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Finite-Temperature Relativistic Electron Beam*

A. J. Toepfer
Sandia Laboratories, Albuquerque, New Mexico 87115
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Moments of the relativistic Vlasov equation were taken over a Maxwellian distribution. The resulting fluid equations were coupled with Maxwell's equations and solved for an axially uniform cylindrically symmetric electron beam in the steady state. Magnetic neutralization and thermal conduction were neglected, and the beam was assumed to be charge neutralized in the observer's reference frame. The equilibrium was supported by radial gradients in particle pressure balancing against magnetic forces. With no externally applied magnetic field, or for a beam propagating parallel to a magnetic field with no rotation, the total current carried by the beam was found to be significantly less than the Alfvén critical current I_A, the ratio of beam current to I_A increasing with temperature and *decreasing* with increasing β. For a diamagnetic beam in an external field, a rigid-rotator model was assumed. Solutions of the steady-state equations were obtained for various values of rotation frequency ω and beam temperature. The radial density profiles were peaked on axis for small ω, and became hollow as ω increased, because of centrifugal forces. For all ω, the beam density was observed to exhibit radial oscillations which grew in amplitude and decreased in wavelength with decreasing temperature. Equilibria with magnetic field reversal were possible for all values of ω. Purely rotational self-consistent equilibria of the rigid-rotor type were not possible for a charge-neutralized beam of the type considered.

I. INTRODUCTION

The equations of motion for a finite-temperature relativistic fluid are well known.[1] More recently there have been attempts to formulate Lorentz-invariant theories for many-particle systems[2] and for plasmas.[3] It has been shown that if a description of a many-particle system is to be Lorentz invariant, then the particles must interact through fields.[4] The present work utilizes the results of Synge and Kursonoglu[5] to formulate the problem of a finite-temperature electron beam immersed in a plasma. The interactions between the beam electrons and plasma electrons and ions are described by self-consistent electric and magnetic fields determined from Maxwell's equations. All particle-particle correlations are neglected, and the electromagnetic field is treated classically. Thus, pair production and other quantum processes are not considered.

Bennett[6] has previously obtained solutions for a finite-temperature beam in which the axial velocity was independent of radial position. Radial gradients in pressure were balanced by space-charge effects in the beam rest frame. The possibility of a net rotation of the beam, such as can be expected if the beam is injected into an axial magnetic field, was neglected. The present theory includes the Bennett approximation of the problem as a special case. More recent theoretical work by Hammer and Rostoker[7] considered only the case of a zero-temperature electron beam injected into a plasma.

Alfvén[8] has derived an expression for the maximum current which can be carried by a fully space-charge-neutralized beam of constant cross section and velocity V. The total current is limited by the drift of the constituents of the beam in the self-magnetic field of the beam, and is of order

$$I_A \simeq (mc^3/e)\,\beta\gamma = 17\,000\beta\gamma \text{ A} \tag{1}$$

for electrons. In Eq. (1), $m(e)$ is the electron mass (charge), c is the speed of light, $\beta = V/c$, and $\gamma = (1-\beta^2)^{-1/2}$.

Section II contains some results pertaining to the choice of a distribution function for a relativistic many-body system. In Sec. III fluid equations for a finite-temperature system of charged particles are obtained by taking moments of the relativistic Vlasov equation. In Sec. IV the coupled Vlasov-Maxwell equations are solved in the steady state under the assumptions that all variables are functions of radius only and that the background plasma is immobile. Two specific cases are considered. The first is that of a beam in no external magnetic field which is fully space-charge neutralized in the lab frame. In this case there is no beam rotation and $\vec{v}\times\vec{B}$ forces are cancelled by radial gradients in pressure. The second case is for a beam injected into a weak axial magnetic field. Such a beam will tend to rotate about its axis in order to screen out the external field.

II. DISTRIBUTION FUNCTIONS

It will be assumed that one can write down a distribution function $\rho_m(x^\mu, p^\mu)$ which is a Lorentz scalar and whose amplitude is proportional to the probability of finding a particle of mass m in a volume $d\Gamma$ of the phase space of the independent variables

x^μ and p^μ. Following Kursonoglu[5] and Klimintovich[9] we choose[10]

$$d\Gamma = p^\mu d\sigma_\mu d^4p \ , \tag{2}$$

where $d\sigma_\mu$ is a timelike three-dimensional surface in space-time. Also, d^4p is the four-dimensional volume element in momentum-energy.

The normalization of ρ_m is then given by

$$mN = \int \rho_m p^\mu d\sigma_\mu d^4p = \int d\sigma_\mu P^\mu \ , \tag{3}$$

where

$$P^\mu \equiv \int \rho_m p^\mu d^4p \ , \tag{4}$$

and N is a Lorentz scalar. It is easy to see that if N is to be a constant in time and if $\rho_m(\beta_\mu x^\mu, \alpha_\mu p^\mu)$, where α_μ and β_μ are constant four-vectors, then we must have

$$\beta_\mu d\sigma^\mu = 0, \tag{5}$$

and β_μ must be spacelike.

In Fig. 1 let $\vec{\sigma}_1$ denote the timelike surface Σ at time t_1 and $\vec{\sigma}_2$ the same timelike surface at time t_2. In the time interval $t_2 - t_1$, Σ sweeps out a three-dimensional volume element Ω. If N is a constant, and if

$$I_1 = \int_{\vec{\sigma}_1} d\sigma_\mu P^\mu, \quad I_2 = \int_{\vec{\sigma}_2} d\sigma_\mu P^\mu, \tag{6}$$

then

$$\begin{aligned} I_1 - I_2 &= \left(\int_{\vec{\sigma}_1} d\sigma_\mu - \int_{\vec{\sigma}_2} d\sigma_\mu\right) P^\mu \\ &= \oint_{\vec{\sigma}_1+\vec{\sigma}_2} d\sigma_\mu P^\mu \\ &= \int_\Omega d^4x \frac{\partial P^\mu}{\partial x^\mu} = 0, \end{aligned} \tag{7}$$

where the four-dimensional Gauss rule has been invoked. The time interval and therefore Ω is arbitrary, and hence

$$\frac{\partial P^\mu}{\partial x^\mu} = 0 \tag{8}$$

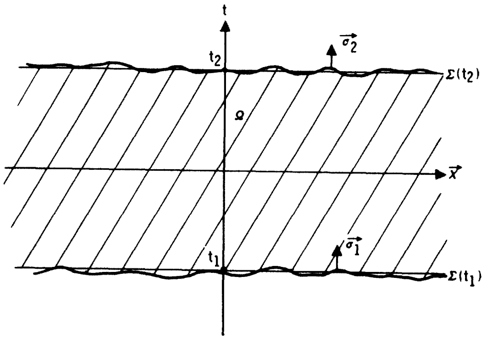

FIG. 1. The four-dimensional volume Ω swept out by the timelike three-dimensional surface Σ in the time interval $(t_2 - t_1)$.

