charge wave.

It has come to our attention that Freund and Levine have performed a related calculation.[10]

ACKNOWLEDGMENT

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Electromagnetic Penetration and Confinement of a Hot Dense Plasma

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For a special model, it is possible to compute the self-consistent confinement and penetration of an energetic plasma by an rf field of large amplitude for arbitrary values of the quantity $(v/c)\,(\omega_P/\omega)$, where v is a characteristic electron thermal speed, ω_P is the plasma frequency deep in the plasma, and ω ($<\omega_P$) is the rf frequency. Previous theories concerning nonlinear behavior of electromagnetic waves in plasmas have required this quantity to be small compared to unity.

I. INTRODUCTION

Previous theories of propagation of large-amplitude electromagnetic waves in plasmas[1-3] are based on assumptions that become invalid when the electromagnetic frequency is much less than the plasma frequency, $\omega \ll \omega_p$ (an overdense plasma). These assumptions are

$$\begin{aligned} &\text{(i)}\ \ vB/c \ll E\ , \quad \text{(Refs. 1–3)} \\ &\text{(ii)}\ \ w \gg (1/T)\ , \quad \text{(Ref. 1)} \end{aligned} \tag{1}$$

where v, B, and E are representative values of particle velocity, and magnetic and electric fields, and T is the characteristic time during which a representative particle samples the scale length of the fields. The second assumption is clearly necessary for the validity of the multiple time-scale method, commonly employed in theories of rf confinement.[3] Also, it is clearly necessary for replacing the Vlasov equation by local (moment) equations in describing the particle dynamics.[3]

In contrast to the present work in which large amplitude circularly polarized waves are confining a semi-infinite plasma, Gibbons and Hartle[4] have considered a case in which large amplitude linearly polarized waves are propagating in an infinite plasma. The $\bar{v}\times\bar{B}$ forces are correctly included there, as they are in the present paper.

Both (i) and (ii) above can be expressed approximately by the following inequality, valid when $\omega \ll \omega_p$:

$$(v/c)(\omega_p/\omega) \ll 1\ . \tag{2}$$

For plasmas of thermonuclear interest (kilovolt energies, densities greater than 10^{13} cm^{-3}), this condition becomes violated for electromagnetic frequencies less than 10^{10} sec^{-1}.

The purpose of the present paper is to solve the problem of self-consistent rf confinement and penetration of a Vlasov plasma under conditions in which inequality (2) is violated. We consider here a simple model problem subject to the following restrictions:

(a) The incident transverse electromagnetic field is circularly polarized.

(b) The electrons have no incident energy transverse to the rf wave vector.

(c) The ions are too heavy to respond to the rf field.

(d) The ions are cold, and therefore adjust themselves so as to exactly cancel out the (time-independent) electron space charge. [Because of the considered geometry, namely, circularly polarized waves propagating parallel to the density gradient, the space-charge potential proves to be time independent. Further investigations that include the ions (to be presented in a later paper) have shown that the present results are approximately correct provided $m_- \ll m_+$ and $T_- \gg T_+$, where m is the mass and T is the temperature.]

We are presently studying more general problems including the removal of some of these four restrictions.

The electromagnetic fields and the plasma distribution are solved for in the following. In Sec. II, we consider a monoenergetic electron-beam plasma. The results for the monoenergetic plasma are then extended in Sec. III to a truncated Maxwellian plasma where we obtain profiles of density, electric field, and magnetic field.

II. MONOENERGETIC PLASMA

A. Derivation of Nonlinear Equation for the Vector Potential

To illustrate the methods to be used, we first consider only monoenergetic electrons deep in the plasma. Suppose the electromagnetic waves are propagating along the z axis, and that all quantities of interest depend only on the z coordinate.

In terms of the vector potential, the fields are

$$\vec{E} = -(1/c)\,\partial_t \vec{A};\quad \vec{B} = \nabla\times\vec{A} \tag{3}$$

and there is no scalar potential Φ because of assumption (d). Then Maxwell's equation can be reduced to

$$[\partial_z^2 - (1/c^2)\,\partial_t^2]\vec{A} = -(4\pi/c)\,\vec{J}\ , \tag{4}$$

where we have used the Lorentz gauge

$$\nabla\cdot\vec{A} = -(1/c)\,\partial_t\Phi = 0;\ \text{hence } A_z = 0\ . \tag{5}$$

(The last condition follows from the assumption that there is no disturbance deep in the plasma at $z\to\infty$.)

