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Transport Theory for a Two-Dimensional Dense Gas

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A kinetic theory has been proposed by several authors with the goal of eliminating the divergences which appear in the density expansion in nonequilibrium systems. Here, it is shown that for a two-dimensional simple gas the theory presents a new divergence, resulting from the fact that correlations propagate over long distances as a result of hydrodynamic transport. This divergence is discussed explicitly for a gas model: the Maxwell model. It will be indicated why the kinetic theory for a perfect Lorentz gas does not exhibit this new divergence.

I. INTRODUCTION

Near the perfect-gas state, equilibrium quantities of a classical fluid can be calculated by means of the virial expansion. In the same manner, nonequilibrium quantities such as viscosity or thermal conductivity can be expanded in powers of density. One can also develop the collision operator of the kinetic theory in powers of the density n. In the lowest order in n, one finds the Boltzmann kinetic equation, then the Choh-Uhlenbeck equation, [1] etc. This second approximation in n implies an explicit solution of the three-body problem, so that any calculation to this order would be difficult. However, it has been shown[2] that the shear viscosity of a gas of hard disks, at the Choh-Uhlenbeck order, leads to a diverging integral, and that presumably the same difficulty arises with the next higher order of density for a gas of hard spheres.

A theory has been proposed[3] with the goal of eliminating these divergencies: The transport coefficients are developed in powers of the density. Summing in each order the most divergent contributions, one is led to a renormalized Choh-Uhlenbeck collision operator, called the "ring-collision operator." This operator brings into effect the collective dynamics for calculation of the long-range correlation. In Sec. II we derive this ring-collision operator in a manner slightly different from those given previously. [3]

The Green's function for the linearized Boltzmann equation appears in the ring-collision operator. In the general case, one cannot find this Green's function explicitly. Nevertheless one knows its properties for those disturbances which propagate at long range, i.e., in the hydrodynamical limit. The

possibility of the long-range propagation of correlation thus appears. In the case of the two-dimensional gas, a new type of divergence originates from this propagation. We will show this new class of divergence in a particular case.

Until now this renormalization has only been studied for the perfect Lorentz gas.[4] We will indicate why for this particular case the renormalized theory does not possess the new divergence.

II. RING KINETIC EQUATION FOR A MONOATOMIC GAS

Following the work of Kawasaki and Oppenheim,[3] several methods[5] have been proposed to renormalize the divergent density expansion of nonequilibrium quantities.

In general, these methods start from an expression for a transport coefficient given by a Kubo integral. In the model to be treated in Sec. IV, the transport coefficient is not given by an already known Kubo formula. We therefore give a direct derivation of the renormalized ring-collision term without starting from a Kubo-type formula.

Let $f_s(1, 2, \ldots, S)$ be the s-body distribution function obtained from the distribution function $D(1, 2, \ldots, N)$ of N identical particles by

$$f_s(1, 2, \ldots, S) = \left(\frac{V}{N}\right)^s \frac{N!}{(N-S)!} \times \int d\Gamma_{s+1} \cdots d\Gamma_N D(1, 2, \ldots, N), \tag{2.1}$$

where V is the volume of the box containing the particles, $d\Gamma_i = d\vec{r}_i\, d\vec{p}_i$, and $D(1, 2, \ldots, N)$ is normalized as $\int d\Gamma_1 \cdots d\Gamma_N\, D(1, 2, \ldots, N) = 1$.

The particles of mass m are supposed to interact through a two-body potential $V(\vec{r}_{ij})$. The Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy provides us with an infinite system of coupled equations defining the evolution of the reduced distribution functions f_s. The general term of this hierarchy reads[6]

$$\frac{\partial f_s}{\partial t} + H_s f_s + n \sum_{i=1}^{s} \int d\Gamma_{s+1}\, \theta_{i,s+1} f_{s+1} = 0, \tag{2.2}$$

where

$$\theta_{i,j} = \frac{\partial V(\vec{r}_{ij})}{\partial \vec{r}_{ij}} \cdot \left(\frac{\partial}{\partial \vec{p}_i} - \frac{\partial}{\partial \vec{p}_j}\right),$$

$$H_s = H_s^0 + \sum_{i<j} \theta_{i,j},$$

where H_s^0 is the Liouville operator of the s noninteracting particles defined as

$$H_s^0 = \sum_{i=1}^{s} \frac{\vec{p}_i}{m} \cdot \frac{\partial}{\partial \vec{r}_i}.$$

In place of the s-body *distribution* functions f_s, we shall use the s-body *correlation* functions ϕ_s. The first functions ϕ_s are defined as

$$f_1(1) = \phi_1(1),$$

$$f_2(1, 2) = \phi_1(1)\,\phi_1(2) + \phi_2(1, 2)$$

$$f_3(1, 2, 3) = \phi_1(1)\,\phi_1(2)\,\phi_1(3) + \phi_2(1, 2)\,\phi_1(3) + \phi_2(2, 3)\phi_1(1) + \phi_2(1, 3)\phi_1(2) + \phi_3(1, 2, 3).$$

The evolution in space and time of the correlation functions ϕ_1, ϕ_2, and ϕ_3 is described by the set of equations[7]

$$\frac{\partial \phi_1(1)}{\partial t} + H_1\phi_1(1) = -n \int d\Gamma_2\, \theta_{1,2}[\phi_2(1, 2) + \phi_1(1)\,\phi_1(2)], \tag{2.3a}$$

$$\frac{\partial \phi_2(1, 2)}{\partial t} + H_2\phi_2(1, 2) + \theta_{1,2}\phi_1(1)\phi_1(2) = -n \int d\Gamma_3(1 + \mathcal{P}_{1,2})\, \theta_{1,3}[\phi_3(1, 2, 3) + \phi_2(2, 3)\phi_1(1) + \phi_2(1, 2)\phi_1(3)], \tag{2.3b}$$

$$\frac{\partial \phi_3(1, 2, 3)}{\partial t} + H_3\phi_3(1, 2, 3) = -(1 + \mathcal{P}_{1,2} + \mathcal{P}_{2,3})\, \theta_{1,3} \times [\phi_2(1, 2)\phi_1(3) + \phi_2(2, 3)\phi_1(1)]. \tag{2.3c}$$

To write (2.3a)–(2.3c) we have used the permutation operator $\mathcal{P}_{ij}$ which acts on a function of $(\vec{r}_i, \vec{p}_i)$ and $(\vec{r}_j, \vec{p}_j)$ as

$$\mathcal{P}_{i,j} f(i, j) \equiv f(j, i).$$

In (2.3c) the terms involving integration are dropped since, as we shall see, they lead to corrections of higher order in the density. The expansion of the right-hand side of (2.3a) in powers of the density is obtained as follows: To get the first term we set the right-hand side of (2.3b) equal to zero and solve to find

$$\phi_2(1, 2) = -G_{1,2}(t)\theta_{1,2}\phi_1(1)\phi_1(2), \tag{2.4a}$$

where $G_{1,2}(t)$ is the Green's function for the two-body Liouville equation. This result is then put into the right-hand side of (2.3a) and, taking the limit of t large, we are led to the Boltzmann equation. Continuing, in the next order we put (2.4a) into the right-hand side of (2.3c) and solve for $\phi_3(1, 2, 3)$. This will clearly involve the three-particle Green's function, and hence a solution of the three-body problem. The second term in the density expansion of $\phi_2(1, 2)$ is now obtained by putting this first approximation for $\phi_3(1, 2, 3)$ together with (2.4a) into the right-hand side of (2.3b), dropping the third

term on the left-hand side, and solving by means of the two-body Green's function. Putting the result into the right-hand side of (2. 3a) and taking the limit of long times, we are led to the Choh-Uhlenbeck correction to the Boltzmann equation. However, there is a grave difficulty with the long-time limit. In the three-dimensional case, the Choh-Uhlenbeck correction exists but in two dimensions it diverges logarithmically. In three dimensions the next correction diverges.

These divergences occur because there is an infinite volume in phase space over which the particular term in the density expansion of the pair correlation function is nonzero. This arises from the unphysical feature that in the formalism the particles are allowed to traverse long distances without interaction. Actually, it appears that the dynamics of the long-range part of ϕ_2 are collective. That is, as the particles traverse long distances they are continually interacting with the other gas particles and this interaction produces a modification of the long-range part of ϕ_2. It has been proposed that this effect can be treated by selecting in each order of the density the terms in ϕ_2 leading to the strongest divergence and adding them all up. The result of this *resummation* is the so-called ring-collision term, which we want to study here.

It turns out that the ring-collision term can be obtained fairly simply from Eqs. (2. 3a)–(2. 3c) if we treat them as a set of coupled equations to be solved without expansion in powers in n. This can be done since we have dropped the term in (2. 3c) which involves ϕ_4. What we must do is to solve (2. 3c) for ϕ_3 and then put the result in (2. 3b), which gives a linear integrodifferential equation for ϕ_2. This equation is too complicated to solve exactly, but for the region of phase space of interest, where the two particles are far apart, we can produce a solution. We now show how this is done to give the ring-collision term.

It is expected that, by putting into the right-hand side of (2. 3b) the correct value of ϕ_3, the corresponding term will create a cutoff effect on the long-range part of $\phi_2(1, 2)$. This cutoff occurs in a region of phase space where $|\vec{r}_1 - \vec{r}_2| \gg r_0$, r_0 being the range of the two-body potential $V(\vec{r}_{i,j})$. Moreover in (2. 3b) the function $\phi_3(1, 2, 3)$ takes place after an operator $\theta_{1,3}$ or an operator $\theta_{2,3}$. The cutoff effect therefore depends on the value of $\phi_3(1, 2, 3)$ in the region of phase space of particles 1, 2, 3 defined by the inequalities $|\vec{r}_1 - \vec{r}_3| \lesssim r_0$ and $|\vec{r}_1 - \vec{r}_2| \gg r_0$ (or $|\vec{r}_2 - \vec{r}_3| \lesssim r_0$ and $|\vec{r}_1 - \vec{r}_2| \gg r_0$). Let $\phi_3^{(1,3)}$ be the value of ϕ_3 in this region. Since in this region we have $|\vec{r}_1 - \vec{r}_2| \gg r_0$ and $|\vec{r}_2 - \vec{r}_3| \gg r_0$, the potentials $V(\vec{r}_{1,2})$ and $V(\vec{r}_{2,3})$ are nearly equal to zero and in order to calculate $\phi_3^{(1,3)}$ one is allowed to replace in (2. 3c) $\theta_{2,3}$ and $\theta_{1,2}$ by zero. Hence $\phi_3^{(1,3)}$ is given by the solution of

$$\left(\frac{\partial}{\partial t} + H_3^0\right) \phi_3^{(1,3)} + \theta_{1,3}[\phi_3^{(1,3)} + \phi_2(1, 2)\phi_1(3) + \phi_2(2, 3)\phi_1(1)] = 0. \quad (2.4b)$$

Similarly, let $\phi_3^{(2,3)}$ be the value of ϕ_3 in the region of phase space of particles 1, 2, 3 where we have $|\vec{r}_1 - \vec{r}_2| \gg r_0$ and $|\vec{r}_2 - \vec{r}_3| \lesssim r_0$, then $\phi_3^{(2,3)}$ is the solution of

$$\left(\frac{\partial}{\partial t} + H_3^0\right) \phi_3^{(2,3)} + \theta_{1,3}[\phi_3^{(2,3)} + \phi_2(1, 2)\phi_1(3) + \phi_2(1, 3)\phi_1(2)] = 0. \quad (2.4c)$$

In Laplace transform, (2. 4b) reads

$$(\epsilon + H_3^0 + \theta_{1,3})\,\phi_3^{(1,3)}(\epsilon) + \theta_{1,3}\int_0^\infty dt\, e^{-\epsilon t}[\phi_2(1, 2; t)\phi_1(3; t) + \phi_2(2, 3; t)\phi_1(1; t)] = \phi_3^{(1,3)}(t = 0). \quad (2.5)$$

It is presumed that the long-range part of ϕ_2 evolves in a time of the order of a mean free-flight time, which in turn is much longer than the duration of a collision. Since $\phi_3^{(1,3)}$ takes its values in a region of phase space where particles 1 and 3 interact, $\phi_3^{(1,3)}$ evolves in a time of the order of the duration of a collision. Consequently, in order to calculate $\phi_3^{(1,3)}$ at a given time, one may assume that ϕ_2 and ϕ_1 do not depend on time and that, after a time of the order of the duration of a collision, $\phi_3^{(1,3)}$ takes a constant value which is equal to $\lim \epsilon \phi_3^{(1,3)}(\epsilon)$, as $\epsilon \to 0_+$. Accounting for this assumption, Eq. (2. 5) is solved, giving

$$\phi_3^{(1,3)}(1, 2, 3; t) = -\lim_{\epsilon \to 0_+}(\epsilon + H_3^0 + \theta_{1,3})^{-1} \times \theta_{1,3}[\phi_2(1, 2; t)\phi_1(3; t) + \phi_2(2, 3; t)\phi_1(1; t)] + \lim_{\epsilon \to 0_+}(\epsilon + H_3^0 + \theta_{1,3})^{-1}\,\epsilon\,\phi_3^{(1,3)}(1, 2, 3; t = 0). \quad (2.6)$$

The last term on the right-hand side of (2. 6) defines the contribution of the initial value of $\phi_3^{(1,3)}$ to the value of $\phi_3^{(1,3)}$ at time t. If $\phi_3^{(1,3)}(1, 2, 3; t = 0)$ is a correlation function with a finite range, it can be shown[8] that this contribution of the initial value of $\phi_3^{(1,3)}$ to $\phi_3^{(1,3)}(1, 2, 3; t)$ vanishes, and since there is no reason for taking as an initial value of $\phi_3^{(1,3)}$ a correlation function with an infinite range, the effect of this initial value will be neglected, and (2. 6) reads

$$\phi_3^{(1,3)}(1, 2, 3; t) = -\lim_{\epsilon \to 0_+}(\epsilon + H_3^0 + \theta_{1,3})^{-1} \times \theta_{1,3}[\phi_2(1, 2; t)\phi_1(3; t) + \phi_2(2, 3; t)\phi_1(1; t)]. \quad (2.7)$$

Similarly, we have

$$\phi_3^{(2,3)}(1, 2, 3; t) \equiv \mathcal{P}_{1,2}\phi_3^{(1,3)}(1, 2, 3; t) .$$

Now, inserting into (2. 3b) the values $\phi_3^{(1,3)}$ of ϕ_3 behind the operator $\theta_{1,3}$, and $\phi_3^{(2,3)}$ behind $\theta_{2,3}$, we obtain

$$\left(\frac{\partial}{\partial t} + H_2\right) \phi_2(1, 2; t) + \theta_{1,2}\phi_1(1; t)\phi_1(2; t)$$

$$= -n\int d\Gamma_3(1 + \mathcal{P}_{1,2})\ T_{1,3}[\phi_2(1, 2; t)\phi_1(3; t) + \phi_2(2, 3; t)\phi_1(1; t)] . \qquad (2.8)$$

In order to simplify the expression of (2. 8), we have used the so-called "binary-collision operator" $T_{1,3}$ defined as[9]

$$T_{1,3} \equiv \lim_{\epsilon \to 0_+} \theta_{1,3}[1 - (\epsilon + H_3^0 + \theta_{1,3})^{-1}\theta_{1,3}]$$

$$\equiv \lim_{\epsilon \to 0_+} \theta_{1,3}(\epsilon + H_3^0 + \theta_{1,3})^{-1}(\epsilon + H_3^0)$$

$$\equiv \lim_{\epsilon \to 0_+} (\epsilon + H_3^0)[(\epsilon + H_3^0)^{-1} - (\epsilon + H_3^0 + \theta_{1,3})^{-1}](\epsilon + H_3^0) . \qquad (2.9)$$

Owing to the presence of H_3^0 in the definitions of $T_{1,3}$, it would seem that $T_{1,3}$ were a three-body operator. Nevertheless, it can be shown[8,9] that $T_{1,3}$ acts only on the dynamical variables of particles 1 and 3 and is a true two-body operator.