for N constant. If P^μ is defined to be the four-momentum associated with a volume element of fluid at the point x^μ, then Eq. (8) is just the continuity equation.

For a system of particles in equilibrium, the form of the distribution function in momentum space has been given by Synge[2]:

$$\rho_m(x^\mu, p^\mu) = 2A(x^\mu)\delta^4(p^\mu p_\mu + m^2)e^{\alpha_\mu p^\mu}. \tag{9}$$

Here α_μ is a timelike vector related to the temperature and the factor of 2 is due to the fact that we only consider particles with positive energy. The δ function confines the particles to a three-dimensional shell in four-momentum space, and $A(x^\mu)$ describes the configuration of the system in real space. The constant N in Eq. (3) is given by

$$N = [4\pi m K_2(\alpha m)/\alpha] \int_S d\vec{r} A(\vec{r}),$$

where S is the reference frame in which $\alpha_\mu = (0, -\alpha)$ and K_2 is the K Bessel function of order 2. The constant α is just $1/\Theta$, where Θ is the kinetic temperature of the gas, and we will define

$$\xi \equiv m\alpha = m/\Theta, \tag{10}$$

the ratio of rest-mass energy to thermal energy.

III. FLUID EQUATIONS

We will assume that the distribution function for a given species obeys a "correlationless" kinetic equation given by

$$p^\mu \frac{\partial \rho}{\partial x^\mu} + f^\mu \frac{\partial \rho}{\partial p^\mu} = 0, \tag{11}$$

where p^μ and x^μ are independent variables, and f^μ is the self-consistent force arising from the distribution of particles in the beam-plasma system. It will also be assumed that the particles interact solely through electromagnetic forces, in which case

$$f^\mu = qF^{\mu\nu}P_\nu , \tag{12}$$

where q is the charge associated with a given species.

$F^{\mu\nu}$, the electromagnetic field tensor, is antisymmetric and obeys the Maxwell equations

$$\begin{aligned} &\frac{\partial F^{\mu\nu}}{\partial x^\nu} = 4\pi \sum_i \frac{q_i}{m_i} \int d^4p p^\mu \rho_i , \\ &\frac{\partial F^{\mu\nu}}{\partial x^\sigma} + \frac{\partial F^{\sigma\mu}}{\partial x^\nu} + \frac{\partial F^{\nu\sigma}}{\partial x^\mu} = 0, \end{aligned} \tag{13}$$

with i summed over all species.

The fluid equations are obtained by taking moments of Eq. (11) in momentum space. The zero moment just recovers the continuity equation (8). The first moment yields (cf. the Appendix)

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = \frac{q}{m} F^{\mu\sigma} P_\sigma , \tag{14}$$

with

$$T^{\mu\nu}=(1/m)\int \rho p^\mu p^\nu d^4p. \tag{15}$$

In order to cast Eq. (14) into a more familiar form, it will be convenient to define a fluid velocity v^σ by

$$P^\sigma = n_0 m v^\sigma,$$

where n_0 is the particle number density in the system where the fluid element is instantaneously at rest. The quantity $n_0 m$ will be called the proper mass density μ.

Define

$$\pi^{\mu\nu}\equiv T^{\mu\nu}-\mu v^\mu v^\nu \tag{16}$$

and Eq. (14) becomes

$$\frac{\partial}{\partial x^\nu}(\mu v^\mu v^\nu+\pi^{\mu\nu})=qn_0F^{\mu\sigma}v_\sigma, \tag{17}$$

which may be rewritten

$$\mu\frac{dv^\mu}{d\tau}+\frac{\partial\pi^{\mu\nu}}{\partial x^\nu}=qn_0F^{\mu\sigma}v_\sigma, \tag{18}$$

where use has been made of the continuity equation, and

$$\frac{d}{d\tau}\equiv v^\sigma\frac{\partial}{\partial x^\sigma}$$

is the proper time derivative. $\pi^{\mu\nu}$ may be identified with the pressure tensor.

Taking successively higher moments of Eq. (11) leads to a hierarchy of equations. One way to terminate the hierarchy is to assume a given form for the distribution function. If the distribution function is a Maxwellian and heat flow is neglected, then (cf. the Appendix)

$$\pi^{\mu\nu}=(p+\epsilon)v^\mu v^\nu+pg^{\mu\nu}, \tag{19}$$

where

$$p=\mu/\xi=n_0\Theta,\qquad \epsilon=-\mu\left\{\frac{d}{d\xi}\left[\ln\frac{K_2(\xi)}{\xi}\right]+1\right\}, \tag{20}$$

and $g^{\mu\nu}$ is the metric tensor.

In the limit $\xi\gg1$, we have $\epsilon\simeq\frac{3}{2}p$, and in the ultrarelativistic limit $\xi\ll1$, $\epsilon\simeq3p$. From Eqs. (16) and (19)

$$T^{\mu\nu}=(\mu+\epsilon+p)v^\mu v^\nu+pg^{\mu\nu}, \tag{21}$$

which is the well-known energy-momentum tensor for a perfect fluid.[1] For a fluid with a more complicated distribution function, the energy-momentum tensor may be calculated from Eq. (15).

