i(\psi, t_0)$ in a plane which contains both $\hat{k}_1$ and $\hat{z}$. The arrow indicates the direction $\hat{k}_1$ and $\hat{z}$ is along the horizontal axis. In each case the plotted points give the incoherent intensity $i(\psi, 0)$.

Regarding the directional character of the emitted radiation, we find that when most of the atoms are excited the far-field radiation pattern is the same as that for a single excited atom. However, after a short time interval the pattern becomes sharply spiked in the forward direction; most of the radiation is emitted into a small cone about the direction of the plane-wave excitation pulse. The larger the dimensions of the confining volume, the smaller the solid angle subtended by the cone.

We emphasize that all of our results have been derived by assuming that the plane-wave excitation pulse leaves each atom with a finite nonzero dipole moment $\langle \vec{D}_l \rangle$. It is the relative phases of these dipole moments which fixes the direction of the single prominent "spike" of the far-field radiation pattern. If all of the dipole moments are zero, as they would be following excitation by a perfect π pulse, there is no preferred direction which is imposed by the excitation pulse, and the direction of the spike, if there is one, can only be determined by the geometry of the atomic system. This is indeed verified by the work of Ernst and Stehle.[14]

The principal advantages of our treatment are that it allows an (almost) arbitrary degree of initial excitation of the atomic system; it incorporates the complications of N cooperating emitters into the single constant μ; and it yields explicit formulas for the radiated intensity as a function of time and angle.

Note (*a*) *added in proof* [*cf. Sec. II*]. There are implications of this assumption that, so far we know, have not previously been stated. Two dipoles, in each other's near field, experience the usual dipole-dipole *antialigning* force. The common assumption is that this force is unimportant compared with the *aligning* force of the electric field which is used to excite the dipoles. A comparison of the antialignment dipole-dipole interaction energy (roughly p^2/r^3, where p is the dipole's moment and r is the dipole-dipole separation) with the aligning field-dipole interaction energy (roughly pE, where E is the exciting electric field strength) is possible. One finds that in order for the dipole-dipole interaction to be negligible in an exciting plane-wave field of strength $E \sim 10^3$ V/cm, the dipoles (assumed to have moment $p \sim ea_0$, where a_0 is the Bohr radius) must be further apart on the average than $\frac{1}{20}\lambda$, where λ is the radiated optical wavelength. Thus the idea that the dipole-dipole force does not disrupt the radiation process, a central feature of all studies of superradiance, is not compatible with another common assumption (which we have explicitly discarded) that the dipoles are all within a wavelength of each other. At least it is not compatible if the number of dipoles exceeds a few thousand. If the exciting plane wave field which aligns the dipoles is not as strong as 10^3 V/cm, then they must be even further apart. We are pleased to thank our colleagues, especially Professor G. S. Sherman and Professor C. R. Stroud, Jr., for provoking and participating in discussions of this subject.

Note (*b*) *added in proof* [*cf. Sec. III B*]. A step toward the inclusion of inhomogeneous broadening (nonidentical resonant frequencies) into the theory of superradiance has recently been taken. See J. H. Eberly, Nuovo Cimento Letters (to be published).

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APPENDIX A

We have implicitly and explicitly restricted ourselves to a physical system in which the following conditions are satisfied:

(i) There is negligible attenuation in the exciting pulse so that every atom in the system is excited to the same degree.

(ii) The spontaneous radiation from one part of the system does not give rise to ordinary stimulated emission in another part of the system.

(iii) The angle θ characterizing the superposition state of each atom evolves in time such that the same function of time $\theta(t)$ effectively describes every atom in the system.

(iv) The decay process is slow on the time scale of an optical period $2\pi/\omega$.

Condition (i) can be made more precise with the help of the "area theorem" of McCall and Hahn.[3] The area theorem is relevant here because the excitation process is imagined to be a coherent one, in which the exciting field has been tuned so that its center frequency corresponds to the resonant frequency of the atoms. Immediately after an excitation of this type, the state of the lth atom may be written as

$$|\theta_0\rangle_l = e^{i\chi_l/2}\sin\tfrac{1}{2}\theta_0|+\rangle_l + e^{-i\chi_l/2}\cos\tfrac{1}{2}\theta_0|-\rangle_l\,, \qquad \text{(A1)}$$

where θ_0 depends upon the field seen by the lth atom. The area theorem states that for a plane-wave exciting pulse, the angle θ_0 is a function of z, the distance into the sample along the path of the exciting pulse, and satisfies

$$\frac{d\theta_0(z)}{dz} = -\frac{\alpha}{2}\sin\theta_0(z)\,, \qquad \text{(A2)}$$

where α is the reciprocal absorption length for the atomic system.

We see that the degree of excitation cannot be the same for every atom unless the difference $\Delta\theta_0$

is very small even for atoms on opposite faces of the confining volume. Thus, in order to ensure that condition (i) is met, we must require

$$\Delta\theta_0 \cong \frac{d\theta_0}{dz}\,\Delta z \ll 1\ , \tag{A3}$$

where Δz is the thickness of the sample as seen by the exciting pulse. This is possible if the system thickness is small, or if $\theta_0(z=0)$ is close to π.