Now we have to deduce from (2. 8) and (2. 3a) a kinetic equation for ϕ_1. This must be done in the following way[6]: At first one assumes that ϕ_1 is constant in time and calculates from (2. 8) the corresponding value of ϕ_2 for large times. Let $\phi_2\{\phi_1(t)\}$ be this assymptotic value of ϕ_2; by inserting this value of the correlation function into (2. 3a), a "Markoffian" kinetic equation is obtained. If we are able to deal in this way, without coming across any divergence, we can apply the Chapman-Enskog method[10] and compute the transport coefficients corresponding to this kinetic equation. The existence of transport coefficients therefore depends on the existence of the functional $\phi_2\{\ \}$. Now we shall try to obtain this functional and the corresponding expression of the kinetic equation.

By means of Laplace transform $\phi_2\{\phi_1(t)\}$ is formally deduced from (2. 8) and is given by

$$\phi_2\{\phi_1(t)\} = -\lim_{\epsilon' \to 0_+}[\epsilon' + H_2 + n\Lambda(1) + n\Lambda(2)]^{-1}\theta_{1,2}\phi_1(1; t)\phi_1(2; t)$$

$$+ \lim_{\epsilon' \to 0_+} \epsilon'[\epsilon' + H_2 + n\Lambda(1) + n\Lambda(2)]^{-1}\phi_2(1, 2; t = 0) . \qquad (2.10)$$

In (2. 10) there appears the $\Lambda(j)$ operator which depends on $\phi_1(t)$ and acts on a function $\psi(j)$ as

$$\Lambda(j)\psi(j) = \int d\Gamma_3\, T_{j,3}[\psi(j)\phi_1(3; t) + \psi(3)\phi_1(j; t)] .$$

Since $T_{j,3}$ is a true two-body operator, $\Lambda(j)$ is a true one-body operator.

Furthermore since $\phi_2\{\phi_1(t)\}$ must be a functional of $\phi_1(t)$ only, its expression (2. 10) is inadequate, since it depends on the value of ϕ_2 at $t = 0$. It can be verified[8] by a density expansion of this contribution of the initial value $\phi_2(1, 2; t = 0)$ that, if $\phi_2(1, 2; t = 0)$ has a finite range, this contribution vanishes and that, within this last restriction, we have

$$\phi_2\{\phi_1(t)\} = -\lim_{\epsilon' \to 0_+}[\epsilon' + H_2 + n\Lambda(1) + n\Lambda(2)]^{-1} \times \theta_{1,2}\phi_1(1; t)\phi_1(2; t). \qquad (2.11)$$

Inserting this value of ϕ_2 into the right-hand side of (2. 3a), we obtain, at least formally, the expected Markoffian kinetic equation. What is the meaning of this kinetic equation with respect of the density expansion of the kinetic theory?

In fact we are looking for a divergence-free correction to the Boltzmann collision operator. In general, [3,5] to obtain this correction one expands the exact kinetic operator (or the exact value of ϕ_2) in powers of the density, and at each order of this expansion one finds secular terms with a maximum divergence, which gets more and more catastrophic when the order in n increases. Since the origin of these secularities has been explained in the literature, [5] we do not duplicate these explanations. These divergencies correspond to the so-called[11] ring dynamical events occurring between a finite number of particles, hence the collision operator which renormalizes these divergencies will be called the "ring-collision operator."

Here we shall not reproduce the whole procedure of renormalization, but shall use an expansion of $\phi_2\{\phi_1(t)\}$ in which the leading term gives the value

of ϕ_2 at the Boltzmann order and the following term gives rise to the renormalized correction corresponding to the ring-collision term. We write the operator $[\epsilon' + H_2 + n\Lambda(1) + n\Lambda(2)]$ as the sum of the three operators $A(\epsilon'; 1, 2)$, $B(\epsilon'; 1, 2)$, and $C(\epsilon'; 1, 2)$ defined as

$$A(\epsilon'; 1, 2) \equiv \epsilon' + H_2 , \tag{2.12a}$$

$$B(\epsilon'; 1, 2) \equiv \epsilon' + H_2^0 + n\Lambda(1) + n\Lambda(2) , \tag{2.12b}$$

$$C(\epsilon'; 1, 2) \equiv -(\epsilon' + H_2^0) . \tag{2.12c}$$

From the algebraic identity

$$(A+B+C)^{-1} = A^{-1} - A^{-1}(B+C) + A^{-1}(B+C) \times B^{-1}(A+C)(A+B+C)^{-1} , \tag{2.13}$$

and from

$$[\epsilon' + H_2 + n\Lambda(1) + n\Lambda(2)]^{-1} \equiv [A(\epsilon'; 1, 2) + B(\epsilon'; 1, 2) + C(\epsilon'; 1, 2)]^{-1} ,$$

the operator $[\epsilon' + H_2 + n\Lambda(1) + n\Lambda(2)]^{-1}$ is expanded as

$$[\epsilon' + H_2 + n\Lambda(1) + n\Lambda(2)]^{-1} = A^{-1}(\epsilon'; 1, 2) - A^{-1}(\epsilon'; 1, 2) \times C(\epsilon'; 1, 2)\,[C^{-1}(\epsilon'; 1, 2) + B^{-1}(\epsilon'; 1, 2)] \times C(\epsilon'; 1, 2) A^{-1}(\epsilon'; 1, 2) + \cdots . \tag{2.14}$$

This development is inserted into (2.11), and after insertion into (2.3a) of the corresponding value of ϕ_2, gives to a series of collision terms. The first few terms of this series are

$$\frac{\partial \phi_1}{\partial t} + H_1 \phi_1 = -n \int d\Gamma_2\, T_{1,2}\, \phi_1(1)\phi_1(2) + n^2 \int d\Gamma_2\, T_{1,2} \lim_{\epsilon' \to 0+} [B^{-1}(\epsilon'; 1, 2) + C^{-1}(\epsilon'; 1, 2)]\, T_{1,2} \phi_1(1)\phi_1(2) + \cdots . \tag{2.15}$$

The first term on the right-hand side of (2.15) is the well-known Boltzmann-Enskog collision term. The second one can be expanded formally in powers of the density n, giving rise at any order to the most divergent terms corresponding to the "ring dynamical events." One may assert that this second term is the actual renormalized ring-collision term.

In Sec. III, this collision term will be studied in detail for the case of an homogeneous system, namely when $\phi_1(1)$ depends on $\vec{p}_1$ only. A simplified form of the ring-collision term may be obtained in this case through the use of Fourier transforms of functions and operators. In this simplified form any operator acting on the dynamical variables of a particle is turned into an operator acting on the momentum of this particle and into a function of a single wave vector $\vec{k}$.

Usually a two-body linear operator $T_{1,2}$ is defined by a function $T(1, 2 | \bar{1}, \bar{2})$, and acts on a function $\phi(1, 2)$ as

$$T_{1,2}\phi(1, 2) = \int d\Gamma_{\bar{1}} d\Gamma_{\bar{2}}\, T(1, 2 | \bar{1}, \bar{2}) \phi(\bar{1}, \bar{2}) .$$

The matrix elements of $T_{1,2}$ act on the momenta $\vec{p}_1$ and $\vec{p}_2$ and are defined as

$$\langle \vec{k}_1, \vec{k}_1' | T_{1,2} | \vec{k}, \vec{k}' \rangle = V^{-1} \int d\vec{r}_1 \int d\vec{r}_2 \int d\vec{r}_{\bar{1}} \int d\vec{r}_{\bar{2}}\, e^{-i\vec{k}\cdot\vec{r}_1} \times e^{-i\vec{k}'\cdot\vec{r}_2}\, e^{i\vec{k}\cdot\vec{r}_{\bar{1}}}\, e^{i\vec{k}\cdot\vec{r}_{\bar{2}}}\, T(1, 2 | \bar{1}, \bar{2}) . \tag{2.16}$$

From the translational invariance of the law of motion, it may be shown[9] that the only nonzero matrix elements of $T_{1,2}$ are of the form $\langle \vec{k}+\vec{q}, \vec{k}' - \vec{q} | T_{1,2} | \vec{k}, \vec{k}' \rangle$. From this last property, and from the elementary properties of the Fourier transformation, one has

$$\int d\vec{r}_2\, T_{1,2}\, e^{i\vec{k}\cdot\vec{r}_2}\, \Phi(\vec{p}_1, \vec{p}_2) = e^{i\vec{k}\cdot\vec{r}_1} \langle \vec{k}, 0 | T_{1,2} | 0, \vec{k} \rangle \Phi(\vec{p}_1, \vec{p}_2) , \tag{2.17}$$

$$\int d\vec{r}_2\, T_{1,2}\, e^{i\vec{k}\cdot\vec{r}_1} \Phi(\vec{p}_1, \vec{p}_2) = e^{i\vec{k}\cdot\vec{r}_1} \langle \vec{k}, 0 | T_{1,2} | \vec{k}, 0 \rangle \Phi(\vec{p}_1, \vec{p}_2) . \tag{2.18}$$

Using (2.17) and (2.18) one can find a simplified form of the ring-collision term for an homogeneous system. For this case, $T_{1,2}\phi_1(\vec{p}_1)\phi_1(\vec{p}_2)$ is given by

$$T_{1,2}\phi_1(\vec{p}_1)\phi_1(\vec{p}_2) = \int [d\vec{k}/(2\pi)^{\nu}]\, e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)} \times \langle \vec{k}, -\vec{k} | T_{1,2} | 0, 0 \rangle \phi_1(\vec{p}_1)\phi_1(\vec{p}_2) , \tag{2.19}$$

where ν is the number of dimensions of the system.

The equality (2.19) results from the definition (2.16) of the matrix elements of $T_{1,2}$ and from the Plancherel-Parseval theorem. From (2.17), (2.18), and from the definition (2.12b) of the operator $\Lambda(j)$, one deduces that the operator Λ is diagonal in the Fourier representation, and that the operator $B(\epsilon'; 1, 2)$ itself is diagonal in this representation, giving

$$B^{-1}(\epsilon';1,2)\,T_{1,2}\phi_1(\vec{p}_1)\phi_1(\vec{p}_2)\equiv[\epsilon'+H_2^0+n\Lambda(1)+n\Lambda(2)]^{-1}\,T_{1,2}\phi_1(\vec{p}_1)\phi_1(\vec{p}_1)$$

$$=\int\frac{d\vec{k}}{(2\pi)^\nu}\,e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\,B^{-1}(\epsilon',\vec{k};1,2)\,\langle\vec{k},-\vec{k}|\,T_{1,2}\,|0,0\rangle\,\phi_1(\vec{p}_1)\phi_1(\vec{p}_2),\qquad(2.20)$$

where $B(\epsilon',\vec{k};1,2)$ acts on functions of $\vec{p}_1$ and $\vec{p}_2$, and is defined as

$$B(\epsilon',\vec{k};1,2)=\epsilon'+[i\vec{k}\cdot(\vec{p}_1-\vec{p}_2)]/m+n\Lambda(1,\vec{k};\phi_1)$$

$$+n\Lambda(2,-\vec{k};\phi_1),\qquad(2.21)$$

where $\Lambda(1,\vec{k};\phi_1)$ is a linear functional of ϕ_1, and an operator acting on a function $\psi(\vec{p}_1)$ as

$$\Lambda(j,\vec{k};\phi_1)\psi(\vec{p}_j)=\int d\vec{p}_l[\langle\vec{k},0|\,T_{j,l}\,|\vec{k},0\rangle\psi(\vec{p}_j)\phi_1(\vec{p}_l)$$

$$+\langle\vec{k},0|\,T_{j,l}\,|0,\vec{k}\rangle\psi(\vec{p}_l)\phi_1(\vec{p}_j)].\qquad(2.22)$$

The ring-collision term is generally a function of $(\vec{r}_1,\vec{p}_1)$ and a functional of the one-body distribution function ϕ_1. From (2.15) the value of this collision term is

$$S_R\{1;\phi_1\}\equiv n\int d\Gamma_2\,T_{1,2}\lim_{\epsilon'\to0_+}[B^{-1}(\epsilon';1,2)$$

$$+C^{-1}(\epsilon';1,2)]\,T_{1,2}\phi_1(1)\phi_1(2).$$

For the special case of an homogeneous system, the ring-collision term depends on $\vec{p}_1$ only, and from (2.17), (2.18), and (2.20) it may be written

$$S_R\{\vec{p}_1;\phi_1\}\equiv n\int d\vec{p}_2\int[d\vec{k}/(2\pi)^\nu]\langle0,0|\,T_{1,2}\,|\vec{k},-\vec{k}\rangle$$

$$\times[B^{-1}(\epsilon',\vec{k};1,2)+C^{-1}(\epsilon';1,2)]$$

$$\times\langle\vec{k},-\vec{k}|\,T_{1,2}\,|0,0\rangle\,\phi_1(\vec{p}_1)\,\phi_1(\vec{p}_2),\qquad(2.23)$$

where

$$C(\epsilon',\vec{k};1,2)=-\epsilon'+\frac{i\vec{k}\cdot\vec{p}_1}{m}-\frac{i\vec{k}\cdot\vec{p}_2}{m}.\qquad(2.24)$$

Section III is devoted to the study of $S_R\{\vec{p}_1;\phi_1\}$ through its expression (2.23).

III. DIVERGENCIES OF THE RING-COLLISION TERM IN TWO DIMENSIONS

It has been conjectured[5] that the ring-collision operator removes the most dangerous secularities which appear in the density expansion of the collision operator. More precisely, it has been emphasized that the renormalization introduces in the theory a new length, the mean free path, and that this length acts as a cutoff on the diverging integrals of the density expansion. Using this sort of argument, one predicts that, for $\nu=2$ (ν is the number of dimensions of the system), the ring-collision term behaves like $n^2\ln n$ in the small-density limit. For a special case of the perfect Lorentz gas, this conjecture has been verified.[4]

Nevertheless, by investigating the case of the two-dimensional simple gas, we will show that a new kind of divergence appears in this renormalized collision theory. This divergence will be exhibited by inspection of the convergence of the integral (2.22) which defines $S_R\{\vec{p}_1;\phi_1\}$. It will be proved that in some cases this integral diverges in the limit $k\to0$, when ϵ' is equal to zero. This ring-collision term is a very complicated quantity, especially because it depends nonlinearly on ϕ_1. Thus we shall consider, instead of the general term $S_R\{\vec{p}_1;\phi_1\}$, its linearized value calculated when ϕ_1 differs from an equilibrium value ϕ_1^0 of a small amount. In particular it must be pointed out that, following the Chapman-Enskog theory,[11] a transport coefficient is determinated from this linearized collision operator.