In terms of fluid quantities, the first of Maxwell's equations in (13) becomes

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu}=4\pi\sum_i n_0^{(i)}q_i v^\mu_{(i)} \tag{22}$$

summed over all species.

It will be assumed that the electron beam and background plasma can be treated as perfect fluids, i.e., fluids in which the pressure tensor is diagonal and spatially isotropic in the rest frame of a fluid element. Heat flow will be neglected. In a given inertial reference frame, the continuity equation (8) may be written

$$\frac{\partial n}{\partial t}+\vec\nabla\cdot(n\vec v)=0, \tag{23}$$

where $n=\gamma n_0$, $\gamma=(1-\vec v\cdot\vec v)^{-1/2}$, and $\vec v$ is the fluid velocity. The equation of motion (18) becomes (cf. the Appendix)

$$n\gamma F\frac{d\vec v}{dt}+\frac{qn}{m}(\vec E\cdot\vec v)\vec v+\vec v\frac{\partial}{\partial t}\frac{n}{\gamma\xi}+\vec\nabla\frac{n}{\gamma\xi}=\frac{qn}{m}(\vec E+\vec v\times\vec B), \tag{24}$$

with $F=K_3(\xi)/K_2(\xi)$ and d/dt the convective derivative. Maxwell's equations are

$$\vec\nabla\cdot\vec E=4\pi\sum_i q_in_i,\qquad \vec\nabla\times\vec B-\frac{\partial\vec E}{\partial t}=4\pi\sum_i q_in_i\vec v_i,\qquad \vec\nabla\cdot\vec B=0,\quad \vec\nabla\times\vec E+\frac{\partial\vec B}{\partial t}=0. \tag{25}$$

IV. STEADY-STATE SOLUTION OF SELF-CONSISTENT EQUATIONS

Equations (23)–(25) will be solved in the steady state. The background plasma will be considered immobile and will serve only to reduce the space charge within the beam. Magnetic neutralization will not be considered. The electron beam will be assumed to be cylindrically symmetric and without variation in the axial direction. Even under these assumptions the resulting equations are grossly nonlinear and must be numerically solved.

In the steady state we have

$$\vec\nabla\cdot(n\vec v)=0, \tag{23$'$}$$

$$n\gamma F\vec v\cdot\vec\nabla\vec v-\frac{en}{m}\vec v(\vec v\cdot\vec E)+\frac{1}{\xi}\vec\nabla\left(\frac{n}{\gamma}\right)=-\frac{en}{m}(\vec E+\vec v\times\vec B), \tag{24$'$}$$

$$\vec\nabla\cdot\vec E=4\pi ne(f-1),\qquad \vec\nabla\times\vec B=4\pi ne(f\vec V-\vec v),\qquad \vec\nabla\times\vec E=0,\quad \vec\nabla\cdot\vec B=0. \tag{25$'$}$$

Here e is the magnitude of the charge on the electron, f is the fractional charge neutralization, and $\vec V$ is the velocity of the reference system relative to the laboratory.

Since $\vec\nabla\times\vec E=0$, let $\vec E=-\vec\nabla\Phi$. Substituting

$$\vec\nabla(1/\gamma)=-\gamma(\vec v\cdot\vec\nabla\vec v+\vec v\times\text{curl}\vec v)$$

into Eq. (24$'$) and dividing through by $n\gamma$, we obtain

$$\left(F-\frac{1}{\xi}\right)\vec{v}\cdot\vec{\nabla}\vec{v}-\frac{1}{\xi}\,\vec{v}\times\mathrm{curl}\vec{v}+\frac{1}{\xi\gamma^2 n}\,\vec{\nabla}n$$
$$+\frac{e}{m\gamma}\left[\vec{v}(\vec{v}\cdot\vec{\nabla}\Phi)-\vec{\nabla}\Phi\right]=-\frac{e}{m\gamma}\,\vec{v}\times\vec{B}\,. \qquad (26)$$

Assuming that all variables are functions of r only, Eq. (26) when written in cylindrical coordinates becomes

$$\frac{F\xi-1}{2\xi}\,\frac{dv^2}{dr}-F\left[v_\theta\left(\frac{dv_\theta}{dr}+\frac{v_\theta}{r}\right)+v_z\,\frac{dv_z}{dr}\right]+\frac{1}{\xi\gamma^2 n}\,\frac{dn}{dr}$$
$$+\frac{e}{m\gamma}\,(v_r^2-1)\,\frac{d\Phi}{dr}=\frac{e}{m\gamma}\,(v_z B_\theta-v_\theta B_z), \qquad (27)$$

$$Fv_r\left(\frac{dv_\theta}{dr}+\frac{v_\theta}{r}\right)+\frac{e}{m\gamma}\,v_\theta v_r\,\frac{d\Phi}{dr}$$
$$=\frac{e}{m\gamma}\,(v_r B_z-v_z B_r), \qquad (28)$$

$$Fv_r\,\frac{dv_z}{dr}+\frac{e}{m\gamma}v_z v_r\,\frac{d\Phi}{dr}=\frac{e}{m\gamma}(v_\theta B_r-v_r B_\theta). \qquad (29)$$

Both v_r and B_r may be eliminated from Eqs. (28) and (29) by taking their ratio. Then we have

$$F\left[v_\theta\left(\frac{dv_\theta}{dr}+\frac{v_\theta}{r}\right)+v_z\,\frac{dv_z}{dr}\right]+\frac{e}{m\gamma}\,(v_\theta^2+v_z^2)\,\frac{d\Phi}{dr}$$
$$=\frac{e}{m\gamma}(v_\theta B_z-v_z B_\theta). \qquad (30)$$

Substituting Eq. (30) into (27), we obtain

$$\frac{F\xi-1}{2\xi}\,\frac{dv^2}{dr}-\frac{e}{m\gamma^3}\,\frac{d\Phi}{dr}+\frac{1}{\xi\gamma^2 n}\,\frac{dn}{dr}=0. \qquad (31)$$

Choosing the reference system such that $\vec{V}\cdot\hat{r}=0$, the $\vec{\nabla}\times\vec{B}$ equation (25′) implies $v_r=0$, and the continuity equation

$$v_r\left(\frac{n}{r}+\frac{dn}{dr}\right)+n\,\frac{dv_r}{dr}=0 \qquad (32)$$

is trivially satisfied. Because of this, the set of equations (30), (31), and (25′) are incomplete. One must specify either $n(r)$ or $\Phi(r)$, and make some assumption on $v^2(r)$ and the relationship between v_θ and v_z in order to close the system of equations.