The equation of motion of a particle of charge q, mass m, can be written as

$$\dot{\vec{v}} = (q/m)[\vec{E} + (1/c)\vec{v}\times\vec{B}] = (q/mc)(-\partial_t\vec{A} + \vec{v}\times\nabla\times\vec{A})$$
$$= (q/mc)(-\partial_t\vec{A} - \vec{v}\cdot\nabla\vec{A} + (\vec{v}\cdot\partial_z\vec{A})\hat{z})\ , \tag{6}$$

where the partial space and time derivatives are evaluated at the current position of the electron, and the dot either over a symbol or as a superscript represents the total time derivative.

The transverse component of Eq. (6) is

$$[\vec{v}_\perp + (q/mc)\vec{A}]^{\boldsymbol{\cdot}} = 0\ . \tag{7}$$

At the "beginning" of the trajectory, deep in the plasma where $z\to+\infty$, it is assumed that $\vec{v}_{\perp\infty}=0$, which corresponds to a plasma with only longitudinal temperature, and that there is no rf field there. Then Eq. (7) can be integrated, and reads

$$\vec{v}_\perp = -(q/mc)\vec{A}\ . \tag{8}$$

The longitudinal component of Eq. (6) is, using Eq. (8),

$$\dot{v}_z = (q/mc)\vec{v}\cdot\partial_z\vec{A} = -\tfrac{1}{2}(q/mc)^2\partial_z A^2\ . \tag{9}$$

We now suppose the rf field to be circularly polarized:

$$\vec{A}(z,t) = A_0(z)(\cos(-\omega t+\phi),\ \sin(-\omega t+\phi))\ . \tag{10}$$

In general, the phase ϕ could be allowed to depend on z. However, it can then be shown that consistent results are obtained only if $\phi=\text{const}$ (see Appendix A). Therefore, we shall henceforth take

$$\phi = \text{const}\ . \tag{11}$$

Since Eq. (10) implies that

$$|\vec{A}(z,t)| \equiv A = A_0(z) = \text{magnitude of } \vec{A}\ , \tag{12}$$

one sees that A depends only on z, so that

$$v_z\partial_z A^2 = (A^2)^{\boldsymbol{\cdot}}\ . \tag{13}$$

Hence, multiplication of Eq. (9) by v_z produces

$$(v_z^2 + (q/mc)^2A^2)^{\boldsymbol{\cdot}} = 0\ . \tag{14}$$

Integration of Eq. (14) shows that the total kinetic energy of each particle is constant. [The results obtained are also relativistically correct as long as one uses the increased (constant) mass]:

$$v_z^2 + (q/mc)^2A^2 = v_\infty^2\ , \tag{15}$$

where v_∞ is the longitudinal velocity of the electron at the "beginning" of the orbit, deep in the plasma ($z\to\infty$), and it has again been assumed that there is no rf field deep in the plasma. Equation (15) shows that

$$v_z = \pm[v_\infty^2 - (q/mc)^2A^2(z)]^{1/2}\ \begin{cases}\text{incident particles: } v_z<0\\ \text{reflected particles: } v_z>0\end{cases} \tag{16}$$

depends explicitly only on z, and not on t.

From Eq. (16), one sees that the turn-around point for particles incident from deep in the plasma is given by

$$A(0) = mc|v_\infty|/|q| \tag{17}$$

and we take $z=0$ at this point.

From Eqs. (8) and (15), one sees immediately that electrons carry away no net energy from the field region for this case of circular polarization and no initial transverse energy. From this, one can draw the following two conclusions:

(i) In order that the particles do carry energy away from the field region, in the case of circular polarization there must be initial transverse energy. Thus, nonlocal conductivity depends on the presence of transverse energy in the distribution function deep in the plasma.

(ii) Since the general solution of Eq. (7) is $\vec{v}_\perp = -(q/mc)\vec{A} + \vec{v}_{\perp\infty}$, it follows that a particle that has been reflected from the field region will be left with the same transverse velocity that it had upon entering the field region. Therefore, any energy gains must be strictly longitudinal. This statement holds for arbitrary polarization and the plasma distribution function.