From the expression (2.23) of $S_R\{\vec{p}_1;\phi_1\}$ and from the definition (2.21) of the operator $B(\epsilon',\vec{k};1,2)$, it may be seen that ϕ_1 figures in $S_R\{\vec{p}_1;\phi_1\}$ in two different places: at first in the product $\phi_1(\vec{p}_1)\phi_1(\vec{p}_2)$ on the extreme right of (2.23), and then in the operator $B(\epsilon',\vec{k};1,2)$, since $\Lambda(j,\vec{k};\phi_1)$ is a functional of ϕ_1. Thus the linear variation δS_R, owing to a variation $\delta\phi_1$ of ϕ_1 around ϕ_1^0, is split up into two parts:

$$\delta S_R\{\vec{p}_1;\delta\phi_1\}=\delta S_R^{(1)}\{\vec{p}_1;\delta\phi_1\}+\delta S_R^{(2)}\{\vec{p}_1;\delta\phi_1\},\qquad(3.1)$$

where

$$\delta S_R^{(1)}\{\vec{p}_1;\delta\phi_1\}=n\int d\vec{p}_2\int[d\vec{k}/(2\pi)^\nu]\langle0,0|\,T_{1,2}\,|\vec{k},-\vec{k}\rangle\lim_{\epsilon'\to0_+}[B_0^{-1}(\epsilon',\vec{k};1,2)+C^{-1}(\epsilon',\vec{k};1,2)]$$

$$\times\langle\vec{k},-\vec{k}|\,T_{1,2}\,|0,0\rangle\,[\delta\phi_1(\vec{p}_1)\phi_1^0(\vec{p}_2)+\delta\phi_1(\vec{p}_2)\phi_1^0(\vec{p}_1)],\qquad(3.2)$$

$$\delta S_R^{(2)}\{\vec{p}_1;\delta\phi_1\}=-n^2\int d\vec{p}_2\int[d\vec{k}/(2\pi)^\nu]\langle 0,0|T_{1,2}|\vec{k},-\vec{k}\rangle \lim_{\epsilon'\to 0_+} B_0^{-1}(\epsilon',\vec{k};1,2)$$

$$\times[\Lambda(1,\vec{k};\delta\phi_1)+\Lambda(2,-\vec{k};\delta\phi_1)]B_0^{-1}(\epsilon',\vec{k};1,2)\langle\vec{k},-\vec{k}|T_{1,2}|0,0\rangle\,\phi_1^0(\vec{p}_1)\,\phi_1^0(\vec{p}_2), \tag{3.3}$$

and where

$$B_0(\epsilon',\vec{k};1,2)=\epsilon'+[i\vec{k}\cdot(\vec{p}_1-\vec{p}_2)]/m$$

$$+n\Lambda(1,\vec{k};\phi_1^0)+n\Lambda(2,-\vec{k};\phi_1^0).$$

In the case of an hard-core potential, there is a simplification, since $\delta S_R^{(2)}\{\vec{p}_1;\delta\phi_1\}=0$. In fact, in this case,[12] the matrix element of $T_{1,2}$ acts on a function $\Phi(\vec{p}_1,\vec{p}_2)$ as

$$\langle\vec{k}+\vec{q},\vec{k}'-\vec{q}|T_{1,2}|\vec{k},\vec{k}'\rangle\,\Phi(\vec{p}_1,\vec{p}_2)=-(1/m)|\vec{p}_1-\vec{p}_2|$$

$$\times\int d\vec{b}\,e^{-i\vec{q}\cdot\vec{\rho}(\vec{b},\vec{p}_1-\vec{p}_2)}[\Phi(\vec{p}_1',\vec{p}_2')-\Phi(\vec{p}_1,\vec{p}_2)], \tag{3.4}$$

where the integral runs over the values of the impact parameter $\vec{b}$, and where $\vec{\rho}(\vec{b},\vec{p}_1-\vec{p}_2)$ is the value of $(\vec{r}_1-\vec{r}_2)$ at the time of the collision which creates the particles 1, 2 with a momentum $\vec{p}_1$ and $\vec{p}_2$, the value of these momenta being $\vec{p}_1'$ and $\vec{p}_2'$ before this collision. Now, since an equilibrium function $\phi_1^0(\vec{p}_1)$ verifies

$$\phi_1^0(\vec{p}_1')\phi_1^0(\vec{p}_2')=\phi_1^0(\vec{p}_1)\phi_1^0(\vec{p}_2),$$

we have

$$\langle\vec{k},-\vec{k}|T_{1,2}|0,0\rangle\,\phi_1^0(\vec{p}_1)\phi_1^0(\vec{p}_2)=0,\quad \delta S_R^{(2)}\{\vec{p}_1;\delta\phi_1\}=0$$

for a hard-core potential.

In what follows, we shall study the linearized ring-collision term $\delta S_R\{\vec{p}_1;\delta\phi_1\}$ in the case of an hard-core potential, where this collision term is equal to $\delta S_R^{(1)}\{\vec{p}_1;\delta\phi_1\}$. More precisely we shall examine the integrand on the right-hand side of (3.2) near $k=0$. In this integrand, the operator $B_0^{-1}(\epsilon',\vec{k};1,2)$ acts on the function $\alpha_{\vec{k}}(\vec{p}_1,\vec{p}_2)$ defined as

$$\alpha_{\vec{k}}(\vec{p}_1,\vec{p}_2)=\langle\vec{k},-\vec{k}|T_{1,2}|0,0\rangle$$

$$\times\,[\phi_1^0(\vec{p}_1)\delta\phi_1(\vec{p}_2)+\phi_1^0(\vec{p}_2)\delta\phi_1(\vec{p}_1)].$$

This function depends on $\vec{k}$ through the matrix element $\langle\vec{k},-\vec{k}|T_{1,2}|0,0\rangle$, and an inspection of (3.4) shows that this matrix element, and thus $\alpha_{\vec{k}}(\vec{p}_1,\vec{p}_2)$, are continuous and well defined around $\vec{k}=0$. But the situation is more complicated in what concerns the operator $\lim_{\epsilon'\to 0_+}B_0^{-1}(\epsilon',\vec{k};1,2)$, as $\epsilon'\to 0_+$, since $B_0(0,0;1,2)$ has zero as an eigenvalue.

In order to state this question more precisely, let us recall the expression of $B_0(\epsilon',\vec{k};1,2)$,

$$B_0(\epsilon',\vec{k};1,2)=\epsilon'+[i\vec{k}\cdot(\vec{p}_1-\vec{p}_2)]/m+n\Lambda(1,\vec{k};\phi_0^1)$$

$$+n\Lambda(2,-\vec{k};\phi_1^0),$$

and define the eigenfunctions $\mu(\vec{k},\vec{p}_1)$ and the eigenvalues $z_\mu(\vec{k})$ of the operator $[(i\vec{k}\cdot\vec{p}_1)/m+n\Lambda(1,\vec{k};\phi_1^0)]$ as

$$z_\mu(\vec{k})\mu(\vec{k},\vec{p}_1)=[(i\vec{k}\cdot\vec{p}_1)/m+n\Lambda(1,\vec{k};\phi_1^0)]\mu(\vec{k},\vec{p}_1). \tag{3.5}$$

The eigenvalues of $B_0^{-1}(\epsilon',\vec{k};1,2)$ are $[\epsilon'+z_\mu(\vec{k})+z_{\mu'}(-\vec{k})]^{-1}$, the corresponding eigenfunctions being the products $\mu(\vec{k},\vec{p}_1)\,\mu'(-\vec{k},\vec{p}_2)$. Now we shall express the function $B_0^{-1}(\epsilon',\vec{k};1,2)\alpha_{\vec{k}}(\vec{p}_1,\vec{p}_2)$ by means of this spectrum, but we have previously to expand $\alpha_{\vec{k}}(\vec{p}_1,\vec{p}_2)$ on the basis of eigenfunctions of $B_0(\epsilon',\vec{k};1,2)$, we will assume that this can be done as

$$\alpha_{\vec{k}}(\vec{p}_1,\vec{p}_2)=\sum_{\mu,\mu'}\alpha_{\vec{k};\mu,\mu'}\,\mu(\vec{k},\vec{p}_1)\,\mu'(-\vec{k},\vec{p}_2). \tag{3.6}$$

Now, from (3.6) and (3.7) the function $B_0^{-1}(\epsilon',\vec{k};1,2)\alpha_{\vec{k}}(\vec{p}_1,\vec{p}_2)$ may be expressed by means of the spectral decomposition of $B_0^{-1}(\epsilon',\vec{k};1,2)$ giving

$$B_0^{-1}(\epsilon',\vec{k};1,2)\alpha_{\vec{k}}(\vec{p}_1,\vec{p}_2)$$

$$=\sum_{\mu,\mu'}\frac{\alpha_{\vec{k};\mu,\mu'}\,\mu(\vec{k},\vec{p}_1)\,\mu'(-\vec{k},\vec{p}_2)}{\epsilon'+z_\mu(\vec{k})+z_{\mu'}(-\vec{k})}\,. \tag{3.7}$$

It is known that the spectrum of the operator $[(i\vec{k}\cdot\vec{p}_1/m)+n\Lambda(1,\vec{k}=0;\phi_1^0)]$ has a mixed structure,[13] with a discrete set of eigenvalues and a continuum, and it may be thought that the spectrum of the operator $[(i\vec{k}\cdot\vec{p}_1)/m]+n\Lambda(1,\vec{k};\phi_1^0)]$ has the same structure. However, as we shall be only concerned with the discrete part of this spectrum, the contribution of the continuum may be neglected.

Let us study now the behavior, near $k=0$, of the denominator $[\epsilon'+z_\mu(\vec{k})+z_{\mu'}(-\vec{k})]$ occuring in (3.7). The linearized collision operator $\Lambda(1,\vec{k}=0,\phi_1^0)$ has zero as a degenerate eigenvalue, the corresponding eigenfunctions being ϕ_1^0, $\vec{p}_1\phi_1^0$, and $\vec{p}_1^2\phi_1^0$, as can be verified from the definition (2.11) of the operator $\Lambda(1,\vec{k};\phi_1^0)$, and from the expression of the matrix element of $T_{1,2}$ given in (3.4). Since the operator $[(i\vec{k}\cdot\vec{p}_1)/m+n\Lambda(1,\vec{k};\phi_1^0)]$ goes to $\Lambda(1,\vec{k}=0;\phi_1^0)$ with k, at the same time some of its eigenvalues $z_\mu(\vec{k})$ go to

the eigenvalue zero of $\Lambda(1, \vec{k}=0; \phi_1^0)$. This statement corresponds to the existence of phenomena such as sound propagation and entropy and vorticity diffusion. In the first case (sound propagation) $z_\mu(k)$ is of order k near $k=0$, in the latter ones (entropy and vorticity diffusion) $z_\mu(k)$ is of order k^2. These statements may be proved in a purely algebraic way. But the details of this proof are not essential for what follows, and they are given in Appendix A. When μ and μ' are the labels corresponding to the diffusion eigenfunctions, the eigenvalues of $B^{-1}(\epsilon'=0, \vec{k}; 1, 2)$, namely $[z_\mu(\vec{k})+z_{\mu'}(-\vec{k})]^{-1}$, are of order k^{-2} near $k=0$. This point is of a crucial importance for the two-dimensional systems: By means of (3.7) the collision term $\delta S_R^{(1)}(\vec{p}_1; \delta\phi_1)$ reads

$$\delta S_R^{(1)}\{\vec{p}_1; \delta\phi_1\} = n\int d\vec{p}_2 \int \frac{d\vec{k}}{(2\pi)^\nu} \langle 0,0|\, T_{1,2}|\vec{k}, -\vec{k}\rangle \lim_{\epsilon'\to 0_+}\left(\sum_{\mu,\mu'} \frac{\alpha_{\vec{k};\mu,\mu'}\, \mu(\vec{k}, \vec{p}_1)\, \mu'(-\vec{k}, \vec{p}_2)}{\epsilon' + z_\mu(\vec{k}) + z_{\mu'}(-\vec{k})}\right.$$

$$\left. + n\int d\vec{p}_2 \int \frac{d\vec{k}}{(2\pi)^\nu} \langle 0,0|\, T_{1,2}|\vec{k}, -\vec{k}\rangle\right) \lim_{\epsilon'\to 0_+} C_0^{-1}(\epsilon', \vec{k}; 1, 2)\alpha_{\vec{k}}(\vec{p}_1, \vec{p}_2). \tag{3.8}$$

Thus, for these diffusion eigenfunctions, the integrand on the right-hand side of (3.8) behaves near $k=0$ as

$$\lim_{\epsilon'\to 0_+} \frac{k^{\nu-1}}{\epsilon' + O(k^2)} \sim k^{\nu-3}$$

(the numerator $k^{\nu-1}$ originates from the volume element $d\vec{k}$ expressed in spherical coordinates), and the corresponding integral, say

$$\int_0 dk \lim_{\epsilon'\to 0_+}\left(\frac{k^{\nu-1}}{\epsilon' + O(k^2)}\right),$$

diverges logarithmically when $\nu=2$. Nevertheless the actual existence of this divergence may be challenged at this stage, since it is submitted to some supplementary conditions:

(i) The diverging term is multiplied by

$$\lim_{k\to 0} \int_0^{2\pi} d\hat{k} \int d\vec{p}_2 \langle 0,0|\, T_{1,2}|\vec{k}, -\vec{k}\rangle\, \mu(\vec{k}, \vec{p}_1) \times \mu'(-\vec{k}, \vec{p}_2)\alpha_{\vec{k};\mu,\mu'},$$

where $\hat{k}$ is the polar angle of the vector $\vec{k}$. The divergence exists only if this quantity differs from zero. This question must be considered very carefully. In fact, for the case of the perfect Lorentz gas, this quantity is equal to zero, and the divergence disappears.

Let us consider this particular case: The only moving particle is the light particle, labeled 1, the other ones making a random array of fixed scatterers. The ring-collision term of this Lorentz gas may be deduced by minor change from the operator $S_R\{\vec{p}_1; \phi_1\}$, calculated for a simple gas. This ring-collision term for a perfect Lorentz gas is known for some time[4]: It is linear in the velocity distribution function $\phi_1(\vec{p}_1)$, and reads in our formalism

$$S_R^l(\vec{p}_1; \phi_1) = n\int \frac{d\vec{k}}{(2\pi)^\nu} \langle 0,0|\, T_{1,2}^l|\vec{k}, -\vec{k}\rangle \times \lim_{\epsilon'\to 0_+}\{[B^l(\epsilon', \vec{k}; 1)]^{-1} + [C^l(\epsilon', \vec{k}; 1, 2)]^{-1}\} \times \langle \vec{k}, -\vec{k}|\, T_{1,2}^l|\, 0,0\rangle \phi_1(\vec{p}_1), \tag{3.9}$$

where n is the density of scatterers, $T_{1,2}^l$ is the value of the operator $T_{1,2}$ when the particle 2 is fixed, and $\langle \vec{k}, -\vec{k}|\, T_{1,2}^l|\, 0,0\rangle$ acts on functions of $\vec{p}_1$,

$$B^l(\epsilon', \vec{k}; 1) = \epsilon' + (i\vec{k}\cdot\vec{p}_1)/m + n\langle \vec{k}, 0|\, T_{1,2}^l|\, \vec{k}, 0\rangle$$

and

$$C^l(\epsilon', \vec{k}; 1) = -[\epsilon' + (i\vec{k}\cdot\vec{p}_1)/m].$$

Let us restrict ourselves to the case of a hard-disks interaction, for which the modulus of the momentum $\vec{p}_1$ is constant in time. In this case the distribution function of the momentum of the light particle may be considered as a function of the polar angle of $\vec{p}_1$, say $\hat{p}_1$.

In order to study the collision term $S_R^l(\vec{p}_1; \phi_1)$ from the point of view of the divergence, we have to define the "hydrodynamical" part of the spectrum of the operator $B^l(\epsilon', \vec{k}; 1)$. An hydrodynamical eigenfunction $d(\vec{k}, \hat{p}_1)$ is defined in the $\vec{k}=0$ limit by

$$B^l(\epsilon'=0, \vec{k}=0; 1)d(\vec{k}=0; \hat{p}_1) = 0, \tag{3.10}$$

which is equivalent to

$$\langle 0,0|\, T_{1,2}^l|\, 0,0\rangle d(\vec{k}=0, \hat{p}_1) = 0. \tag{3.11}$$

But, since the matrix element $\langle 0,0|\, T_{1,2}^l|\, 0,0\rangle$ acts on a function $\psi(\hat{p}_1)$ as

$$\langle 0,0|\, T_{1,2}^l|\, 0,0\rangle\psi(\hat{p}_1) = (|\vec{p}_1|/m)\int db[\psi(\hat{p}_1') - \psi(\hat{p}_1)], \tag{3.12}$$

therefore the only function $d(\vec{k}=0, \hat{p}_1)$ which satisfies

(3.10) and (3.11) is just a constant, independent of $\hat{p}_1$.

More generally, since the operator $B^I(\epsilon', \vec{k}=0;1)$ is isotropic, its eigenfunctions are the circular eigenfunctions $e^{im\hat{p}_1}$, and the expansion of a function $\alpha(\hat{p}_1)$ on the basis of the eigenfunctions of $B^I(\epsilon', \vec{k}=0;1)$ is simply its Fourier expansion. The diffusion eigenfunction $d(\vec{k}=0, \hat{p}_1)$ being a constant, e.g., $d(\vec{k}=0, \hat{p}_1)=1$, the component α_d of a function $\alpha(\hat{p}_1)$ on this eigenfunction is

$$\alpha_d = (1/2\pi)\int_0^{2\pi} d\hat{p}_1\, \alpha(\hat{p}_1) \ . \tag{3.13}$$

Now we may consider the expression (3.9) of the ring-collision term for the perfect Lorentz gas. In this term the operator $[B_0^I(\epsilon', \vec{k};1)]^{-1}$ acts on the function

$$\alpha_{\vec{k}}^I(\hat{p}_1) = \langle \vec{k}, -\vec{k}|\, T_{1,2}^I\, |0,0\rangle \phi_1(\hat{p}_1) \ . \tag{3.14}$$

The divergence occurs if, and only if, this function has a nonzero component on the diffusion eigenfunction $d(\vec{k}, \hat{p}_1)$ in the limit $\vec{k}=0$, namely, if

$$\alpha_{\vec{k}=0;d}^I \equiv (1/2\pi)\int_0^{2\pi} d\hat{p}_1 \langle 0,0|\, T_{1,2}\, |0,0\rangle \phi_1(\hat{p}_1)$$

differs from zero. But from (3.12) we have

$$\alpha_{\vec{k}=0;d}^I = 0 \ .$$

We may now assume that the logarithmic divergence occurring in the general expression (3.8) of $\delta S_R^{(1)}\{\vec{p}_1; \delta\phi_1\}$ is absent in $S_R^I(\vec{p}_1; \phi_1)$, since the factor k^{-2} due to the diffusion eigenvalue of $B^I(\epsilon', \vec{k};1)$ is just multiplied by the function $\alpha_{\vec{k},d}^I$ which goes to zero with $\vec{k}$.