The assumption of Bennett was that $v^2=\text{const}$. Under this assumption γ is constant, and for electrons, Eq. (31) gives

$$n=n_c e^{e\xi\Phi/m\gamma}\,. \qquad (33)$$

Substituting Eq. (33) into Poisson's equation given in (25′), one obtains the Bennett solution:

$$n=n_c(1+br^2)^{-2},\quad b=\omega_p^2\,\xi(f_b-1)/8c^2. \qquad (34)$$

Here ω_p is the plasma frequency corresponding to the beam density on axis in the rest frame of the beam,

$$\omega_p^2=4\pi n_c e^2/m\gamma\,, \qquad (35)$$

f_b is the charge neutralization in the *beam rest frame*, and c is the speed of light.

For the Bennett distribution, the total current carried by the beam is

$$I_B=\frac{2\pi n_c e\beta}{\omega_p^2}\,c^3\int_0^\infty\frac{x\,dx}{(1+bc^2x^2/\omega_p^2)^2}=\frac{mc\beta\gamma\omega_p^2}{4be}.$$

The ratio of I_B to the Alfvén critical current I_A is

$$I_B/I_A=2/[\xi(f_b-1)].$$

Thus for $f_b\to 1$, the Bennett distribution can carry an arbitrarily large current.

Another class of solutions is possible under the assumption $f=1$, i.e., the beam is completely neutralized *in the observer's* frame of reference. In that case, $\nabla^2\Phi=0$ and $\Phi(0)=\text{finite}$ requires $\Phi\equiv 0$. Then the solution of Eq. (31) is

$$n=n_c[(1-v^2)/(1-v_c^2)]^{(F\xi-1)/2}, \qquad (36)$$

with $n(0)=n_c$, $v(0)=v_c$. In the nonrelativistic limit, Eq. (36) reduces to $n\simeq n_c\exp[-m(v^2-v_c^2)/2\Theta]$.

With $\Phi=0$, Eq. (30) becomes

$$v_\theta\left(\frac{dv_\theta}{dr}+\frac{v_\theta}{r}\right)+v_z\,\frac{dv_z}{dr}=\frac{e}{m\gamma F}(v_\theta B_z-v_z B_\theta). \qquad (37)$$

The self-consistent magnetic field is given by

$$B_z(r)=B_0+4\pi e\int_0^r nv_\theta\,dr',$$
$$B_\theta(r)=(4\pi e/r)\int_0^r n(fV-v_z)r'\,dr', \qquad (38)$$

with n computed from Eq. (36). Equations (36)–(38) constitute a closed set of nonlinear integro-differential equations for a space-charge-neutralized axially uniform cylindrically symmetric electron beam of finite temperature. These equations will be solved for two cases.

Case I: No Axial Magnetic Field

In this case there is no net rotation of the beam and therefore $v_\theta=0$. The observer's frame will be taken to be the laboratory. The self-consistent equation of motion is

$$\frac{dw}{dx}=\frac{1}{xF}(1-w^2)^{1/2}\int_0^x\left(\frac{1-w^2}{1-\beta^2}\right)^\alpha wx'dx', \qquad (39)$$

with $w\equiv v_z$, $\alpha=\frac{1}{2}(F\xi-1)$, $x=r\omega_{pc}$, and ω_{pc} the plasma frequency corresponding to the beam density on axis in the lab frame. The boundary condition is $w(0)=\beta$. Fixing ξ and β completely specifies the problem.

For β fixed, the width of the beam increases with increasing temperature. Figure 2 is a plot of normalized current $\iota=nw/n_c\beta$ as a function of radius for $\beta=0.94$, corresponding to a 1 MeV beam, and values of ξ ranging between 5 and 100. Also in Fig. 2, the beam profiles for $\xi=5$ and 10 have been compared to the Bennett distribution given in Eq. (33). The charge neutralization parameter f_b was chosen by fitting the Bennett distribution to the solution of Eq. (38) at $\iota = 0.5$. Although the two solutions correspond to completely different physical assumptions, the profiles are indistinguishable except in the outer edges of the beam.

The total current carried by the beam is

$$I = \frac{mc^3}{2e}\int_0^\infty \left(\frac{1-w^2}{1-\beta^2}\right)^\alpha wx\,dx. \tag{40}$$

The ratio of I to the Alfvén critical current I_A is plotted in Fig. 3 as a function of ξ for various values of β. It is interesting to note that for fixed temperature the ratio I/I_A decreases with increasing β. This is because the radially inward $v_z B_\theta$ force is balanced in the steady state by radial gradients in pressure. A larger v_z requires larger pressure gradients and hence a narrower beam for fixed ξ. Only for extremely high temperatures can the beam transport a considerable fraction of the Alfvén current.

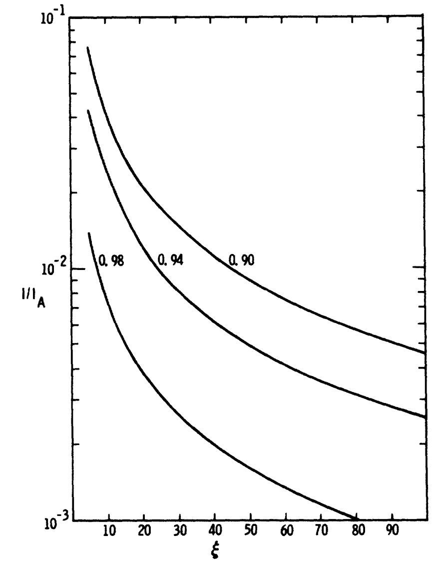

FIG. 3. Ratio of total current carried by beam in case I to the Alfvén critical current. The curves are parametrized by β.