Since particles are conserved on the incident beam, the continuity equation for this beam holds. Now regarding v_z to be the fluid velocity of this beam and n its density, one has

$$\dot{n} + n\,\partial_z v_z = \dot{n} + n(v_z \partial v_z)/v_z = \dot{n} + n\dot{v}_z/v_z = 0 \ .$$

Multiplication by v_z then gives $(nv_z)^{\cdot} = 0$, so that

$$n = -n_\infty v_\infty \left[v_\infty^2 - \left(\frac{q}{mc}\right)^2 A^2\right]^{-1/2}, \quad \text{with } v_\infty < 0 \qquad (18)$$

for the density on the incident beam, where n_∞ is the density on the incident beam at $z = \infty$.

Since particles are conserved, the flux on the reflected beam deep in the plasma must be the same as the incident flux. But the reflected-beam speed is the same as the incident-beam speed at $z = \infty$. Therefore the density on the reflected beam is also the same as the incident density at $z = \infty$. Therefore, application of the continuity equation to the reflected beam again gives exactly expression (18) for the density anywhere on the reflected beam. Therefore, the total density at any z must be

$$N = 2n = 2n_\infty[1 - (q/mcv_\infty)^2 A^2]^{-1/2}$$
$$= 2n_\infty[1 - A^2/A^2(0)]^{-1/2} \ , \qquad (19)$$

where we have noted Eq. (17). Although the density becomes infinite for this monoenergetic beam at the boundary, it is integrable there.

From the foregoing, it is clear that the particle fluxes on the incident and reflected beams exactly cancel at each point, so there can be no net longitudinal currents.

The transverse currents are obtained from Eqs. (19) and (8) as

$$\vec{J} = Nq\vec{v}_\perp = -\frac{\omega_p^2}{4\pi c}\frac{\vec{A}(z,t)}{[1 - A^2(z)/A^2(0)]^{1/2}} \ , \qquad (20)$$

where

$$\omega_p^2 \equiv 4\pi 2n_\infty q^2/m \qquad (21)$$

defines the plasma frequency deep in the plasma.

Equation (20) shows that no harmonics are generated in the current density. All the nonlinear effects are in the amplitude dependence for a circularly polarized wave.

Substitution of the current density (20) into Maxwell's equations (4) gives

$$\partial_z^2 A + \frac{\omega^2}{c^2}\{1 - (\omega_p^2/\omega^2)[1 - A^2/A^2(0)]^{-1/2}\}A = 0 \quad (22)$$

for the amplitude of the vector potential.

B. Solution

Let

$$z \equiv (c/\omega_p)\zeta, \qquad a(\zeta) \equiv A(\zeta)/A(0), \qquad \Omega = \omega/\omega_p \ . \quad (23)$$

Then Eq. (22) becomes

$$a'' + (\Omega^2 - (1 - a^2)^{-1/2})a = 0, \text{ with } a(0) = 1 \ , \qquad (24)$$

where prime represents differentiation with respect to ζ.

If the rf wave is to vanish deep in the plasma as has been assumed, then it is clear from Eq. (24) that Ω is restricted to

$$\Omega < 1 \quad \text{or} \quad \omega < \omega_p \ . \qquad (25)$$

Further, it can be seen from Eq. (19) that the plasma density is given by

$$N/N_\infty \equiv \eta = [1 - a^2(\zeta)]^{-1/2} \ , \qquad (26)$$

where

$$N_\infty = 2n_\infty \qquad (27)$$

is the total density deep in the plasma $z = \infty$.

The electric and magnetic fields are determined from Eq. (1) to be

$$\vec{E} = (\omega/c)A(0)a(\zeta)(-\sin(-\omega t + \phi),\ \cos(-\omega t + \phi)) \ ,$$
$$(28)$$
$$\vec{B} = (\omega_p/c)A(0)a'(\zeta)(-\sin(-\omega t + \phi),\ \cos(-\omega t + \phi)) \ .$$

Since $\vec{E}$ and $\vec{B}$ are colinear, it is clear that the Poynting vector $\sim \vec{E} \times \vec{B}$ vanishes and no energy flows into the plasma, consistent with the statement following Eq. (17).