(ii) Up to now the limit $\epsilon'=0_+$ has been taken before the integration over $\vec{k}$ has been carried out, and one may wonder whether the existence of the divergence depends on the mutual order of the integration and of the limit.

In order to give an indication about this question, let us consider the following problem: Given two functions $A(\vec{k})$ and $z(\vec{k})$ of a two-dimensional vector $\vec{k}$ such that $A(\vec{k})$ is continuous and differentiable at $\vec{k}=0$ and $A(0)\neq 0$, and $z(k)\simeq Dk^2$ as $k\to 0$ $(D>0)$, do both the quantities

$$\int d\vec{k} \lim_{\epsilon'\to 0_+} \left(\frac{A(\vec{k})}{\epsilon' + z(\vec{k})}\right) \quad \text{and} \quad \lim_{\epsilon'\to 0_+} \int d\vec{k} \left(\frac{A(\vec{k})}{\epsilon' + z(\vec{k})}\right)$$

actually diverge?

The answer is *yes*, if one assumes furthermore that, for any given value of $k_0>0$, we have

$$\left| \lim_{\epsilon'\to 0_+} \int_{k>k_0} \frac{d\vec{k} A(\vec{k})}{\epsilon' + z(\vec{k})} \right| < \infty \ . \tag{3.15}$$

Proof. From the properties of $A(\vec{k})$ and $z(\vec{k})$, there exists a number k_0, strictly positive, such that for $0<k<k_0$

$$\frac{A(0)}{\epsilon' + D_1 k^2} < \frac{A(\vec{k})}{\epsilon' + z(\vec{k})} < \frac{A(0)}{\epsilon' + D_2 k^2} \ , \tag{3.16}$$

D_1 and D_2 being two positive constants which depend on D and k_0. Hence, we have

$$\ln\left(\frac{\epsilon' + D_2 k_0^2}{\epsilon'}\right) < \frac{1}{2A(0)} \int_{k<k_0} d\vec{k} \frac{A(\vec{k})}{\epsilon' + z(\vec{k})}$$

$$< \ln\left(\frac{\epsilon' + D_1 k_0^2}{\epsilon'}\right) .$$

These last inequalities, together with (3.15), prove that both the quantities

$$\int d\vec{k} \lim_{\epsilon'\to 0_+} \frac{A(\vec{k})}{\epsilon' + z(\vec{k})} \quad \text{and} \quad \lim_{\epsilon'\to 0_+} \int d\vec{k} \frac{A(\vec{k})}{\epsilon' + z(\vec{k})}$$

diverge logarithmically.

From this general result, one can conclude that the divergence found in $\delta S_R^{(1)}\{\vec{p}_1; \delta\phi_1\}$ does not depend on the mutual order of the integration over $\vec{k}$ and of the limit $\epsilon'=0_+$. Nevertheless, since this question is a major source of controversy,[14] we shall examine it carefully in the model to be studied in Sec. IV.

IV. RING KINETIC EQUATION FOR THE MAXWELL MODEL IN TWO DIMENSIONS

In this section, we shall study the ring-collision term in a special model, and we shall *prove* in this model the actual existence of the divergence examined in Sec. III. This model has various advantages:

(i) An eigenvalue of the linearized collision operator can be calculated exactly, without any reference to an approximation like the "first Enskog" approximation.

(ii) This calculation may be related to a transport coefficient which is *not* the mobility of a light particle in a perfect Lorentz gas. This latter point is here of a crucial importance, since the predicted divergence disappears in the perfect Lorentz gas.

Generally, if we try to obtain an explicit value of any quantity at the ring approximation, we have to calculate the operator $B_0^{-1}(\epsilon', \vec{k};1,2)$. In the general case, we cannot do this, since we do not have an explicit expression for the Green's function for the linearized Boltzmann-Enskog equations. The main obstacle stems from the continuous character of the operator $\Lambda(j, \vec{k}; \phi_1^0)$. It then appears natural to have recourse to a model in which the momentum space has a discrete and finite character. The problem of inversion of the continuous operators is thus transformed into an elementary problem of the inversion of finite matrices. We shall thus study a well-known model of a simple gas (i.e., with a single component) with a discrete

momentum space: the Maxwell model.[15]

This section is devoted to the calculation of an eigenvalue Ω of the linearized ring-collision operator $\delta S_R\{\vec{p}_1;\}$ in the Maxwell model. With a view to this calculation, this section is arranged as follows: In Sec. IV A, elementary properties of the model are given, many proofs are omitted, since they are not essential here and will be the subject of another publication; in Sec. IV B it is explained how the eigenvalues $z_\mu(\vec{k})$ and the eigenfunctions $\mu(\vec{k},\vec{p})$ are obtained for this model by using methods of elementary algebra; in Sec. IV C the eigenvalue Ω of the linearized ring-collision operator $\delta S_R\{\vec{p}_1;\}$ is explicitly defined and it is shown that Ω is given by a divergent expression.

A. Definition and Properties of Model

In this model, identical particles move parallel to perpendicular axes in a two-dimensional Cartesian plane. The modulus of the velocity is always equal to unity. Since the velocity can be aimed at four different perpendicular directions, this velocity j is one of the four elements of a discrete set: $j\in\{\mathrm{I, II, III, IV}\}$. This set is provided with an addition law; e.g., $\mathrm{I+IV=II+III=I}$ (mod IV). Let $\vec{e}_j$ be the unit vector of the j direction, the directions are labeled in order to give

$$\vec{e}_j+\vec{e}_{j+\mathrm{II}}=0\ ,\quad \vec{e}_j\cdot\vec{e}_{j+\mathrm{I}}=0\ .$$

This simple case of the Maxwell model is provided with a two-body collision law. Since the motion of the center of gravity of two colliding particles 1 and 2 is unaffected by the collision, this collision is wholly determined by the trajectory of the particle with the relative position $\vec{r}_1-\vec{r}_2$. Furthermore we assume that there are only head-to-head binary collisions, i.e., that two particles may collide only if their velocities are respectively equal to j and $j+\mathrm{II}$, and thus the corresponding "relative" particle moves parallel to the axis of the Cartesian plane, the modulus of its velocity being 2. The velocity of this relative particle is characterized by a discrete index j_r [$j_r\in\{\mathrm{I, II, III, IV}\}$], exactly as the velocity of the actual particles is. When this particle goes close to the origin, it is deflected following a "hard-core" law, i.e., at any time its motion is rectilinear, and the deflection is instantaneous. Deflection occurs when this relative particle crosses over a bisector of the axis, at a distance from the origin smaller than $\sqrt{2}$. In no time its velocity index becomes equal to $j_r-\mathrm{I}$ or to $j_r+\mathrm{I}$, the precise value being determined by the condition that the intersected bisector of the axis is the internal bisector of the right angle drawn by the trajectory. They are two possible choices of interaction law, since the perpendicular axes of the Cartesian plane possesses two bisectors. To complete the specification of the interaction law we

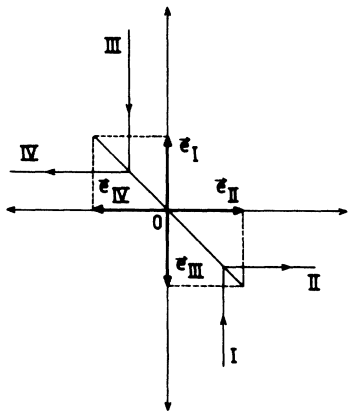

FIG. 1. Trajectory of relative particle during a two-body interaction.

stipulate that one of these two bisectors has been chosen. The interaction law defines a permutation in the set of velocities j_r through the one-to-one correspondence between the values of j_r before and after the collision. Let $j_r\to p(j_r)$ be this permutation which verifies $p[p(j_r)]=j_r$. In Fig. 1 the trajectory of the relative particle is drawn during a two-body interaction, where the choice of the bisector defining the "hard core" corresponds to the nilpotent permutation p defined by $p(\mathrm{I})=\mathrm{II}$. This choice of the interaction law will be used in what follows.

This model has been extensively studied with respect to the Boltzmann kinetic equation, and the question arises: Can the ring-collision term be fairly applied to this model, in order to calculate a transport coefficient at the ring order? For that purpose, one demands of this model the three following properties: (i) There is a kinetic theory, with an equilibrium ensemble. (ii) A hydrodynamic theory can be constructed, and a transport coefficient can be deduced from the kinetic theory, by the Chapman-Enskog method. (iii) The collision operator (and therefore the transport coefficient) may be expanded near its Boltzmann value in a formal way by considering events occurring between a finite number of particles moving in their entire phase space, and this finite-number-of-particles-colliding expansion (FNCE) leads to diverging integrals.

Let us determine these three points more precisely:

(i) In this model a momentum distribution function $\phi_1(\vec{p}_1)$ is replaced by a vector ϕ_j of a four-dimensional space. In order to avoid confusion in the labels the index 1 occurring in a one-body distribution function $\phi_1(\vec{p})$ will be omitted for this model. An s-body distribution function is turned into a tensor $f_{s;j_1,\ldots,j_S}(\vec{r}_1,\ldots,\vec{r}_S)$. In an equilibrium

situation[8,16] any s-body distribution "function" $f_{s;j_1,j_2,\ldots,j_S}(\vec{r}_1,\ldots,\vec{r}_S)$ is factorized as $f_{s;j_1,j_2,\ldots,j_S}(\vec{r}_1,\ldots,\vec{r}_S)=\phi^0_{j_1}\ldots\phi^0_{j_S}$, and the equilibrium distribution function ϕ^0_j satisfies

$$\phi^0_j\phi^0_{j+II}=\phi^0_{j+I}\phi^0_{j-I} \quad . \tag{4.1}$$

An equilibrium situation is characterized by the density n and the total current $\vec{y}$, which is equal to

$$\vec{y}=\sum_{j=I}^{IV}\vec{e}_j\,\phi^0_j n \quad ,$$

$\vec{e}_j$ being the unit vector of the j direction. The explicit value of ϕ^0_j corresponding to given values of $\vec{y}$ and n is determined by

$$\sum_{j=I}^{IV}\phi^0_j=1 \quad , \quad \sum_{j=I}^{IV}\phi^0_j\,\vec{e}_j n=\vec{y}$$

and by

$$\phi^0_I\,\phi^0_{III}=\phi^0_{II}\,\phi^0_{IV} \quad .$$

Thus, after some algebraic manipulations, we have

$$\phi^0_j=\frac{1}{4}\left(1+\frac{2y_j}{n}+\frac{y_j^2-y_{j+1}^2}{n^2}\right) \quad , \tag{4.2}$$

where

$$y_j=\vec{y}\cdot\vec{e}_j \quad .$$

The most simple equilibrium situation is defined by the absence of macroscopic current, i.e., by $\vec{y}=0$; the corresponding value of ϕ^0_j is $\frac{1}{4}$.

The kinetic theory of this model can be derived, as usual, from a master equation or from the BBGKY hierarchy. Since the "binary collision expansion" of the exact collision operator is known in general,[17] the same expansion can be used for the collision operator of this model. The basic element of this expansion is the matrix element of $T_{1,2}$, $\langle\vec{k}+\vec{q},\vec{k}'-\vec{q}\,|\,T_{1,2}|\vec{k},\vec{k}'\rangle$, a linear operator acting on a function $\Phi(\vec{p}_1,\vec{p}_2)$. This function $\Phi(\vec{p}_1,\vec{p}_2)$ is turned into a 4×4 tensor Φ_{j_1,j_2}, and the matrix element of $T_{1,2}$ into a linear operator acting on this tensor

$$[\langle\vec{k}+\vec{q},\vec{k}'-\vec{q}\,|\,T_{1,2}|\vec{k},\vec{k}'\rangle\,\Phi(1,2)]_{j_1,j_2}$$
$$=\sum_{j_{\bar1},j_{\bar2}}[\langle\vec{k}+\vec{q},\vec{k}'-\vec{q}\,|\,T_{1,2}\,|\vec{k},\vec{k}'\rangle]^{j_{\bar1},j_{\bar2}}_{j_1,j_2}\Phi_{j_{\bar1},j_{\bar2}} \quad .$$

The explicit form of the matrix element of $T_{1,2}$ is deduced from the general formula (3.4) and reads

$$\langle\vec{k}+\vec{q},\vec{k}'-\vec{q}|\,T_{1,2}\,|\vec{k},\vec{k}'\rangle^{j_{\bar1}j_{\bar2}}_{j_1j_2}$$
$$=-4\delta_{j_1,j_2+II}\,\delta_{j_{\bar1},j_{\bar2}+II}\,a(\vec{q})\,[\delta_{j_{\bar1},p(j_1)}-\delta_{j_{\bar1},j_1}] \quad , \tag{4.3}$$

where

$$a(\vec{q})=\frac{\sin(q_{II}-q_I)}{(q_{II}-q_I)} \quad .$$

(ii) The model possesses a continuous set of equilibrium states determined by the values of the density and of the current. A kinetic theory can be constructed, showing that, in a homogeneous system, the total density and the total current are conserved.[8] Accordingly, the evolution of slowly varying perturbations of the local density $n(\vec{r},t)$ and of the local current $\vec{y}(\vec{r},t)$ is described by a system of hydrodynamical equations relating the partial derivatives of $n(\vec{r},t)$ and of $\vec{y}(\vec{r},t)$. The Chapman-Enskog theory[10] shows how the hydrodynamical equations can be deduced from the kinetic theory. Without going into the details of this calculation, let us indicate that at the first order in the gradients a set of hydrodynamical equations of the "perfect fluid" are deduced. At the following order, terms of viscous "pressure" are introduced, and a transport coefficient appears, analogous to a viscosity. As usual, this transport coefficient is defined from an eigenvalue of the linearized collision operator.[8] For this model, this eigenvalue problem is well suited; it is defined through the following property of the collision operator. Let

$$\frac{\partial\phi_j}{\partial t}=S\{j;\phi\}$$

be the kinetic equation of this model, written for a homogeneous system. The collision term $S\{j;\phi\}$ is known, at least formally by its density expansion.[17] We may define the linear variation of $S\{j;\phi\}$ owing to a variation $\delta\phi$ of ϕ near an equilibrium value ϕ^0. This variation $\delta S\{j;\delta\phi\}$ may be considered as the result of the action of the linear operator $\delta S\{j;\}$ on the function $\delta\phi$.

From the conservation properties of S, it can be shown[8] that the distribution function χ, defined by $\chi_j=(-)^j$, is the only eigenfunction of the linear operator $\delta S\{j;\}$, with a nonzero eigenvalue. This statement will be verified for the ring-collision term. However, it must be pointed out that, at this state, the *existence* of this nonzero eigenvalue of $\delta S\{j;\}$ has only been *postulated*, since we need only the formal properties of S to define this eigenvalue.

(iii) Closed cycles of collision between three, four, and five bodies cannot occur for this model. Let us study now the event occurring between six bodies shown in Fig. 2. Particle 3 may be chosen on a collision "cylinder" of surface $AB\times1$, 4 on $AC\times1$, 5 on $CF\times1$, and AB, AC, and CF are as large as desired; particle 6 must be chosen on a surface of order 1, to allow the recollision E.

The total extension of the phase space in which particles 3–6 can be chosen is infinite, of order

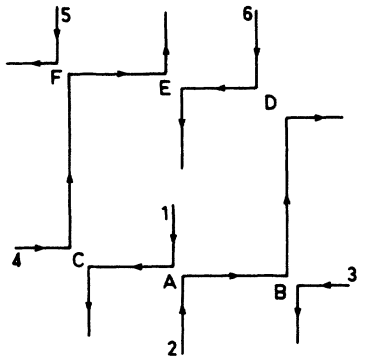

FIG. 2. Closed cycle of collisions between six bodies.

xyz, $x,y,z \to \infty$. If the ring-collision operator gives a convergent result, more or less equivalent to a cutoff of x, y, and z to the mean free path, then the ring-collision operator would be of order n^2.