Case II: Beam Injected into an Axial Magnetic Field

1. *Rigid Rotator*

An electron beam injected into an axial magnetic field B_0 will rotate in the azimuthal direction so as to reduce the field inside the beam. It is interesting to consider the case of a rigid rotator $v_\theta = r\Omega$. The case where Ω is large enough so that the external magnetic field is reversed on the axis of the beam is of relevance to proposed controlled thermonuclear devices such as Astron,[11] which seek to confine high-temperature ions inside a magnetic bottle produced by a rotating shell of relativistic electrons.

The self-consistent equations for the rigid rotator are

$$\frac{dw^2}{dx} = \frac{2}{F}(1 - x^2\omega^2 - w^2)^{1/2}(x\omega\omega_z - w\omega_\theta) - 4x\omega^2,$$

$$\frac{d\omega_z}{dx} = x\omega\left(\frac{1 - x^2\omega^2 - w^2}{1-\beta^2}\right)^\alpha, \tag{41}$$

$$\frac{d\omega_\theta}{dx} + \frac{\omega_\theta}{x} = -w\left(\frac{1 - x^2\omega^2 - w^2}{1-\beta^2}\right)^\alpha,$$

with w, x, and α defined as before, and $\omega = \Omega/\omega_{pc}$, $\omega_z = eB_z/m\omega_{pc}$, and $\omega_\theta = eB_\theta/m\omega_{pc}$. With ω and ξ fixed, the problem is completed by specifying $w(0) = \beta$, $\omega_z(0) = B$, and $\omega_\theta(0) = 0$.

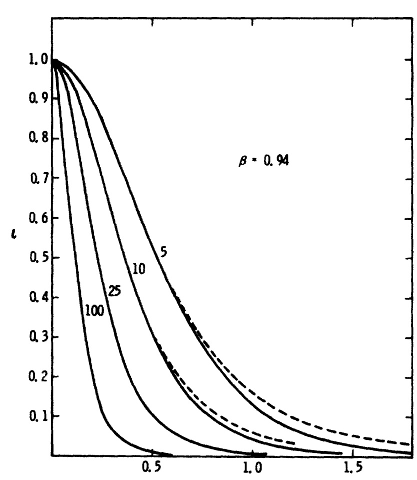

FIG. 2. Normalized beam current ι as a function of radius x for case I when $\beta = 0.94$. The solid curves are labeled by the value of $\xi = m/\Theta$. The dashed curves correspond to the Bennett distributions fit to the curves for $\xi = 5$ and 10 at $\iota = 0.5$.

The case $\omega = 0$ reduces to that considered in case I. For small ω such that

$$(1-\beta^2)^{1/2}(\beta^2 + 2\omega B)/(4F) \geq \omega^2,$$

we have the derivative $dw^2/dx|_{x=0} > 0$ and the beam current density $\iota(x) = nw/n_c\beta$ is peaked on axis. As ω increases, centrifugal forces on the beam increase and the beam current density develops a hole on axis. This behavior is shown in Fig. 4, where the parameters were taken to be $\xi = 5$, $\beta = 0.94$, and $B = 0$. The case $B = 0$ corresponds to the reduction of the external magnetic field to zero on the beam axis.

The line density N of electrons in the beam can be obtained by evaluating $B_z(r)$ in Eq. (38) for a rigid rotor. Defining

$$\nu = Ne^2/mc^2,$$

the number of electrons per classical electron radius of beam length, one obtains

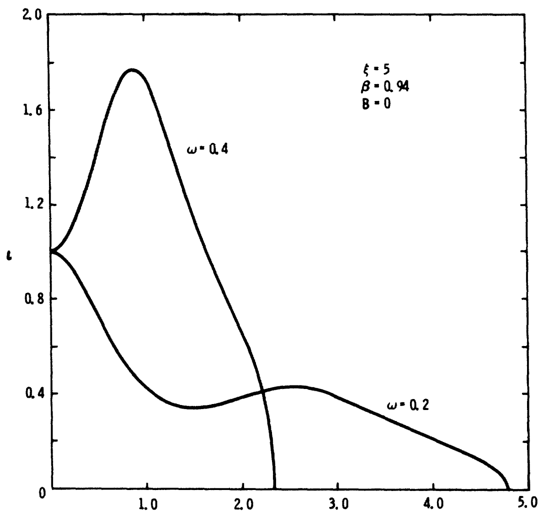

FIG. 4. Variation of radial beam current profile with ω.

$$\nu = [\omega_z(a) - B]/2\omega,$$

where a is the radius of the beam. Table I exhibits the calculated values of ν for various values of angular frequency ω and inverse temperature ξ, with $\beta = 0.98$ and $B = 0$.

For fixed ω, β, and B, as the temperature decreased, radial oscillations were observed in the beam density. Figures 5 and 6 show this behavior reflected in $\iota(x)$, for $\omega = 0.1$ and $\omega = 10$. For beams with field reversal, the same behavior was observed as is shown in Fig. 7, where the normalized density n/n_c is plotted vs x for $\beta = 0.98$, $B = -10$, and $\omega = 10$. This behavior can be understood by examining Eqs. (41). As temperature decreases, $\alpha = \frac{1}{2} \times (F\xi - 1)$ gets very large and small variations in w can lead to large variations in $d\omega_z/dx$ and $d\omega_\theta/dx$. Making the change of variable $\eta^2 = 1 - x^2\omega^2 - w^2$ in Eqs. (41), one obtains

$$\frac{d\eta^2}{dx} = 2x\omega^2 - \frac{2\eta}{F}[x\omega\omega_z - (1 - x^2\omega^2 - \eta^2)^{1/2}\omega_\theta],$$

$$\frac{d\omega_z}{dx} = \frac{xw}{(1-\beta^2)^\alpha}\eta^{2\alpha}, \qquad (41')$$

$$\frac{d\omega_\theta}{dx} + \frac{\omega_\theta}{x} = -\frac{1 - x^2\omega^2 - \eta^2}{(1-\beta^2)^\alpha}\eta^{2\alpha}.$$

Differentiating the first equation in (41′) gives

$$\frac{d^2\eta^2}{dx^2} = -\frac{2\eta}{F}\left[x\omega\frac{d\omega_z}{dx} - (1 - x^2\omega^2 - \eta^2)^{1/2}\frac{d\omega_\theta}{dx}\right] + (\text{slowly varying terms}). \qquad (42)$$

Ignoring the slowly varying terms in Eq. (42) and substituting for $d\omega_z/dx$ and $d\omega_\theta/dx$ leads to

TABLE I. Value of ν as a function of ω and ξ for $\beta = 0.98$ and $B = 0$.