Equation (24) can be integrated once, and then reads [with $a(0) \equiv 1$]

$$[a'(\zeta)]^2 - [a'(0)]^2 = -2[1 - a^2(\zeta)]^{1/2} - \Omega^2[a^2(\zeta) - 1] \ . \qquad (29)$$

If the fields are to vanish deep within the plasma, then a and $a' \to 0$ as $\zeta \to \infty$ according to Eq. (28), and Eq. (29) implies a condition on $a'(0)$ at the boundary, namely,

$$[a'(0)]^2 = 2 - \Omega^2 \ . \qquad (30)$$

What is the meaning of this condition? From

Eqs. (17), (28), and (30), we calculate the radiation pressure at the boundary to be

$$\frac{E^2(0)+B^2(0)}{8\pi}=\left(\frac{\omega^2}{8\pi c^2}+\frac{\omega_p^2}{8\pi c^2}\,[a'(0)]^2\right)A^2(0)$$

$$=\left[\frac{\omega^2}{8\pi c^2}+\frac{\omega_p^2}{8\pi c^2}\left(2-\frac{\omega^2}{\omega_p^2}\right)\right]\frac{m^2c^2v_\infty^2}{q^2}$$

$$=\frac{\omega_p^2}{4\pi c^2}\,\frac{m^2c^2v_\infty^2}{q^2}=Nmv_\infty^2\ . \qquad (31)$$

This just states that the radiation pressure at the boundary is balanced by the kinetic pressure deep within the plasma. Thus, Eq. (30) is the normalized expression of pressure balance.

Substitution of Eq. (30) into Eq. (29) gives

$$a'(\zeta)=-\sqrt{2}\,[1-\tfrac{1}{2}\Omega^2a^2-(1-a^2)^{1/2}]^{1/2}\ , \qquad (32)$$

where the decaying solution has been chosen. This equation for the amplitude of the vector potential is valid for arbitrary rf intensity at the plasma boundary, for arbitrary (relativistic) electron energies, and for all frequencies such that $\omega<\omega_p$. It may be integrated numerically for given values of Ω (analytically for $\Omega=0$) so as to obtain the function $\zeta(a)$.

The normalized vector potential $a(\zeta)$ is plotted in Fig. 1. According to Eq. (28), this also gives the electric-field variation. Also shown are the magnetic-field variation $a'(\zeta)$ and the density variation $(1-a^2)^{-1/2}\equiv\eta$.

It is interesting that the spatial variations of the fields and the density are rather insensitive to the value of ω over the wide range $\omega\ll\omega_p$–$\omega=0.999\,\omega_p$. In the low-frequency case, the field drops to about one-third of its surface value in a distance of c/ω_p. When $\omega=0.999\,\omega_p$, the field drops to one-third of its surface value in a distance of only about $3.5\,c/\omega_p$, whereas, according to linear theory, it should be about $20\,c/\omega_p$. Thus, the nonlinear behavior of the plasma apparently leads to better shielding out (smaller effective skin depth) of the radiation than the linear theory would predict.

C. Boundary Conditions

Suppose that the radiation field on the left of the boundary consists of a circularly polarized incident wave and an unknown reflected wave:

$$\vec{A}(z<0)\equiv\vec{A}_<=A_i(\cos(-\omega t+k_0z),\ \sin(-\omega t+k_0z))+\vec{A}_r(z,t);$$

$$k_0\equiv(\omega/c)\ . \qquad (33)$$

Since the vector potential in vacuum obeys the undriven wave equation, the reflected wave must have the form

$$\vec{A}_r(z,t)=\vec{A}_r(z+ct)\ . \qquad (34)$$

There is no loss of generality in supposing the phase of the incident wave to be known, as we have done in Eq. (33). All phases may be measured with respect to the phase of this wave at $z=0$.