B. Hydrodynamical Modes of the Maxwell-Model Gas in the Boltzmann-Enskog Approximation

With a view to an explicit calculation of the nonzero eigenvalue of the linearized ring-collision operator, we shall seek in this section the spectrum of the operator $\{(i\vec{k}\cdot\vec{p}_1)/m + n\Lambda(1,\vec{k};\phi^0)\}$ for the Maxwell model in the case $\phi_j^0 = \frac{1}{4}$. In order to use the formula (3.8), we must calculate $z_\mu(\vec{k})$ and $\mu(\vec{k},\vec{p})$ and give the explicit decomposition of any function $\Phi(\vec{p})$ on the basis of eigenfunctions $\mu(\vec{k},\vec{p})$.

According to the definition (2.21) of the operator $\Lambda(j,\vec{k};\phi)$ and to the value (4.3) of the matrix element of $T_{1,2}$, Eq. (3.6) reads for this model

$$[z_\mu(\vec{k}) - ik_j]\mu_j(\vec{k}) + n[\mu_{p(j)}(\vec{k}) - \mu_j(\vec{k})] + na(\vec{k})[\mu_{p(j)+\mathrm{II}}(\vec{k}) - \mu_{j+\mathrm{II}}(\vec{k})] = 0 \ . \tag{4.4}$$

The linear homogeneous system (4.4) defines the diagonalization of a 4×4 matrix, the eigenvalues being $z_\mu(\vec{k})$ and the eigenvectors $\mu_j(\vec{k})$. Since this diagonalization can be done by means of the standard methods of linear algebra, the details of calculation are omitted here. The characteristic equation of this diagonalization is

$$x^4 - 4nx^3 + x^2[4n^2 + k^2 - 4n^2a^2(\vec{k})] - 2nxk^2 + k_{\mathrm{I}}^2k_{\mathrm{II}}^2 + n^2[1 - a^2(\vec{k})](k_{\mathrm{I}} + k_{\mathrm{II}})^2 = 0 \ , \tag{4.5}$$

and $z_\mu(\vec{k})$ is one of the four roots of this algebraic equation.

The components $\mu_j(\vec{k})$ of an eigenvector are given, up to an undetermined constant N_μ, by

$$N_\mu\mu_{\mathrm{I}}(\vec{k}) = -a(\vec{k})[z_\mu^2(\vec{k}) - iz_\mu(\vec{k})(k_{\mathrm{II}} - k_{\mathrm{I}}) + k_{\mathrm{I}}k_{\mathrm{II}}] \ ,$$

$$N_\mu\mu_{\mathrm{II}}(\vec{k}) = a(\vec{k})[z_\mu^2(\vec{k}) + k_{\mathrm{I}}^2] \ ,$$

$$N_\mu\mu_{\mathrm{III}}(\vec{k}) = -z_\mu^2(\vec{k}) + z_\mu(\vec{k})[i(k_{\mathrm{I}} + k_{\mathrm{II}}) + n - na^2(\vec{k})] + k_{\mathrm{I}}k_{\mathrm{II}} - in[1 - a^2(\vec{k})](k_{\mathrm{I}} + k_{\mathrm{II}}) \ , \tag{4.6}$$

$$N_\mu\mu_{\mathrm{IV}}(\vec{k}) = z_\mu^3(\vec{k})/n - [z_\mu^2(\vec{k})/n](3n + ik_{\mathrm{II}}) + [z_\mu(\vec{k})/n] \times [k_{\mathrm{I}}^2 + 2nik_{\mathrm{II}} + 2n^2 - 2n^2a^2(\vec{k})] - ik_{\mathrm{II}}k_{\mathrm{I}}^2/n - k_{\mathrm{I}}^2 - ni(k_{\mathrm{I}} + k_{\mathrm{II}})[1 - a^2(\vec{k})] \ .$$

Nevertheless, we do not require an exact solution of (4.4), since our attention is mainly devoted to the values of $z_\mu(\vec{k})$ and $\mu_j(\vec{k})$ near $k=0$. In this limit (Appendix B) there exist three hydrodynamical eigenfunctions, each of them corresponding to an hydrodynamical "mode": the diffusion of the vorticity, labeled d, and the two sound waves, labeled "s^+" and "s^-," propagating in two opposite directions for a given value of $\vec{k}$. The diffusion of the entropy is absent from this model, which is obviously free from any thermal effect. The fourth mode, labeled "χ," corresponds to a purely damped perturbation, since $z_\chi(\vec{k}=0)$ differs from zero.

These eigenvalues and eigenvectors are given in the Table I with a proper choice of the undetermined constants N_μ.

Furthermore, in order to deal with expression (3.8), we must be able to expand a distribution "function" f_j on the basis of eigenvectors $\mu_j(\vec{k})$ such that

$$f_j = f_\chi\chi_j + f_d d_j + f_{s^+} + S_j^+ + f_{s^-} - S_j^- \ ,$$

where f_χ, f_d, f_{s^+}, and f_{s^-} are some linear combinations of the f_j's, which depend on $\vec{k}$. Near $k=0$, these combinations can be found by using the following combinatorial properties of the eigenfunctions in the $\vec{k}=0$ limit:

$$\sum_j(-)^j\mu_j = 4\delta_{\mu,\chi} \ , \quad \sum_j(k_{j+\mathrm{I}}/k)\mu_j = \delta_{\mu,d} \ , \quad \sum_j(k_j/k)\mu_j = \delta_{\mu,s^+} + \delta_{\mu,s^-} \ , \quad \sum_j\mu_j = \sqrt{2}(\delta_{\mu,s^+} - \delta_{\mu,s^-}) \ . \tag{4.7}$$

These properties proceed from the values of the eigenfunctions given in Table I. Thus from (4.7), one deduces the values of f_χ, f_d, f_{s^+}, and f_{s^-} near $k=0$,

$$f_\chi = \sum_j f_j(-)^j \ ,$$

$$f_{s^\pm} = \pm\sqrt{2}\sum_j f_j + (1/k)\sum_j k_j f_j \ , \tag{4.8}$$

$$f_d = (1/k)\sum_j k_{j+\mathrm{I}} f_j \ .$$

Now, since we know fully the spectrum of the operator $[(i\vec{k}\cdot\vec{p}_1)/m + n\Lambda(1,\vec{k};\phi^0)]$ near $k=0$, we can succeed in an explicit calculation of the nonzero eigenvalue of the linearized ring-collision operator.

C. Divergence of the Ring-Collision Term in the Maxwell Model

As announced, we shall demonstrate the divergence of the ring-collision term in two dimensions by calculating the nonzero eigenvalue of the linearized ring-collision operator. Let Ω be this eigenvalue. First it will be shown that the corresponding eigenfunction is χ_j. Since the interaction is of the hard-core type, the ring-collision term, linearized around an equilibrium value ϕ^0 of ϕ, is equal to $\delta S_R^{(1)}\{j_1;\delta\phi\}$. After some straightforward modifications, owing to the discrete character of the velocity space in the model, from (3.2) $\delta S_R\{j_1;\delta\phi\}$ is given by

$$\delta S_R\{j_1;\delta\phi\}=\sum_{j_2}\sum_{\bar{j}_1,\bar{j}_2}\int\frac{d\vec{k}}{(2\pi)^2}\,[\langle 0,0|\,T_{1,2}|\vec{k},-\vec{k}\rangle]^{\bar{j}_1,\bar{j}_2}_{j_1,j_2}\times\lim_{\epsilon'\to 0_+}\psi_{\bar{j}_1,\bar{j}_2}(\epsilon',\vec{k};\delta\phi)\,, \tag{4.9}$$

where

$$\psi_{\bar{j}_1,\bar{j}_2}(\epsilon',\vec{k};\delta\phi)=\sum_{j_1',j_2'}\{[B_0^{-1}(\epsilon',\vec{k})]^{j_1',j_2'}_{\bar{j}_1,\bar{j}_2}+[C^{-1}(\epsilon',\vec{k})]^{j_1',j_2'}_{\bar{j}_1,\bar{j}_2}\}\times\sum_{j_1'',j_2''}(\langle\vec{k},-\vec{k}|\,T_{1,2}|0,0\rangle)^{j_1',j_2'}_{j_1'',j_2''}\times(\phi^0_{j_1''}\delta\phi_{j_2''}+\phi^0_{j_2''}\delta\phi_{j_1''})\,. \tag{4.10}$$

In order to prove that χ_j is an eigenfunction of the linearized collision operator, it must be verified that, taking $\delta\phi_j=(-)^j$ in (4.9), we have $\delta S_R\{j_1;\delta\phi\}$ equal to $\Omega(-)^{j_1}$, where Ω is a constant independent of j_1. For that purpose, one remarks that it is true of the function $\psi_{\bar{j}_1,\bar{j}_2}(\epsilon',\vec{k};\delta\phi)$ that

$$\psi_{\bar{j}_1,\bar{j}_2}(\epsilon',\vec{k};\delta\phi)=\psi_{\bar{j}_2,\bar{j}_1}(\epsilon',-\vec{k};\delta\phi)\,. \tag{4.11}$$

This equality (4.11) originates from the following properties of symmetry of the operators B_0^{-1}, C^{-1}, and $\langle\vec{k},-\vec{k}|\,T_{1,2}|0,0\rangle$, which are obvious consequences of their definitions:

$$B_0^{-1}(\epsilon',\vec{k};1,2)=B_0^{-1}(\epsilon',-\vec{k};2,1)\,,$$

$$C^{-1}(\epsilon',\vec{k};1,2)=C^{-1}(\epsilon',-\vec{k};2,1),$$

$$\langle\vec{k},-\vec{k}|\,T_{1,2}|0,0\rangle=\langle-\vec{k},\vec{k}|\,T_{2,1}|0,0\rangle\,.$$

Inserting now into (4.9) the explicit value of the matrix elements $[\langle 0,0|\,T_{1,2}|\vec{k},-\vec{k}\rangle]^{\bar{j}_1,\bar{j}_2}_{j_1,j_2}$ given in (4.3), one obtains

$$\delta S_R\{j_1;\delta\phi\}=n\int\frac{d\vec{k}}{(2\pi)^2}\lim_{\epsilon'\to 0}a(\vec{k})\times[\psi_{p(j_1),p(j_1)+\mathrm{II}}(\epsilon',\vec{k};\delta\phi)-\psi_{j_1,j_1+\mathrm{II}}(\epsilon',\vec{k};\delta\phi)]\,, \tag{4.12}$$

where $a(\vec{k})\equiv[\sin(k_\mathrm{I}-k_\mathrm{II})]/(k_I-k_{II})$ is an even function of $\vec{k}$. Using this last property, together with (4.11), one shows at once that $\delta S_R\{j_1;\delta\phi\}$ depends upon j_1 as $(-)^{j_1}$. Thus the eigenfunction of the operator $\delta S_R\{j_1;\ \}$ is $\chi_j=(-)^j$, and the corresponding eigenvalue is

$$\Omega=-n\int\frac{d\vec{k}}{(2\pi)^2}\lim_{\epsilon'\to 0_+}a(\vec{k})[\psi_{\mathrm{II},\mathrm{IV}}(\epsilon',\vec{k};\chi)-\psi_{\mathrm{I},\mathrm{III}}(\epsilon',\vec{k};\chi)]\,. \tag{4.13}$$

Let us find now the other eigenfunctions of $\delta S_R\{j_1;\ \}$, and show that the corresponding eigenvalues are naught. For that purpose one remarks that $\psi_{\bar{j}_1,\bar{j}_2}(\epsilon',\vec{k};\delta\phi)$ is given by the action of a linear operator on the tensor $\alpha_{\vec{k};j_1,j_2}$ defined as

$$\alpha_{\vec{k};j_1,j_2}=\sum_{j_1',j_2'}(\langle\vec{k},-\vec{k}|\,T_{1,2}|0,0\rangle)^{j_1',j_2'}_{j_1,j_2}\times(\delta\phi_{j_1'}\phi^0_{j_2'}+\delta\phi_{j_2'}\phi^0_{j_1'})\,. \tag{4.14}$$

From (4.3) this tensor is explicited as

$$\alpha_{\vec{k};j_1',j_2'}=a(\vec{k})\delta_{j_1',j_2'+\mathrm{II}}[\delta\phi_{p(j_1')}\phi^0_{p(j_1')+\mathrm{II}}+\delta\phi_{p(j_1')+\mathrm{II}}\phi^0_{p(j_1')}-\delta\phi_{j_1'}\phi^0_{j_1'+\mathrm{II}}-\delta\phi_{j_1'+\mathrm{II}}\phi^0_{j_1'}]\,,$$

or, by setting $\delta\phi_j=\delta\bar{\phi}_j\phi^0_j$, and accounting for (4.2),

TABLE I. Hydrodynamical eigenmodes in the Maxwell model.

	Eigenvalue	Eigenfunction
Vorticity diffusion	$z_d(\vec{k})=\dfrac{k_\mathrm{I}^2k_\mathrm{II}^2+\frac{1}{3}n^2(k_\mathrm{I}^2-k_\mathrm{II}^2)^2}{2nk^2}+O(k^4)$	$d_j(\vec{k})=\dfrac{k_j+\mathrm{I}}{k}+O(k)$
Sound waves	$s^\pm(\vec{k})=\pm\dfrac{ik}{\sqrt{2}}+\dfrac{1}{8nk^2}[k_\mathrm{I}^4+k_\mathrm{II}^4+\frac{2}{3}n^2(k_\mathrm{I}-k_\mathrm{II})^4]+O(k^4)$	$s_j^\pm(\vec{k})=\dfrac{k_j}{2k}\dfrac{\sqrt{2}}{4}+O(k)$
Purely damped perturbation	$z_\chi=+4n+O(k)$	$\chi_j=(-)^j+O(k)$

$$\alpha_{\vec{k};j_1',j_2'} = a(\vec{k})\delta_{j_1',j_2'+\mathrm{II}}(-)^{j_1'+\mathrm{I}}$$

$$\times \phi^0_{\mathrm{II}}\phi^0_{\mathrm{IV}}(\delta\bar{\phi}_{\mathrm{II}} + \delta\bar{\phi}_{\mathrm{IV}} - \delta\bar{\phi}_{\mathrm{I}} - \delta\bar{\phi}_{\mathrm{III}}) \ . \quad (4.15)$$

Thus $\alpha_{\vec{k};j_1',j_2'}$ and $\delta S_R\{j_1;\ \delta\phi\}$ are equal to zero for any choice of $\delta\phi$ such as the combination $\delta\bar{\phi}_{\mathrm{II}} + \delta\phi_{\mathrm{IV}} - \delta\bar{\phi}_{\mathrm{I}} - \delta\phi_{\mathrm{III}}$ is equal to zero. This is realized when $\delta\bar{\phi}_j$ is a linear combination of the following three functions of the velocity: 1, $\delta_{j+\mathrm{II}} - \delta_j$, and $\delta_{j+\mathrm{I}} - \delta_{j+\mathrm{III}}$. Thus the three eigenfunctions of the operator $\delta S_R\{j_1;\ \}$ with the eigenvalue 0 are ϕ^0_j, $(\delta_{j+\mathrm{II}} - \delta_j)\phi^0_j$, and $(\delta_{j+\mathrm{I}} - \delta_{j+\mathrm{III}})\phi^0_j$.