ω	ξ	ν
0.1	5	4.7
	10	3.4
	50	2.6
1.0	5	2.5
	10	2.4
	50	2.4
10.0	5	1.9
	10	1.7
	50	2.2

$$\frac{d^2\eta^2}{dx^2} \simeq -\frac{2}{F(1-\beta^2)^\alpha}(1-\eta^2)\eta^{2\alpha+1}. \qquad (43)$$

In the low-temperature limit, we have $F \simeq 1$ and $\alpha \sim \xi \gg 1$, and therefore may expand $\eta = \eta_0 + \delta n$, and the dominant term in Eq. (43) is proportional to ξ:

$$\frac{d^2\delta\eta}{dx^2} \simeq -\frac{\xi(1-\eta_0^2)\eta_0^{2\alpha-1}}{(1-\beta^2)^\alpha}\delta\eta,$$

so that $\delta\eta \simeq e^{i\kappa x}$, with

$$\kappa^2 = \xi(1-\eta_0^2)\eta_0^{2\alpha-1}/(1-\beta^2)^\alpha. \qquad (44)$$

Equation (44) agrees well with the wave number of the oscillations observed in the numerical solutions to Eqs. (41).

The physical mechanism of the radial density oscillations can be somewhat clarified by noting that

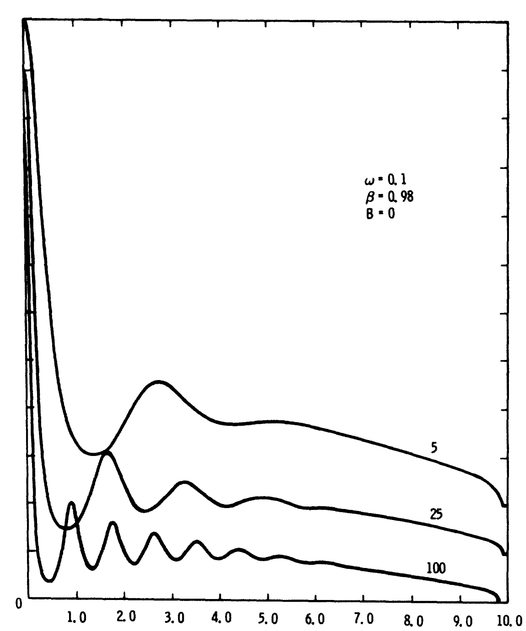

FIG. 5. Radial beam current profiles for $\omega = 0.1$. Each curve is labeled by the parameter $\xi = m/\Theta$, with the ordinates displaced by one unit for purposes of clarity.

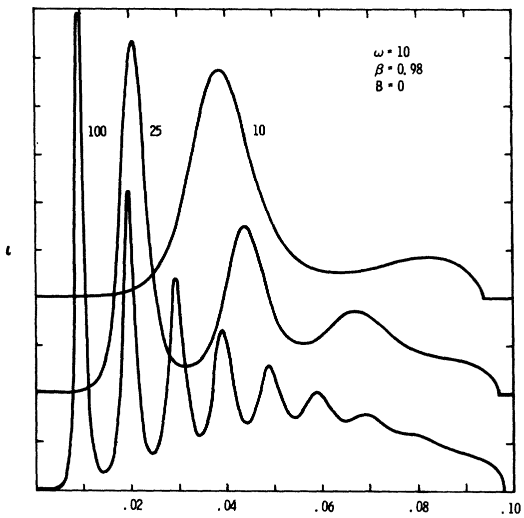

FIG. 6. Radial beam current profiles for $\omega = 10$. Each curve is labeled by the parameter $\xi = m/\Theta$, with the ordinates displaced by two units for purposes of clarity.

the wave number κ varies inversely with temperature and that the dominant term in Eq. (41) which gives rise to the oscillatory solutions is the $\vec{v}\times\vec{B}$ force. Hence, the interaction is between magnetic pressure and kinetic pressure. We note, however, that such oscillations did not arise for $v_\theta = 0$, thus implying that centrifugal force effects also contribute to the process.

2. *Constant Axial Velocity*

For w constant, the system of equations (41) is overdetermined and no rigid-rotor solution to the equations exists for real temperature. A self-consistent solution for v_θ does exist and is given by the solution to the equation (written in the inertial frame where $w = 0$)

$$\frac{du}{dx} + \frac{u}{x} = \frac{1}{F}(1-u^2)^{1/2}\left(B + \int (1-u^2)^{\alpha} u\, dx'\right), \quad (45)$$

with $u \equiv v_\theta$, and the other parameters as defined above. The boundary condition is $u(0) = 0$. Fixing the beam temperature ξ and B completely specifies the problem.

The case $B = 0$ in Eq. (45) leads to the trivial solution $u = 0$. Note that this limit does not correspond to case I considered above; rather, it corresponds to the limit of the Bennett solution, Eq. (34), with $f_b = 1$. As B is increased in magnitude, the radius of the beam decreases and is peaked on the axis. The solutions to Eq. (45) do not admit a hollow beam since the sign of du/dx at $x = 0$ is determined by B. For B finite and fixed, the width of the electron beam increases with temperature as shown in Fig. 8.

V. DISCUSSION

The self-consistent equations of motion for a finite-temperature collisionless relativistic plasma have been derived by taking moments of the Lorentz-covariant Vlasov equation. The special case of an axially uniform cylindrically symmetric electron beam in an immobile background of positive ions has been considered in the steady state. In order to close the set of equations, it was necessary to make some assumptions on either the density, or the electric field, or the fluid velocity.