The electric and magnetic fields on the vacuum side are then given by

$$\vec{E}_<=-c^{-1}\partial_t\vec{A}_<=(\omega/c)$$

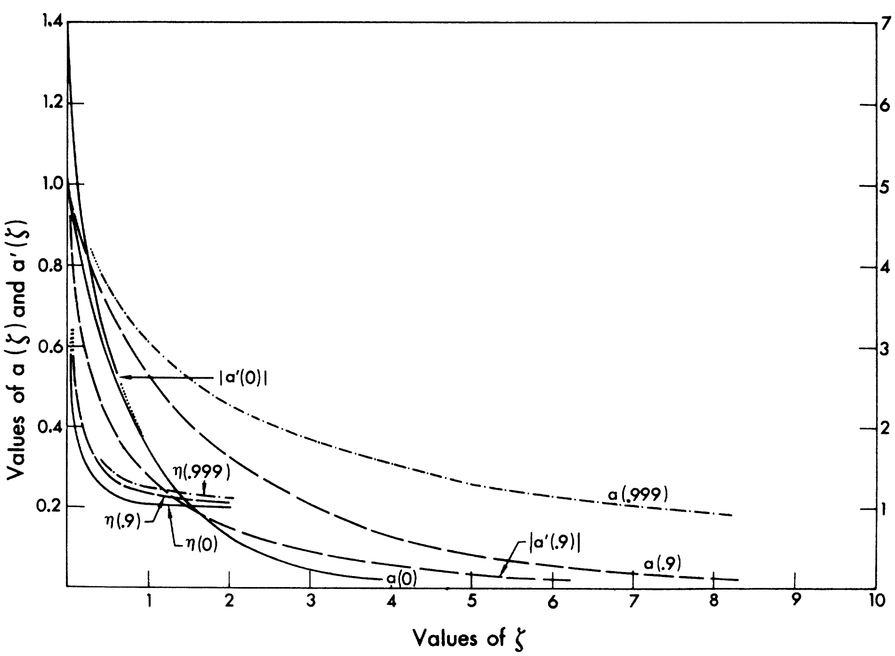

FIG. 1. Plots of $E\propto a$, $B\propto a'$, and $\eta=(1-a^2)^{-1/2}$ vs $\zeta=\omega_pz/c$ for a monoenergetic plasma. Numbers in parentheses give values of $\Omega=\omega/\omega_p$. The normalized density is η. All curves indicate only relative variations of physical quantities.

$$\times A_i(-\sin(-\omega t+k_0z),\ \cos(-\omega t+k_0z)) - c^{-1}\,\partial_t\vec{A}_r\,,$$
$$\vec{B}_< = \nabla\times\vec{A}_< = (\omega/c) \times A_i(-\cos(-\omega t+k_0z), -\sin(-\omega t+k_0z)) + (-\partial_z A_{ry},\ \partial_z A_{rx})\ . \tag{35}$$

At the boundary $z=0$, $\vec{E}$ must be continuous. Also the plasma density there is not singular enough to produce surface currents, so $\vec{B}$ is also continuous at the boundary. Matching the vector potential (electric field) at $z=0$ gives

$$\vec{A}_r(0) = -A_i[\cos\omega t, -\sin\omega t] + A(0)[\cos(\omega t-\phi), -\sin(\omega t-\phi)] \tag{36}$$

and, since $A_r(z,t)$ can depend only on the combination $(\omega t+k_0z)$,

$$\vec{A}_r(z,t) = -A_i[\cos(\omega t+k_0z), -\sin(\omega t+k_0z)] + A(0)[\cos(\omega t+k_0z-\phi), -\sin(\omega t+k_0z-\phi)]\ . \tag{37}$$

Matching magnetic fields at $z=0$ yields

$$-(\omega/c)A_i\cos\omega t - \partial_z A_{ry} = (\omega_p/c)A(0)a'(0)\sin(\omega t-\phi)\ ,$$
$$(\omega/c)A_i\sin\omega t + \partial_z A_{rx} = (\omega_p/c)A(0)a'(0)\cos(\omega t-\phi)\ . \tag{38}$$

Substituting Eq. (37) into (38) yields the two conditions

$$-2\omega A_i\cos\omega t = A(0)[-\omega\cos(\omega t-\phi)+\omega_p a'(0)\sin(\omega t-\phi)]\ ,$$
$$2\omega A_i\sin\omega t = A(0)[\omega\sin(\omega t-\phi)+\omega_p a'(0)\cos(\omega t-\phi)]\ . \tag{39}$$

From either of Eqs. (39) one finds

$$\tan\phi = (\omega_p/\omega)a'(0)\ , \quad A(0) = 2A_i\cos\phi\ . \tag{40}$$

For the monoenergetic-beam plasma one obtains from Eqs. (30) and (17)

$$\tan\phi = -((2\omega_p^2/\omega^2)-1)^{1/2}\ ,$$
or
$$\cos\phi = (1/\sqrt{2})\,\omega/\omega_p\ , \tag{41}$$

and

$$A_i = \frac{mc|v_\infty|}{|q|\sqrt{2}}\,\frac{\omega_p}{\omega}\ . \tag{42}$$

Knowing A_i and ϕ one can, of course, also determine the reflected fields from Eq. (37).