Now we shall succeed in an explicit calculation of Ω, and prove the divergence of the integral (4.13). As announced, we shall restrict ourselves to the equilibrium defined by the absence of macroscopic current, i.e, by $\phi^0_j = \frac{1}{4}$. In that case, and when $\delta\phi_j = \chi_j$, one has

$$\alpha_{\vec{k};j_1',j_2'} = -a(\vec{k})\delta_{j_1',j_2'+\mathrm{II}}(-)^{j_1} \ . \quad (4.16)$$

Now, by combining (4.10), (4.13), and (4.16), Ω is given as

$$\Omega = n\sum_{j_1',j_2'}\sum_{j=\mathrm{I},\mathrm{II}}\int\frac{d\vec{k}}{(2\pi)^2}a^2(\vec{k})\lim_{\epsilon'\to 0_+}\{[B_0^{-1}(\epsilon',\vec{k})]^{j_1',j_2'}_{j,j+\mathrm{II}}$$

$$+ C^{-1}(\epsilon',\vec{k})]^{j_1',j_2'}_{j,j+\mathrm{II}}\}(-)^{j+j_1'}\delta_{j_1',j_2'+\mathrm{II}} \ . \quad (4.17)$$

First one calculates the part of Ω proportional to the operator $C^{-1}(\epsilon',\vec{k})$. From the general definition of this operator we have

$$[C^{-1}(\epsilon',\vec{k})]^{j_1',j_2'}_{j,j+\mathrm{II}} = -\delta_{j,j_1'}\delta_{j_2',j+\mathrm{II}}(\epsilon' + 2ik_j)^{-1} \ .$$

Thus we have

$$n\sum_{j_1',j_2'}\sum_{j=\mathrm{I},\mathrm{II}}\int\frac{d\vec{k}}{(2\pi)^2}a^2(\vec{k})\lim_{\epsilon'\to 0_+}[C^{-1}(\epsilon',\vec{k})]^{j_1',j_2'}_{j,j+\mathrm{II}}(-)^{j+j_1'}\delta_{j_1',j_2'+\mathrm{II}}$$

$$= -\frac{n}{2\pi^2}\int_{-\infty}^{+\infty}dk_{\mathrm{I}}\int_{-\infty}^{+\infty}dk_{\mathrm{II}}\lim_{\epsilon'\to 0_+}\left(\frac{1}{\epsilon' + 2ik_{\mathrm{II}}}\right)\frac{\sin^2(k_{\mathrm{I}} - k_{\mathrm{II}})}{(k_{\mathrm{I}} - k_{\mathrm{II}})^2} \ . \quad (4.18)$$

And since

$$\lim_{\epsilon'\to 0_+}\frac{1}{\epsilon' + 2ik_{\mathrm{II}}} = \frac{1}{2i}P\frac{1}{k_{\mathrm{II}}} + \frac{\pi}{4}\delta(k_{\mathrm{II}})$$

in the sense of the distributions, it is easily shown that the contribution to Ω originating from $C^{-1}(\epsilon,\vec{k})$ is equal to $-\frac{1}{8}n$.

To proceed further, we must invert the linear operator $B_0(\epsilon',\vec{k};1,2)$ for this model. Owing to the discrete character of the velocity space, this inversion is equivalent to the solution of a system of $4^2 = 16$ linear equations. The corresponding algebra is presumably quite heavy; however, let us recall that we are seeking the value of the integrand of (4.17) near $k = 0$, and in this limit we are aware of the spectrum of the operator $[(i\vec{k}\cdot\vec{p}_1)/m + n\Lambda(1,\vec{k};\phi^0)]$ and thus of the operator $B_0(\epsilon',\vec{k};1,2)$. In order to express Ω by starting from this spectral expansion, we split Ω into the various contributions of the eigenfunctions of $B_0(\epsilon',\vec{k};1,2)$:

$$\Omega = \sum_{\mu,\mu'}\Omega_{\mu,\mu'} - \frac{n}{8} \ , \quad (4.19)$$

where

$$\Omega_{\mu,\mu'} = -4n\int\frac{d\vec{k}}{(2\pi)^2}a(\vec{k})$$

$$\times\lim_{\epsilon'\to 0_+}\frac{\mu_{\mathrm{I}}(\vec{k})\mu'_{\mathrm{III}}(-\vec{k}) - \mu_{\mathrm{II}}(\vec{k})\mu'_{\mathrm{IV}}(-\vec{k})}{\epsilon' + z_\mu(\vec{k}) + z_{\mu'}(-\vec{k})}\alpha_{\vec{k};\mu,\mu'} \ , \quad (4.20)$$

and where the term $(-\frac{1}{8}n)$ originates from the term calculated in (4.18). In (4.20) the function $\alpha_{\vec{k};\mu,\mu'}$ is the function of two velocities $\alpha_{\vec{k};j_1',j_2'}$ given in (4.16) and written on the basis of eigenfunctions $\mu_{j_1'}(\vec{k})\,\mu_{j_2'}(-\vec{k})$. This change of basis can be done near $k = 0$, by means of the formulas (4.8), and gives for example

$$\alpha_{\vec{k};d,d} = -\frac{1}{k^2}\sum_{j_1',j_2'}k_{j_1'+\mathrm{I}}k_{j_2'+\mathrm{I}}\alpha_{\vec{k};j_1',j_2'}$$

$$= (k_{\mathrm{I}}^2 - k_{\mathrm{II}}^2)/k^2 \ . \quad (4.21)$$

As has been explained in Sec. III, we are mainly interested in those values of $\Omega_{\mu,\mu'}$ for which $[z_\mu(\vec{k}) + z_{\mu'}(-\vec{k})]$ is of order k^2 near $k = 0$. This occurs in two cases: (i) when both μ and μ' are the label of the diffusion mode defined in Table I, and (ii) when μ and μ' are, respectively, equal to s^+ and s^- or to s^- and s^+, since $z_{s^\pm}(\vec{k}) = \pm i(k/\sqrt{2}) + O(k^2)$.

Let us consider these three contributions (namely $\Omega_{d,d}$, Ω_{s^+,s^-} and Ω_{s^-,s^+}) to the eigenvalue Ω. Using again the results of the Table I, and the expansion of a function of the velocity on the basis of eigenfunction $\mu_j(\vec{k})$ given in (4.8), one obtains the values

of the integrand of (4.20) near $k=0$. For the case $\mu=\mu'=d$, one has

$$\frac{d_I(\vec{k})\,d_{III}(-\vec{k})-d_{II}(\vec{k})\,d_{IV}(-\vec{k})}{\epsilon'+z_d(\vec{k})+z_d(-\vec{k})}\,\alpha_{\vec{k};d,d}$$

$$=\left(\frac{1}{4}\,\frac{(k_I^2-k_{II}^2)^2}{k^4}+O(k^2)\right)$$

$$\times\left(\epsilon'+\frac{k_I^2k_{II}^2+\frac{1}{3}n^2(k_I^2-k_{II}^2)^2}{nk^2}+O(k^4)\right)^{-1}a(\vec{k})\,. \quad (4.22)$$

Now we are able to calculate the contribution of the domain of small values of $\vec{k}$ to the integral which defines $\Omega_{d,d}$. Let $\Omega_{d,d}^{k<k_0}$ be this contribution, which is exactly defined in Appendix C. In the same appendix it is shown that this contribution diverges logarithmically in the limit $\epsilon'\to 0_+$, giving

$$\Omega_{d,d}^{k<k_0}\underset{\epsilon'\to 0_+}{\approx}\frac{8n^2}{\pi(\frac{1}{2}\sqrt{3}+n)}\ln\epsilon'\,. \quad (4.23)$$

Among similar lines, it is shown in Appendix C that the integrals defining the two contributions $\Omega_{s^\pm,s^\mp}$ diverge logarithmically in the limit $\epsilon'=0_+$. The exact result is quite complicated and is not reproduced here.

At this stage we have proved that some contributions to the eigenvalue Ω are logarithmically divergent. However, in order to demonstrate the nonexistence of Ω we will have to consider the other contributions to Ω, in order to rule out the (improbable) possibility of divergences which would just compensate the divergences of $\Omega_{d,d}$ and of $\Omega_{s^\pm,s^\mp}$.

First, the convergence for the large value of k of the integral defining Ω may be questioned. This convergence is proved in Appendix D. Moreover, one has to prove this convergence in the domain of finite values of k, and in the domain of small values of k for those contributions $\Omega_{\mu,\mu'}$ which have not yet been examined, namely, for those values of $\Omega_{\mu,\mu'}$ which differ from $\Omega_{d,d}$ and from $\Omega_{s^\pm s^\mp}$.

Let us glance over these two latter points:

(i) A simple inspection of Table I, and of the definition (4.20) of $\Omega_{\mu,\mu'}$, shows that the factor $\lim[\epsilon'+z_\mu(\vec{k})+z_{\mu'}(-\vec{k})]^{-1}$ occuring in this definition is of order k or constant near $k=0$, except when $(\mu,\mu')=(d,d)$ or $(s^\mp,s^\pm)$. Hence the integrand, written in polar coordinates, remains finite when $\epsilon'=0_+$ near $k=0$, and in the same limit the integral defining $\Omega_{\mu,\mu'}$ is well defined, except when $(\mu,\mu')=(d,d)$ or $(s^\mp,s^\pm)$.

(ii) Let us prove the absence of any singularity of the integrand occuring in (4.20) for a finite value of $\vec{k}$. As has been pointed out, this integrand is determined by the inversion of the operator $B_0(\epsilon',\vec{k})$ and this inversion is equivalent to the solution of a set of 16 linear inhomogeneous equations. The inhomogeneous terms of these equations are the elements of the tensor $\alpha_{\vec{k};j_1',j_2'}$ which are everywhere finite. Thus the solution of this system, namely the 4×4 tensor

$$\lim_{\epsilon'\to 0_+}\sum_{j_1',j_2'}[B_0^{-1}(\epsilon',\vec{k})]_{j_1',j_2'}^{j_1',j_2'}\,\alpha_{\vec{k};j_1',j_2'} \quad (4.24)$$

is finite, except when the corresponding determinant is naught. Since the eigenvalues of the linear operator $[B_0(\epsilon'=0,\vec{k})]_{j_1',j_2'}^{j_1',j_2'}$ are $[z_\mu(\vec{k})+z_{\mu'}(-\vec{k})]$, the corresponding determinant is zero when two eigenvalues verify

$$z_\mu(\vec{k})+z_{\mu'}(-\vec{k})=0. \quad (4.25)$$

These eigenvalues are the roots of the same characteristic Eq. (4.6), which is invariant under the transformation $\vec{k}\to-\vec{k}$. Thus the condition (4.25) implies the existence of a root of (4.6) such that

$$\mathrm{Re}\{z_\mu(\vec{k})\}\le 0. \quad (4.26)$$

This possibility may be excluded for any finite value of $\vec{k}$. In fact Table I shows that *near* $k=0$, $\mathrm{Re}\{z_\mu(\vec{k})\}>0$, whatever $\vec{k}$ will be.

Hence, given this last property of $\mathrm{Re}\{z_\mu(\vec{k})\}$ near $k=0$, in order to satisfy (4.26) for a finite value of $\vec{k}$, say $\vec{k}'$, there must exist a vector $\vec{k}''$ such that

$$0<k''\le k',\quad \mathrm{Re}\{z_\mu(\vec{k}'')\}=0.$$

This possibility is excluded by inspection of (4.6), and thus the absence of singularity of the integral on the right-hand side of (4.20) occuring for a finite value of $\vec{k}$ is proved.

We can assert now that the integral defining Ω has an uncompensated logarithmic divergence when ϵ' goes to zero, and we have shown that this divergence originates from the hydrodynamical long-range propagation of disturbances. Hence the study of this model provides us with a strong indication in favor of the existence of the divergence in any two-dimensional simple dense gas.

V. CONCLUSION

We have shown that, for a particular case, the "renormalization" program in the nonequilibrium density expansion leads to a new class of divergences. In fact, it seems that this divergence is a manifestation of a fundamental property of fluids,[18] i.e., in a nonequilibrium state there is a long-range long-time hydrodynamical propagation of correlations, which may render invalid the Bogolubov synchronization hypothesis for a certain class of many-body systems.

It appears legitimate to think that in two-dimensional systems the ring-collision term is divergence free only in the case of the perfect Lorentz gas.

Thus for the case of a two-dimensional gas, there

remain the following problems to solve:

(i) Does there exist a transport theory analogous to that of the three-dimensional gas? For example, can we say that the Kubo integrals which define the transport coefficients are convergent? Indeed, the existence of a new divergence does not preclude the existence of transport coefficients which could be found by another type of development than that which leads to the ring approximation.

(ii) If the Kubo integrals diverge for two-dimensional systems, what sort of law connects the fluxes to the gradients, replacing the laws of Fourier and of Newton?

APPENDIX A

In this appendix it will be proved that the operator

$$L(1, \vec{k}) \equiv (i\vec{k}\cdot\vec{p}_1)/m + n\Lambda(1, \vec{k}; \phi_1^0) \tag{A1}$$

has $(\nu+2)$ eigenvalues $z_\mu(k)$ such that $z_\mu(\vec{k}=0)=0$, two eigenvalues $z_\mu(\vec{k})$ being of order k and the remaining ones of order k^2. This result is known for some time in what concerns the operator

$$(i\vec{k}\cdot\vec{p}_1)/m + n\Lambda(1, \vec{k}=0; \phi_1^0) \quad \text{for } \nu=3 .$$

Hence we shall prove that this property of the operator $L(1, \vec{k})$ is of a very general character, since we shall only use quite general hypotheses. An essential element of this proof will be the property of isotropy and of parity invariance of the space in which the particles behave. Hence this proof is not valid for the Maxwell model studied in Sec. IV of this paper. But for this model, the eigenvalues $z_\mu(\vec{k})$ are the roots of an algebraic equation, given in (4.5), and it may be proved in a straightforward way that one of these roots is of order k^2 near $k=0$.

For the sake of the general proof, the following assumptions will be needed:

(i) The operator $L(1, \vec{k})$ is a *scalar* quantity. In order to define this property, let us consider an element U of the group of rotations and space reflections in the Euclidean space R^ν; in the terminology of group theory $U \in O(\nu)$. From U one defines the operator $\tilde{U}$ acting on functions of some sets of vectors of R^ν as

$$\tilde{U}\psi(\vec{V}_1, \ldots, \vec{V}_r) = \psi(U\vec{V}_1, \ldots, U\vec{V}_r) .$$

The vectors $\vec{V}_1, \ldots, \vec{V}_r$ must belong to one of the following sets of vectors: (a) the dynamical variables of a particle, or (b) the variable of the Fourier transform of a function of the position.

The operator $\Lambda(1, \vec{k}; \phi_1^0)$ is scalar if it commutes with any operator $\tilde{U}$,

$$[\tilde{U}, \Lambda(1, \vec{k}; \phi_1^0)] = 0 .$$

In order to verify this property, one notices at once that the operator $T_{1,2}$ defined in (2.9) is scalar since it is an algebraic combination of the obviously scalar Green's functions $(\epsilon + H_2^0)^{-1}$ and $(\epsilon + H_2^0 + \theta_{1/2})^{-1}$. The matrix elements of $T_{1,2}$, namely $\langle \vec{k}+\vec{q}, \vec{k}' - \vec{q} | T_{1,2} | \vec{k}, \vec{k}' \rangle$ are scalar too, and the operator $\Lambda(1, \vec{k}; \phi_1)$ defined in (2.22) is scalar when $\phi_1(\vec{p}_1)$ is $\tilde{U}$ invariant, namely, when

$$\tilde{U}\phi_1(\vec{p}_1) = \phi_1(\vec{p}_1) .$$

This is precisely the case when $\phi_1(\vec{p}_1)$ is an equilibrium distribution function $\phi_1^0(\vec{p}_1)$. Thus $\Lambda(1, \vec{k}; \phi_1^0)$ is scalar and so is the operator $L(1, \vec{k})$ defined in (A1).

(ii) A scalar product of two functions of $\vec{p}_1$ is defined as

$$(\phi, \psi) = \int d\vec{p}_1 \, \phi(\vec{p}_1)\psi(\vec{p}_1) . \tag{A2}$$

The result of this scalar product may depend on the choice of the coordinate system, since ϕ and ψ depend actually on $\vec{p}_1$ and on other vectors. Thus the $\tilde{U}$ invariance of this scalar product reads

$$(\tilde{U}\phi, \tilde{U}\psi) = \tilde{U}(\phi, \psi) . \tag{A3}$$

(iii) The operator $L^0 \equiv L(1, \vec{k}=0)$ has $(\nu+2)$ eigenfunctions with the eigenvalue 0. For further convenience, one writes these eigenfunctions as

$$\begin{aligned}
&\phi_n(\vec{p}_1) \equiv \phi_1^0(\vec{p}_1) = (\beta/\pi)^{\nu/2} e^{-\beta p_1^2} , \\
&\phi_\theta(\vec{p}_1) = \frac{(\nu+2)\nu}{(2\beta)^2}\left(p_1^2 - \frac{\nu}{2\beta}\right)\phi_1^0(\vec{p}_1) , \\
&\phi_j(\vec{p}_1) = \frac{\vec{e}_j\cdot\vec{p}_1}{2\beta}\,\phi_1^0(\vec{p}_1) \quad (j=1, 2, \ldots, \nu) .
\end{aligned} \tag{A4}$$

In the terminology of group theory, the $(\nu+2)$ functions $\phi_a(\vec{p}_1)\,(a=n, \theta; 1, 2, \ldots, \nu)$ may be considered as a basis of the kernel K of L^0. Any function $f(\vec{p}_1)$ such that

$$L^0 f(\vec{p}_1) = 0$$

may be written as

$$f(\vec{p}_1) = \sum_{a=n,\theta;1,2\ldots,\nu} f_a \phi_a(\vec{p}_1) ,$$

the numbers f_a being the components of the vector f in K.