Under the assumption that the fluid velocity was independent of radius, the solutions of Bennett were obtained. The equilibrium consisted of radial gradients in pressure balanced against a radial elec-

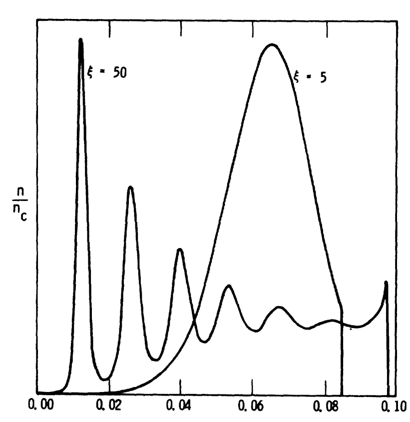

FIG. 7. Radial beam density profiles for $\omega = 10$ and $B = -10$ corresponding to field reversal. Each curve is labeled by the parameter $\xi = m/\Theta$. The ordinate scaling is arbitrary.

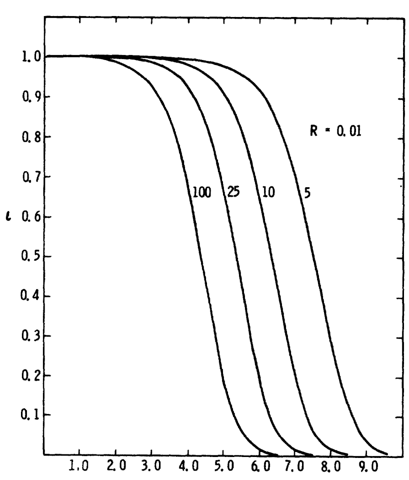

FIG. 8. Normalized beam current ι as a function of radius x for case II with w = const and $B = 0.01$. The curves are labeled by the parameter ξ.

tric field in the rest frame of the beam. For an electron beam with density n_0 on axis, the two parameters which determine the Bennett distribution are the temperature ξ^{-1} and the charge neutralization parameter in the rest frame of the beam f_b. For $f_b \simeq 1$, the total current carried by the beam can exceed the Alfvén critical current.

For an electron beam which had a radially dependent velocity, solutions were obtained for the case where the beam was charge neutralized in the laboratory, $f_l = 1$. In this case, the equilibria were supported by radial gradients in pressure balanced against magnetostatic forces.

In the absence of an external magnetic field, the free parameters were the velocity of the beam on axis β, and the beam temperature ξ^{-1}. For all values of the parameters considered, the ratio of beam current to Alfvén critical current was less than 1. Ratios $I/I_A > 0.1$ implied extremely high temperatures. For example, a 2-MeV beam ($\beta = 0.98$) with $I/I_A \simeq 0.01$ implied a beam temperature of about 500 keV in the rest system. Such temperatures could only be produced by some anomalous randomization of beam energy, possibly due to turbulence. As β was increased, the ratio I/I_A was found to decrease for a given temperature.

For a warm beam injected into an axial magnetic field, radial motion of the beam electrons induces a net rotation of the beam, causing the beam to behave diamagnetically. The equations of motion for the beam were numerically integrated assuming a rigid-rotor equilibrium $v_\theta = x\omega$. The variable parameters were angular velocity ω, temperature ξ^{-1}, velocity of the beam on axis β, and the value of the magnetic field on axis B. The qualitative behavior of the equilibria was not sensitive to the assumed values for β and B. For arbitrary temperature, the density profiles were peaked on axis for small ω and hollow for large ω due to centrifugal forces. Field reversal on axis was possible with both types of equilibria.

Due to the nonlinear interaction between pressure, centrifugal, and magnetic forces, large amplitude oscillations in the radial density profiles were obtained for all values of ω. The wavelength of the oscillations was proportional to temperature. At high temperatures, the wavelengths were comparable to the beam radius and the density oscillations did not appear. One would expect that radial oscillations in density would lead to macroinstabilities and, therefore, it would appear that stable rigid-rotor equilibria would be more probable for a high-temperature beam. From Table I it is apparent that electron beams which have sufficient rotation to exclude an external magnetic field from the axis have appreciable line densities ν. The above results indicate that high-temperature beams with $I/I_A = \nu/\gamma \sim 1$ should be considered for injection into machines like Astron.

Within the present model, purely rotational ($v_z = 0$) self-consistent equilibria of the rigid-rotor type were not possible for a charge-neutralized beam. Self-consistent solutions for $v_\theta(x)$ with $v_z = 0$ were obtained, but did not admit to field reversal.

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APPENDIX

The first moment of the Vlasov equation is

$$\int d^4p\left(p^\nu p^\mu \frac{\partial \rho}{\partial x^\mu} + q p^\nu F^{\mu\sigma} p_\sigma \frac{\partial \rho}{\partial p^\mu}\right) = \frac{\partial}{\partial x^\mu}\int d^4p\, p^\nu p^\mu \rho + qF^{\mu\sigma}\int d^4p\, p^\nu p_\sigma \frac{\partial \rho}{\partial p^\mu} = 0. \quad \text{(A1)}$$

Integrating the second term by parts, we have

$$F^{\mu\sigma}\int d^4p\, p^\nu p_\sigma \frac{\partial p}{\partial p^\mu} = F^{\mu\sigma}\int d^4p\left(\frac{\partial}{\partial p^\mu}(p^\nu p_\sigma \rho) - p_\sigma \delta^\nu_\mu \rho - p^\nu g_{\mu\sigma}\rho\right). \quad \text{(A2)}$$

The first term is 0 because the distribution function vanishes at infinity in momentum space, and the last term is 0 because $F^{\mu\sigma}$ is an antisymmetric tensor. Then, we have

$$F^{\mu\sigma}\int d^4p\, p^\nu p_\sigma \frac{\partial p}{\partial p^\mu} = -F^{\nu\sigma}P_\sigma. \quad \text{(A3)}$$

Multiplying (A1) by $1/m$ and substituting from (A3) give Eqs. (14) and (15).