Conditions (ii)–(iv) will be examined more carefully in Appendix B by making use of the analytical results of Sec. V.

APPENDIX B

Given the analytical results of Sec. V we reconsider conditions (ii)–(iv) of Appendix A.

Condition (ii) requires that the superradiant spontaneous emission of the system as a whole occur faster than the time required for the radiation from one part of the system to cause significant stimulated emission in another part of the system. It is stimulated emission which gives rise to such effects as the "ringing" recently predicted[18] by Burnham and Chiao.

Consider a right circular cylinder densely populated with two-level atoms which has been excited by a $\frac{1}{2}\pi$ pulse (i.e., $\bar{W}=0$) moving in a direction parallel to its axis. Then Eq. (5.8) predicts a superradiant half-life, when $N\mu \gg 1$, of

$$T_1 = \tau_0 \ln(3/N\mu)\ . \tag{B1}$$

Burnham and Chiao have calculated a characteristic time τ_R for ringing to occur which is given by

$$\tau_R = 8\pi\tau_0/3\mathfrak{N}\lambda^2 z\ . \tag{B2}$$

Here $\mathfrak{N}$ is the density of active atoms and z is the distance into the medium at which an observer is recording the ringing.

If we require that the system decay spontaneously before significant ringing occurs within the cylinder then we must have

$$T_1 < \tau_R\big|_{z=h}\ , \tag{B3}$$

where h is the length of the cylinder. (This condition can be relaxed by replacing τ_R by about $10\tau_R$ if the excitation angle is close to π. However, then T_1 becomes larger also.) Equations (B1) and (B2) and the inequality above lead to

$$3\ln(3/2A^2) < \mu\ , \tag{B4}$$

where A is $2\pi/\lambda$ times the radius of the cylinder. Computer calculations indicate that, for a cylinder of length $\lambda H/2\pi$ and radius $\lambda A/2\pi$,

$$\mu \leq (3/2A^2)[1+\sin^2(H)/H^2] \tag{B5}$$

and that the equal sign in Eq. (B5) applies only for cases of large Fresnel number, i.e., $A^2/H \gg 1$. Since the inequality (B3) is not well satisfied even for large Fresnel numbers, we must expect that stimulated emission will always perturb the solutions we have presented to a certain extent. Such perturbation will be minimized for geometries with large Fresnel numbers.

If conditions (i) and (ii) of Appendix A are satisfied, the greatest change in $\theta(t)$ from one atom to another is given by $|\theta(t+\Delta t)-\theta(t)|_{\max}$, where Δt is the time required for light to pass between the two most widely separated atoms. This is because Δt is the time delay which an exciting pulse would suffer in going along the line between these two atoms. Condition (iii) will then be satisfied providing

$$\left|\theta(t+\Delta t)-\theta(t)\right|_{\max} \approx \left|\dot{\theta}\right|_{\max}\Delta t \ll 1\ . \tag{B6}$$

Notice that since $\theta(t)$ is the only time-dependent parameter in the interaction evolution operator u, this inequality is sufficient also to ensure that the approximation leading to Eq. (3.15) is valid.

Since

$$\dot{\theta} = -\left(\frac{d}{dt}\,(\cos\theta)\right)\Big/\sin\theta\ ,$$

$\cos\theta = -2W/N$ [cf. Eq. (4.10)], and $\dot{W}\neq 0$, then no matter how small Δt is, Eq. (B6) will fail if θ is in a sufficiently small neighborhood of π. For this reason (and for the reason mentioned in Sec. IV) our results are not valid for initially complete inversion of the atomic system except in the trivial case $N=1$. Speaking physically, this means that our treatment is invalid when the exciting pulse leaves the system without some finite dipole moment. It is this preexisting dipole moment, however small, which leads in all cases to maximum spontaneous emission in the direction of the exciting pulse. In this respect, our results are complementary to those of Ernst and Stehle,[14] which are valid only in the case of initially *complete* inversion.[29]

Realistic physical parameters may easily be found which ensure that (B6) is satisfied. For example, if $\Delta t = \Delta x/c$ (Δx being the separation between the two most widely separated atoms) and $N\mu \gg 1$, it may easily be shown that (B6) will be satisfied given[30]

$$\Delta x \ll 2c\tau_0/N\mu \tag{B7a}$$

and the following condition on the initial excitation angle:

$$\theta_0 \leq \pi - 4/N\mu\ . \tag{B7b}$$

Finally, we reconsider the approximation leading up to Eqs. (3.26) and (3.27), i.e., condition (iv). If $|\dot{u}| \ll \omega|u|$, this implies

$$1/T_1 \ll \omega\ . \tag{B8}$$

T_1 is always of order $\tau_0/N\mu$ or larger [cf. Eq. (5.9)]. Thus the condition implied by (B8) will always be less restrictive than that implied by (B7a) providing $\Delta x \geq \lambda/\pi$ (which is clearly true for most optical problems). If $\Delta x < \lambda/\pi$, then $\mu \approx 1$ and (B8) becomes

$$N \ll \tau_0 \omega \approx 10^{12} \quad \text{(for ruby)} . \tag{B9}$$

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†NASA Predoctoral Trainee, 1966-69.

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