(iv) Let us define the operator $[L^0]^T$ transposed of L^0 as

$$(\psi, L^0\phi) = \{[L^0]^T\psi, \phi\} .$$

The operator $[L^0]^T$ has a nonempty kernel K^T. In K^T, we may define a basis of functions $\phi_a(\vec{p}_1)(a=n, \theta; 1, 2, \ldots, \nu)$ as

$$\psi_n(\vec{p}_1) = 1 , \quad \psi_\theta(\vec{p}_1) = p_1^2 , \quad \psi_j(\vec{p}_1) = \vec{e}_j\cdot\vec{p}_1 . \tag{A5}$$

The bases of K and K^T, respectively, defined in (A4) and (A5), are mutually orthogonal

$$(\psi_a, \phi_b) = \delta_{a,b} \quad (a, b = n, \theta; 1, 2, \ldots, \nu) .$$

Properties (iii) and (iv) are elementary properties[19] of the linearized Boltzmann collision operator.

(v) In what follows we will have to consider the problem of the inversion of the operator L^0, namely, we will have to solve an integral equation

$$L^0 f(\vec{p}_1) = g(\vec{p}_1) . \tag{A6}$$

Firstly, since $[L^0]^T$ has a nonempty kernel, $g(\vec{p}_1)$ must fulfill the following conditions:

$$(\psi_a, g) = 0 \quad (a = n, ; 1, 2, \ldots, \nu) . \tag{A7}$$

Furthermore, since the kernel K of L^0 is nonempty [property (iii)], the solution of (A6) is determined up to the addition of an arbitrary vector of K. Among all the solutions of (A6), let us call $[L^0]^{-1} g$ the solution which is orthogonal to K^T. Thus the general solution of (A6) reads

$$f(\vec{p}_1) = [L^0]^{-1} g(\vec{p}_1) + \sum_a M_a \phi_a(\vec{p}_1) . \tag{A8}$$

The quantities M_a are the components of an arbitrary vector of K.

The operator $[L^0]^{-1}$ is known explicitly in the particular case of the Maxwellian molecules.[19] In the general case, $[L^0]^{-1}$ is calculated by means of the so-called "Enskog expansion."[10,20]

Now we are able to find the eigenfunctions $\mu(\vec{k}, \vec{p}_1)$ and the eigenvalues $z_\mu(\vec{k})$ of $L(1, \vec{k})$ near $k = 0$. These quantities are defined by

$$L(1, \vec{k}) \mu(\vec{k}, \vec{p}_1) = z_\mu(\vec{k}) \mu(\vec{k}, \vec{p}_1) , \quad z_\mu(\vec{k} = 0) = 0 . \tag{A9}$$

Moreover, we shall assume that the various quantities occurring in (A9) can be expanded near $k = 0$ as

$$L(1, \vec{k}) = L^0 + \vec{k} \cdot \vec{L}^1 + \vec{k} : \vec{k} \cdot \vec{L}^2 + \cdots , \tag{A10}$$

where

$$L^0 \equiv L(1, \vec{k} = 0) , \qquad \vec{L}^1 = \frac{\partial}{\partial \vec{k}} L(1, \vec{k})\big|_{\vec{k}=0} ,$$

$$\vec{L}^2 = \frac{1}{2} \frac{\partial}{\partial \vec{k}} : \frac{\partial}{\partial \vec{k}} L(1, \vec{k})\big|_{\vec{k}=0} ;$$

as

$$z_\mu(\vec{k}) = z^1_\mu + z^2_\mu + \cdots , \tag{A11}$$

where

$$z^1_\mu = k \lim_{k \to 0} \left(\frac{z_\mu(k)}{k} \right) , \quad z^2_\mu = k^2 \lim \left(\frac{z_\mu(k) - k z^1_\mu}{k^2} \right) , \ldots ;$$

and as

$$\mu(\vec{k}, \vec{p}_1) = \mu^0(\hat{k}, \vec{p}_1) + \mu^1(\vec{k}, \vec{p}_1) + \cdots , \tag{A12}$$

where

$$\mu^0(\hat{k}, \vec{p}_1) = \lim_{k \to 0} \mu(\vec{k}, \vec{p}_1), \ldots .$$

In (A12) the dependence of $\mu^0(\hat{k}, \vec{p}_1)$ on $\hat{k} = \vec{k}/k$ has been indicated since, as will be shown, $\lim \mu(\vec{k}, \vec{p}_1)$ as $k \to 0$ depends actually on $\hat{k}$.

From (iii) one deduces at once that $\mu^0(\hat{k}, \vec{p}_1)$ is a vector of the kernel K. This vector is defined by its components μ^0_a on the basis $\{\phi_a\}$ of K. The components μ^0_a are to be determined by expending (A9) in powers of k. In this way, one has at the first order

$$L^0 \mu^1(\vec{k}, \vec{p}_1) = [z^1 - \vec{k} \cdot \vec{L}^1] \mu^0(\hat{k}, \vec{p}_1) . \tag{A13}$$

This equation has been studied in (v). First, one has to account for condition (A7) of orthogonality of the right-hand side of (A13) with K^T:

$$z^1_\mu \mu^0_a = \sum_{b=n,\theta;1,2,\ldots,\nu} (\psi_a, \vec{k} \cdot \vec{L}^1 \phi_b) \mu^0_b . \tag{A14}$$

This set of conditions defines the diagonalization of the matrix $(\psi_a, \vec{k} \cdot \vec{L}^1 \phi_b)$ acting on vectors of K. This diagonalization is a very simple one, since many elements of this matrix are equal to zero. The operator $\vec{L}^1$ is the gradient with respect to $\vec{k}$ of the scalar operator $L(1, \vec{k})$, thus $\vec{L}^1$ is a *vectorial* operator and we have

$$\tilde{U} \vec{L}^1 = (U \vec{L}^1) \tilde{U} , \tag{A15}$$

where U is any element of $O(\nu)$ and $(U \vec{L}^1)$ is the result of the transformation U on $\vec{L}^1$ considered as an ordinary vector of R^ν. Let us consider now the matrix element $(\psi_n, \vec{k} \cdot \vec{L}^1 \phi_n)$ and the space inversion P: $P \vec{V}_r = - \vec{V}_r$. From (A3) we have

$$(\psi_n, \vec{L}^1 \phi_n) = (\tilde{P} \psi_n, \tilde{P} \vec{L}^1 \phi_n) , \tag{A16}$$

and from (A4) and (A5) we have

$$\tilde{P} \psi_n = \psi_n \quad \text{and} \quad \tilde{P} \phi_n = \phi_n .$$

Applying now (A15) and (A16), one obtains

$$(\psi_n, \vec{L}^1 \phi_n) = (\psi_n, P \vec{L}^1) \phi_n = - (\psi_n, \vec{L}^1 \psi_n) = 0. \tag{A17}$$

One shows along similar lines that the only nonzero matrix elements $(\psi_a, \vec{k} \cdot \vec{L}^1 \phi_b)$ are those for which a (or b) is equal to n or θ; and b (or a) is the value of the vectorial index j such that $e_j = \hat{k}$. This result is nothing else but the "selection rules" of the operator $\vec{k} \cdot \vec{L}^1$. Let us call now "l" the index j such that $e_j = \hat{k}$, and "t" $[t = 1, 2, \ldots, (\nu - 1)]$ the $(\nu - 1)$ remaining vectorial indices. Thus from the selection rules (A14) yields

$$z^1_\mu \mu^0_t = 0 \quad [t = 1, 2, \ldots, (\nu - 1)] , \tag{A18}$$

$$z^1_\mu \mu^0_l = (\psi_l, \vec{k} \cdot \vec{L}^1 \phi_n) \mu^0_n + (\psi_l, \vec{k} \cdot \vec{L}^1 \phi_\theta) \mu^0_\theta , \tag{A19}$$

$$z^1_\mu \mu^0_\theta = (\psi_\theta, \vec{k} \cdot \vec{L}^1 \phi_l) \mu^0_l , \tag{A20}$$

$$z^1_\mu \mu^0_n = (\psi_n, \vec{k} \cdot \vec{L}^1 \phi_l) \mu^0_l . \tag{A21}$$

Now, it is obvious that the eigenvalue $z^1_\mu = 0$ is ν-fold degenerate, the corresponding eigenvectors

being determined from

$$\mu_l^0 = 0 \quad \text{and} \quad (\psi_l, \vec{k}\cdot\vec{L}^1\phi_n)\mu_n^0 + (\psi_l, \vec{k}\cdot\vec{L}^1\phi_\theta)\mu_\theta^0 = 0.$$

The two remaining eigenvalues of the matrix $(\psi_a, \vec{k}\cdot\vec{L}^1\phi_b)$ are given by

$$z_\mu^1 = \pm[(\psi_n, \vec{k}\cdot\vec{L}^1\phi_l)(\psi_l, \vec{k}\cdot\vec{L}^1\phi_n) + (\psi_\theta, \vec{k}\cdot\vec{L}^1\phi_l)(\psi_l, \vec{k}\cdot\vec{L}^1\phi_\theta)]^{1/2}.$$

The components μ_n^0, μ_θ^0, and μ_l^0 of the corresponding eigenvectors are determined from (A20) and (A21), the components μ_t^0 are equal to zero. Now, instead of the functions $\phi_a(\vec{p}_1)$ defined in (A4), we shall use as a new basis in K a set of vectors $\mu^0(\vec{p}_1)$ which are eigenvectors of the matrix $(\psi_a, \vec{k}\cdot\vec{L}^1\phi_b)$. This choice of eigenvectors of $(\psi_a, \vec{k}\cdot\vec{L}^1\phi_b)$ remains quite undetermined, since this matrix has zero as a ν-fold degenerate eigenvalue. From this basis $\{\mu^0\}$ of K one defines a new basis $\{\psi_\mu\}$ in K^T such that

$$(\psi_\mu, \mu^0) = \delta_{\mu,\mu'}. \tag{A22}$$

The functions $\psi_\mu(\vec{p}_1)$ and $\mu^0(\hat{k}, \vec{p}_1)$ verify

$$(\psi_n, \vec{k}\cdot\vec{L}^1\mu^0) = z_\mu^1\delta_{\mu,\mu'}, \tag{A23}$$

The functions $\mu^0(\hat{k}, \vec{p}_1)$ will be determined subsequently from the calculation of $\mu^2(\vec{k}, \vec{p}_1)$.

Now we are able to solve the integral equation (A9) which is a particular case of (A6). The solvability conditions (A7) have been written in (A14), and (A9) yields

$$\mu^1(\vec{k}, \vec{p}_1) = [L^0]^{-1}[z_\mu^1 - \vec{k}\cdot\vec{L}^1]\,\mu^0(\hat{k}, \vec{p}_1) + \sum_{\mu'} M_\mu^{\mu'}\mu'^0(\hat{k}, \vec{p}_1) \tag{A24}$$

with the quantities $M_\mu^{\mu'}$ being undetermined at this stage. Expanding now (A9) up to the second order in $\vec{k}$, one has

$$L^0\mu^2(\vec{k}, \vec{p}_1) = (z_\mu^1 - \vec{k}\cdot\vec{L}^1)\mu^1(\vec{k}, \vec{p}_1) + [z_\mu^2 - \vec{k}:\vec{k}\cdot\vec{L}^2]\,\mu^0(\hat{k}, \vec{p}_1). \tag{A25}$$

The condition of solvability (A9) together with (A24) and (A25) yields

$$(z_\mu^1 - z_{\mu'}^1)M_\mu^{\mu'} + z_\mu^2\delta_{\mu,\mu'} = (\psi_{\mu'}, \vec{k}:\vec{k}\cdot\vec{L}^2\mu^0) + \{\psi_\mu, [\vec{k}\cdot\vec{L}^1][L^0]^{-1}[\vec{k}\cdot\vec{L}^1 - z_\mu^1]\mu^0\}. \tag{A26}$$

From (A26) one deduces a set of conditions which define the eigenfunctions $\mu^0(\hat{k}, \vec{p}_1)$. In fact, let us consider (A26) when applied to the case $z_\mu^1 = z_{\mu'}^1 = 0$:

$$(1-\delta_{\mu,\mu'})(\psi_{\mu'}, \vec{k}:\vec{k}\cdot\vec{L}^2\mu^0) + (1-\delta_{\mu,\mu'}) \times \{\psi_{\mu'\mu}, [\vec{k}\cdot\vec{L}^1][L^0]^{-1}[\vec{k}\cdot\vec{L}^1]\mu^0\} = 0. \tag{A27}$$

It may be verified from the "selection rules" of the operators occurring in (A27) that the functions $\mu^0(\hat{k}, \vec{p}_1)$ which verify (A27) and $z_\mu^1 = 0$ are

$$-t^0(\hat{k}, \vec{p}_1) = [(\vec{e}_t\cdot\vec{p}_1)/2\beta)]\,\phi_1^0(\vec{p}_1).$$

e_t being any of the $(\nu-1)$ unit vectors perpendicular to $\hat{k}$. This set of eigenvectors corresponds to the vorticity diffusion

$$-d^0(\hat{k}, \vec{p}_1) = -(\psi_l, \vec{k}\cdot\vec{L}^1\phi_n)\,\phi_\theta(\vec{p}_1) + (\psi_l, \vec{k}\cdot\vec{L}^1\phi_\theta)\phi_n(\vec{p}_1).$$

This eigenfunction corresponds to the entropy diffusion.[21]

From (A26), z_μ^2 is given as

$$z_\mu^2 = (\psi_\mu, \vec{k}:\vec{k}\cdot\vec{L}^2\mu^0) + \{\psi_\mu, [\vec{k}\cdot\vec{L}^1][L^0]^{-1} \times [\vec{k}\cdot\vec{L}^1 - z_\mu^1]\mu^0\}. \tag{A28}$$

Owing to the form of the operators $\vec{k}$: $\vec{k}\vec{L}^2$ and $[\vec{k}\cdot\vec{L}^1][L^0]^{-1}[\vec{k}\cdot\vec{L}^1 - z_\mu^1]$ occuring in (A28) it is not difficult to see that the selection rules never imply that z_μ^2 is equal to zero. However, it does not really prove that z_μ^2 is a well-behaved quantity, namely, that z_μ^2 exists and differs from zero. A general and exact study of this question would require some knowledge about the operator $[L^0]^{-1}$, and to our knowledge, the general problem of the existence of $[L^0]^{-1}$ has not yet been solved rigorously However, if one leaves the requirement of mathematical rigor, it may be shown quite simply that z_μ^2 is well behaved. Let us consider the low-density limit of z_μ^2 when $z_\mu^1 = 0$. Since $\lim_{n\to 0}\vec{L}^1 = i\vec{p}_1/m$ as $n \to 0$, we have

$$z_\mu^2 \underset{n\to 0}{\approx} \{\psi_\mu\,\vec{k}\cdot\vec{p}_1/m,\ [L^0]^{-1}\vec{k}\cdot\vec{p}_1/m\ \mu^0\}. \tag{A29}$$

Now it is an elementary task to verify that this value of z_μ^2 is related in a simple way to the transport coefficients of the gas. In the case of the vorticity diffusion for example, one has

$$z_\mu^2 \underset{n\to 0}{\approx} \eta k^2/nm,$$

η being the shear viscosity of the gas. The shear viscosity of a gas is exactly known for the case of Maxwellian molecules; for other types of two-body interaction, the Enskog expansion[10,20] provides us with means to calculate η; η is well defined, except for very long-range potentials like the Coulomb one for $\nu = 3$. But this kind of interaction law has been tacitly excluded, since we have assumed that L^0 exists.