The pressure tensor is

$$\pi^{\mu\nu} = T^{\mu\nu} - (1/\mu)P^\mu P^\nu. \quad \text{(A4)}$$

For the Maxwellian given in Eq. (9), we identify

$$mN \equiv \int_S d\vec{r}\,\mu(\vec{r}) = [4\pi m^3 K_2(\xi)/\xi]\int_S d\vec{r}\,A(\vec{r}), \quad \text{(A5)}$$

$$A(\vec{r}) = \mu(\vec{r})\,\xi/4\pi m^3 K_2(\xi). \quad \text{(A6)}$$

In order to evaluate P^μ and $\pi^{\mu\nu}$, we convert to spherical coordinates in momentum space through the transformation

$$\begin{aligned} p^1 &= \kappa \sinh\chi \sin\theta \cos\varphi, & 0 &\le \kappa < \infty \\ p^2 &= \kappa \sinh\chi \sin\theta \sin\varphi, & 0 &\le \chi < \infty \\ p^3 &= \kappa \sinh\chi \cos\theta, & 0 &\le \theta \le \pi \\ p^0 &= \kappa \cosh\chi, & 0 &\le \varphi \le 2\pi \\ d^4p &= \kappa^3 \sinh^2\chi \sin\theta\, d\kappa\, d\chi\, d\theta\, d\varphi. \end{aligned} \quad \text{(A7)}$$

In this coordinate system $p^\mu p_\mu = -\kappa^2$ and the δ function in ρ restricts the integrand to a surface $\kappa = m$.

Then we have

$$P^{\mu} = \int d\omega\, p^{\mu} A(x^{\mu})\, e^{\alpha^{\sigma} p_{\sigma}}, \tag{A8}$$

with $d\omega = m^2 \sinh^2\chi \sin\theta\, d\chi\, d\theta\, d\varphi$. In the Lorentz frame S, we have

$$P_S^{\mu} = A(\vec{r}) \int d\omega\, p^{\mu}\, e^{-\xi \cosh\chi}. \tag{A9}$$

Evaluating the integral and using (A6) one obtains

$$P_S^i = 0, \quad i = 1, 2, 3; \quad P_S^0 = \mu(\vec{r}). \tag{A10}$$

Hence S is the rest system of the fluid and μ is the proper mass density at position $\vec{r}$. In any other reference frame we have

$$P^{\mu} = (\mu\gamma\vec{v}, \mu\gamma) = \mu v^{\mu}, \tag{A11}$$

with $\gamma = (1 - \vec{v}\cdot\vec{v})^{-1/2}$.

In a similar way we evaluate $T^{\mu\nu}$ in the reference frame S:

$$T_S^{\mu\nu} = [A(\vec{r})/m] \int d\omega\, p^{\mu} p^{\nu}\, e^{-\xi \cosh\chi}. \tag{A12}$$

All off-diagonal terms are zero and we have

$$T_S^{\mu\nu} = \begin{pmatrix} \mu/\xi & 0 & 0 & 0 \\ 0 & \mu/\xi & 0 & 0 \\ 0 & 0 & \mu/\xi & 0 \\ 0 & 0 & 0 & -\mu \dfrac{d}{d\xi} \ln\left(\dfrac{K_2(\xi)}{\xi}\right) \end{pmatrix}; \tag{A13}$$

hence,

$$\pi_S^{\mu\nu} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, \tag{A14}$$

with p and ϵ as defined in Eq. (20).

A Lorentz transformation on (A14) to the system S' moving with velocity $-\vec{v}$ gives Eq. (19).

The covariant equation of motion is

$$\mu \frac{dv^{\mu}}{d\tau} + \frac{\partial}{\partial x^{\nu}} (p + \epsilon) v^{\mu} v^{\nu} + \frac{\partial p}{\partial x_{\mu}} = q n_0 F^{\mu\sigma} v_{\sigma}, \tag{A15}$$

or

$$(\mu + p + \epsilon) \frac{dv^{\mu}}{d\tau} + v^{\mu} \frac{\partial}{\partial x^{\nu}} [(p + \epsilon) v^{\nu}] + \frac{\partial p}{\partial x_{\mu}} = q n_0 F^{\mu\sigma} v_{\sigma}. \tag{A16}$$

Taking the scalar product of (A16) with v_{μ} results in

$$\frac{\partial}{\partial x^{\nu}} [(p + \epsilon) v^{\nu}] = \frac{dp}{d\tau}, \tag{A17}$$

where we have used $v_{\mu} v^{\mu} = -1$, $F^{\mu\sigma} = -F^{\sigma\mu}$, and the definition of $d/d\tau$. Now from Eq. (20) we have

$$\mu + p + \epsilon = \mu \left[\frac{1}{\xi} - \frac{d}{d\xi} \ln\left(\frac{K_2(\xi)}{\xi}\right) \right] = \mu F(\xi), \tag{A18}$$

with $F(\xi) = K_3(\xi)/K_2(\xi)$. Hence, the equation of motion becomes

$$\mu F \frac{dv^{\mu}}{d\tau} + v^{\mu} \frac{dp}{d\tau} + \frac{\partial p}{\partial x_{\mu}} = q n_0 F^{\mu\sigma} v_{\sigma}. \tag{A19}$$

In a particular Lorentz frame, (A19) becomes

$$\mu F \gamma \frac{d}{dt}(\gamma \vec{v}) + \gamma^2 \vec{v} \frac{dp}{dt} + \vec{\nabla} p = q n_0 \gamma (\vec{E} + \vec{v} \times \vec{B}) \tag{A20}$$

for the space component, and

$$\mu F \gamma \frac{d\gamma}{dt} + \gamma^2 \frac{dp}{dt} - \frac{\partial p}{\partial t} = q n_0 \gamma (\vec{E} \cdot \vec{v}) \tag{A21}$$

for the time component. Defining $n = n_0 \gamma$ and substituting $p = n/\gamma\xi$, we multiply (A21) by $\vec{v}$ and substitute the result into (A20) to obtain Eq. (24).

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