Hence we may assert, within the restriction due to our actual knowledge about the operator $[L^0]^{-1}$, that z_μ^2 is a nonzero and finite quantity near $n = 0$, the value $n = 0$ being obviously excluded. Thus the

initial assumption is proved.

APPENDIX B

In this appendix, we derive the expressions of the eigenvalues $z_\mu(\vec{k})$ and the eigenfunctions $\mu_j(\vec{k})$ near $\vec{k}=0$. The calculations are still elementary, and the main object of this appendix is to clarify the notion "near $k=0$." From the characteristic Eq. (4.5), it may be deduced that "near $k=0$" a root $z_d(\vec{k})$ exists, proportional to k^2, its expansion being given as

$$z_d(\mathrm{k}) = \frac{k_I^2 k_{II}^2 + \frac{1}{3}n^2(k_I^2 - k_{II}^2)^2}{2nk^2} - \frac{n^2}{180k^2}(k_I + k_{II})^2(k_I - k_{II})^4 + \left(\frac{1}{8n^3k^4}\right)[k_I^2 k_{II}^2 + \tfrac{1}{3}n^2(k_I^2 - k_{II}^2)^2]$$

$$\times [k_I^4 + k_{II}^4 - \tfrac{4}{3}n^2(k_I - k_{II})^2(k_I^2 + k_{II}^2 + 4k_I k_{II})] + O(k^6). \tag{B1}$$

In (B1) the terms of increasing order depend actually on k and on the rate k_I/k_{II} (or on the angle $\hat{k}$), and we are seeking the leading term near $k=0$, whatever the rate k_I/k_{II} will be, since we are interested in the whole domain of small values of $|\vec{k}|$.

The successive terms on the right-hand side of (B1) are of the type $P_m(k_I, k_{II})/k^{m-2-m'}$ (m' integer ≥ 0), where $P_m(k_I, k_{II})$ is a homogeneous polynomial in k_I and k_{II} of degree m. Hence each term is bounded as

$$P_m(k_I, k_{II})/k^{m-2-m'} \leq A_{m'} k^{m'+2},$$

$A_{m'}$ is a constant determined by the particular form of $P_m(k_I, k_{II})$. Moreover the leading term of this expansion, namely,

$$\frac{k_I^2 k_{II}^2 + \frac{1}{3}n^2(k_I^2 - k_{II}^2)^2}{2nk^2},$$

is bounded from below as

$$\frac{k_I^2 k_{II}^2 + \frac{1}{3}n^2(k_I^2 - k_{II}^2)^2}{2nk^2} \geq \frac{k^2}{2n}\sup(\tfrac{1}{4}, \tfrac{1}{3}n^2). \tag{B2}$$

Since this leading term has a lower bound proportional to k^2, and since the terms of higher order are bounded from above by $A_{m'} k^{m'+2}$ ($m' > 0$), we may assert that this term is truly the leading term of $z_d(\vec{k})$ near $k=0$, independent of the rate k_I/k_{II}. Thus, given a small number ξ, there exists a positive number k_0 such that for $k \leq k_0$ we have

$$(1-\xi)\frac{k_I^2 k_{II}^2 + \frac{1}{3}n^2(k_I^2 - k_{II}^2)^2}{2nk^2} \leq z_d(\vec{k}) \leq (1+\xi) \times \frac{k_I^2 k_{II}^2 + \frac{1}{3}n^2(k_I^2 - k_{II}^2)^2}{2nk^2}. \tag{B3}$$

Let us examine now the corresponding eigenfunction, namely $d_j(\vec{k})$. According to (4.6), the component $d_I(\vec{k})$ of this eigenfunction may be expanded near $k=0$ as

$$N_d d_I(k) = -k^2 \sin\hat{k}\cos\hat{k} + O(k^3), \tag{B4}$$

$\hat{k}$ being defined by $\mathrm{tg}\hat{k} = k_{II}/k_I$.

Near $\hat{k}=0$, $N_d d_I(\vec{k})$ is of order k^2, except when $\cos\hat{k}$ or $\sin\hat{k}$ is equal to zero; for these latter cases, $N_d d_I(\vec{k})$ is of order k^3. Thus the leading term in the expansion of $N_d d_I(\vec{k})$ is of order k^2, except when the angle $\hat{k}$ lies in small intervals around $0, \frac{1}{2}\pi, \pi$ and $\frac{3}{2}\pi$. The width of these intervals is determined by the equality of terms of order k^2 and k^3 in (B4), and thus is of order k. Consequently the contribution of this range of values of $\hat{k}$ to an integral over $\vec{k}$ is negligible in the limit $k \to 0$, and, in an integral over $\vec{k}$, $N_d d_I(\vec{k})$ may be replaced near $k=0$ by $k^2 \sin\hat{k}\cos\hat{k}$. Taking $N_d = -k/k_I$, one obtains more generally

$$d_j(\vec{k}) = k_{j+1}/k \quad \text{near } k=0.$$

This approximation is valuable in an integration over $\vec{k}$, but not for any value of $\hat{k}$.

The other quantities defining the spectrum are calculated near $k=0$ along very similar lines, giving rise to the results of the Table I.

APPENDIX C

In this Appendix it will be shown that the integrals defining $\Omega_{d,d}$ and $\Omega_{s^\pm,s^\mp}$ behave like $\ln\epsilon'$ when ϵ' goes to O_+.

From the definition (4.20) of $\Omega_{\mu,\mu'}$, and from the value of

$$\frac{d_I(\vec{k})d_{III}(-\vec{k}) - d_{II}(\vec{k})d_{IV}(-\vec{k})}{\epsilon' + z_d(\vec{k}) + z_d(-\vec{k})}\,\alpha_{k;d,d}$$

given in (4.25), the contribution of the small values of k to $\Omega_{d,d}$ reads

$$\Omega_{d,d}^{k<k_0} = -4n \lim_{\epsilon' \to 0_+} \int_{k<k_0} \frac{d\vec{k}}{(2\pi)^2}\left(\frac{1}{4}\frac{k_I^2 - k_{II}^2}{k^4} + O(k^2)\right) \times \left(\epsilon' + \frac{k_I^2 k_{II}^2 + \frac{1}{3}n^2(k_I^2 - k_{II}^2)^2}{nk^2} + O(k^4)\right)^{-1}. \tag{C1}$$

In order to define k_0, namely, the domain of small values of $\vec{k}$, let us consider a number k_0 such that, for $k < k_0$, the two factors occuring in the integrand in (C1) are bounded as

$$\frac{(k_I^2-k_{II}^2)^2}{4k^4}+E_1k^2\leq\tfrac{1}{4}\frac{k_I^2-k_{II}^2}{k^4}+O(k^2)\leq\tfrac{1}{4}\frac{k_I^2-k_{II}^2)^2}{k^4}+E_2k^2\,, \tag{C2}$$

$$\left[\epsilon'+\frac{1+\xi}{nk^2}\left(k_I^2k_{II}^2+\frac{n^2}{3}(k_I^2-k_{II}^2)^2\right)\right]^{-1}\leq\left[\epsilon'+\frac{1}{nk^2}\left(k_I^2k_{II}^2+\frac{n^2}{3}(k_I^2-k_{II}^2)^2\right)+O(k^4)\right]^{-1}$$

$$\leq\left[\epsilon'+\frac{1-\xi}{nk^2}\left(k_I^2k_{II}^2+\frac{n^2}{3}(k_I^2-k_{II}^2)^2\right)\right]^{-1}. \tag{C3}$$

E_1 and E_2 are some numbers which depend on the precise form of the term $O(k^2)$ occurring in (4.25) and ξ is a given small number. The important point is that k_0 depends on E_1, E_2, and ξ, but not on ϵ'. By means of the inequalities (C2) and (C3), one finds upper and lower bounds of $\Omega_{d,d}^{k<k_0}$ which are equal up to a constant independent of ϵ', and one may obtain the main part of $\Omega_{d,d}^{k<k_0}$ near $\epsilon'=0_+$, which is given by

$$\Omega_{d,d}^{k<k_0}\underset{\epsilon'\to0_+}{\approx}-4n\lim_{\epsilon'\to0_+}\int_{k<k_0}\frac{d\vec{k}}{(2\pi)^2}\frac{(k_I^2-k_{II}^2)^2}{k^4}$$

$$\times\left(\epsilon'+\frac{k_I^2k_{II}^2+\frac{1}{3}n^2(k_I^2-k_{II}^2)^2}{nk^2}\right)^{-1}. \tag{C4}$$

This integral may be carried out by using polar coordinates. After the integration over the angle, one obtains

$$\Omega_{d,d}^{k<k_0}\underset{\epsilon'\to0_+}{\approx}\frac{-8n}{\pi(1-\frac{4}{3}n^2}\int_0^{k_0}\frac{dk}{k}\left(\frac{(k^2/2n+2\epsilon')^{1/2}}{(\frac{2}{3}nk^2+2\epsilon')^{1/2}}-1\right). \tag{C5}$$

Although this last integral may be explicitly performed, we shall not give the exact result, but only its value in the limit $\epsilon'\to0_+$. We may notice that, by taking as a new variable of integration $\bar{k}=(\epsilon')^{1/2}k$, the quantity ϵ' lies only in the upper bound of the domain of integration, and it is an elementary task to find the behavior of the integral when ϵ' goes to 0_+. Thus

$$\Omega_{d,d}^{k<k_0}\underset{\epsilon'\to0_+}{\approx}\frac{4n^2}{\pi(\frac{1}{2}\sqrt{3}+n)}\ln\epsilon'. \tag{C6}$$

Following the same procedure, the other divergent contributions to Ω, namely $\Omega_{s^\pm,s^\mp}$, are calculated by evaluating the corresponding integrals in the domain $k<k_0'$, k_0' being a small number. One obtains

$$\Omega_{s^-,s^+}^{k<k_0'}+\Omega_{s^+,s^-}^{k<k_0'}\underset{\epsilon'\to0_+}{\approx}-8n\int_{k<k_0'}\frac{d\vec{k}}{(2\pi)^2}\frac{(k_I^2-k_{II}^2)^2}{k^4}$$

$$\times\left(\epsilon'+\frac{1}{4nk^2}\left\{k_I^4+k_{II}^4+\tfrac{2}{3}n^2(k_I-k_{II})^4\right\}\right)^{-1}. \tag{C7}$$

After a quite long but straightforward calculation, it may be shown that the right-hand side of (C7) behaves as $\ln\epsilon'$ near $\epsilon'=0_+$, giving

$$\Omega_{s^-,s^+}^{k<k_0'}+\Omega_{s^+,s^-}^{k<k_0'}\underset{\epsilon'\to0_+}{\approx}-\frac{16n^2\ln\epsilon'}{\pi(1-\frac{4}{3}n^2)}$$

$$\times\left(1-\frac{y_++y_-}{(y_+^2-1)^{1/2}+(y_-^2-1)^{1/2}}\right),$$

with

$$y_\pm=-\frac{2n^2\pm2(1/2n-\frac{1}{3}n)^{1/2}}{(1-\frac{4}{3}n^2)}.$$

APPENDIX D

In this appendix, it is proved that the integral (4.17) which defines Ω is convergent for the large values of k. Many calculations are elementary and only the general lines are given here.

At first we shall examine the behavior of $a(\vec{k})$ in the limit $k\to\infty$. Since

$$a(\vec{k})=\frac{\sin(k_{II}-k_I)}{k_{II}-k_I}=\frac{\sin[k\sqrt{2}\sin(\hat{k}-\frac{1}{4}\pi)]}{k\sqrt{2}\sin(\hat{k}-\frac{1}{4}\pi)}, \tag{D1}$$

where $\mathrm{tg}\hat{k}=k_{II}/k_I$, thus $a(\vec{k})$ goes to zero when k goes to infinity, except when $\hat{k}$ is nearly equal to $\frac{1}{4}\pi$ or $\frac{5}{4}\pi$. In order to define more precisely the range of values of $\hat{k}$ where, for a given value of k, $a(\vec{k})$ differs noticeably from zero, let us consider the inequality $|a(\vec{k})|>\xi$, ξ being a given small number. From (D1) this inequality may be replaced by the less restrictive ones

$$[k\sqrt{2}\,|\sin(\hat{k}-\tfrac{1}{4}\pi)|\,]^{-1}>\xi\,.$$

Accordingly, when k goes to infinity, $a(\vec{k})$ differs from zero only in a domain of value of $\hat{k}$ around $\frac{1}{4}\pi$ and $\frac{5}{4}\pi$ with a width of order k^{-1}. Thus the contribution of this range of value of k may be neglected in the domain of large values of K, and we shall consider in this domain $a(\vec{k})$ as being equal to zero with respect of a quantity of order unity. This remark is useful when applied to $\Lambda(1,\vec{k};\phi^0)$. According to the general definition of $\Lambda(1,\vec{k};\phi^0)$ one has

$$\sum_{j_1'}[\Lambda(1,\vec{k};\phi^0)]_{j_1}^{j_1'}\Phi_{j_1'}=-[\Phi_{p(j_1')}-\Phi_{j_1}]-a(\vec{k})$$

$$[\Phi_{p(j_1)+II} - \Phi_{j_1+II}] \,. \tag{D2}$$

In the domain of large values of k, $a(\vec{k})$ may be neglected; thus we have

$$\sum_{j_1'} [\Lambda(1, k\to\infty; \phi^0)]^{j_1'}_{j_1} \Phi_{j_1'} = -[\Phi_{p(j_1)} - \Phi_{j_1}] \,. \tag{D3}$$

Using this asymptotic value of $\Lambda(1, \vec{k}; \phi^0)$, it is not difficult to invert the operator $B_0(\epsilon', \vec{k}; 1, 2)$ in the limit $k\to\infty$, since the equivalent set of 16 linear equations is converted into four independent sets of four linear equations which are solved at once. Defining now the domain of large values of k by $k>\kappa$, κ being a given number, as large as wanted, we may evaluate the contribution of the domain $k>\kappa$ to Ω as

$$\Omega^{k>\kappa} = -8n \lim_{\epsilon'\to 0_+} \int_{k>\kappa} \frac{d\vec{k}}{(2\pi)^2} [a(\vec{k})]^2 \frac{2(\epsilon'+ik_I+ik_{II})(\epsilon'+ik_I+ik_{II}+4n)}{[(\epsilon'+2ik_I)(\epsilon'+2ik_{II})+4ni(k_I+k_{II})][\epsilon'+2n+i(k_I+k_{II})]} - \frac{1}{\epsilon'+2ik_I} - \frac{1}{\epsilon'+2ik_{II}} \Big) \,. \tag{D4}$$

In (D4) the integrand has been evaluated from (4.17) by using the asymptotic expression of the operator $[B_0^{-1}(\epsilon', \vec{k})]^{j_1', j_2'}_{j_1, j_2}$ deduced from (C3). Using the variables of integration $k_+ = \frac{1}{2}(k_I + k_{II})$, $k_- = \frac{1}{2}(k_I - k_{II})$, and carrying out in (D4) the integration over k_+, one obtains

$$\Omega^{k>\kappa} = \frac{-16n}{(2\pi)^2} \int_{4n}^{\kappa} \frac{dk_-}{7n^2-k_-^2} \frac{\sin^2(k_-)}{k_-^2} \Big\{ n^2 \Big[\pi - 2\arctan\Big(\frac{\gamma}{n}\Big)\Big] + (6n^2 - k_-^2 - k_-^2)\Big[\pi - \arctan\Big(\frac{\gamma-\sqrt{\Delta}}{4n}\Big) - \arctan\Big(\frac{\gamma+\sqrt{\Delta}}{4n}\Big)\Big]$$
$$+ \frac{n}{2\sqrt{\Delta}} (k_-^2 + 50n^2) \ln\Big(\frac{\kappa^2 - 2\gamma\sqrt{\Delta}}{\kappa^2 + 2\gamma\sqrt{\Delta}}\Big) \Big\} \,. \tag{D5}$$

To obtain (D5) we have supposed, as allowed, that $\kappa > 4n$, and have written $\gamma = (\kappa^2 - k_-^2)^{1/2}$, $\Delta = k_-^2 - 16n^2$.

The integral on the right-hand side of (D5) defines the contributions to Ω of the domain of large values of k. It cannot be performed explicitly; a simple inspection, however, shows that the whole integrand is well defined and continuous in the domain of integration, and thus that $\Omega^{k>\kappa}$ is not divergent in the limit $\epsilon' = 0_+$.

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