The actual incident electric and magnetic fields required for equilibrium are found from Eqs. (35) and (42) to have the amplitudes

$$|\vec{E}_i| = \frac{m|v_\infty|\omega_p}{q\sqrt{2}} = |\vec{B}_i|,\ \text{independent of } \omega. \tag{43}$$

The amplitudes of the total $\vec{E}$ and $\vec{B}$ fields at the plasma boundary are determined from Eqs. (17), (28), and (30) to be

$$|\vec{E}(0)| = \frac{m|v_\infty|\omega}{|q|}\ ;\quad |\vec{B}(0)| = \frac{m|v_\infty|\omega_p}{|q|}\left(2-\frac{\omega^2}{\omega_p^2}\right)^{1/2}\ . \tag{44}$$

One sees that when $\omega \ll \omega_p$ the electric field is much smaller than the magnetic field in the plasma. Then Eq. (31) indicates essentially a balance between kinetic and magnetic pressure, in contrast to the usual rf confinement pictures.

Furthermore, it is not difficult to show that the reflected vector potential has the form

$$\vec{A}_r = A_i[\cos(\omega t+k_0z-2\phi), -\sin(\omega t+k_0z-2\phi)]\ . \tag{45}$$

III. EXTENSION TO LONGITUDINAL TEMPERATURE

A. Equation Governing the Vector Potential

The extension of the previous results from the case of an incident monoenergetic electron beam to an incident distribution of beam energies is perfectly straightforward. Let $u=|v_{z\infty}|$ be the *magnitude* of the incident velocities deep within the plasma, and dn be the incident density for particles in the velocity range (du) around the velocity u. The total density is the sum of the densitites of each incremental beam du, and one has, from Eq. (19)

$$N(z) = 2\int_{(qA/mc)}^{\infty}\frac{(dn/du)du}{[1-(qA/mcu)^2]^{1/2}} = 2\int_{(qA/mc)}^{\infty}\frac{f(u)du}{[1-(qA/mcu)^2]^{1/2}}\ , \tag{46}$$

for the density at any point z, where $f(u)=dn/du$ is the velocity distribution of the incident particles. The significance of the lower limit of integration is that only particles with high enough incident energies can penetrate the vector potential to reach the point z.

If we define a normalized velocity distribution by

$$F(u) = f(u)\Big/\int_0^\infty f(u)du,$$

where

$$N_{\rm inc} = \int_0^\infty f(u)du\ , \tag{47}$$

then Eq. (46) becomes

$$N(z) = N_\infty\int_{(qA/mc)}^{\infty}\frac{F(u)du}{[1-(qA/mcu)^2]^{1/2}}\ , \tag{48}$$

where

$$N_\infty = 2N_{\text{inc}} \tag{49}$$

is the total density at $z=\infty$, since the density on each incremental reflected beam equals the density on that same incident beam at each point.

From Eq. (8), the current density at point z is

$$\vec{J} = \int dn(z) q\vec{v}_\perp = -(q^2\vec{A}/mc)\int dn(z)$$

$$= -N(z)(q^2\vec{A}/mc)\ , \tag{50}$$

where $dn(z)$ is the total density at the point z (incident + reflected) due to the incremental beam du. Substitution of Eq. (48) into (50) produces

$$\vec{J} = -\frac{\omega_p^2}{4\pi c}\vec{A}\int_{(qA/mc)}^{\infty}\frac{F(u)du}{[1-(qA/mcu)^2]^{1/2}}\ , \tag{51}$$

where $\omega_p^2 = 4\pi N_\infty q^2/m$. Again, no harmonics occur in the current density. Although longitudinal thermal effects have been exactly included here, the transverse current density nevertheless depends only on the local value of $\vec{A}$ (or $\vec{E}$). The are no nonlocal effects in the conductivity.

Expressions (48) and (51) are valid for arbitrary incident distribution functions. Equation (51) is to be substituted into Maxwell's equations (4). Again, the arguments outlined in Appendix A can be used to show that the phase ϕ, in $\vec{A} = A(\cos(-\omega t+\phi), \sin(-\omega t+\phi))$, is independent of z. Then Maxwell's equation (4) becomes

$$\partial_z^2 A + \frac{\omega^2}{c^2}A = \frac{\omega_p^2}{c^2}A\int_{(qA/mc)}^{\infty}\frac{F(u)du}{[1-(qA/mcu)^2]^{1/2}}\ , \tag{52}$$

which involves again only the magnitude of the vector potential $A(z)$.

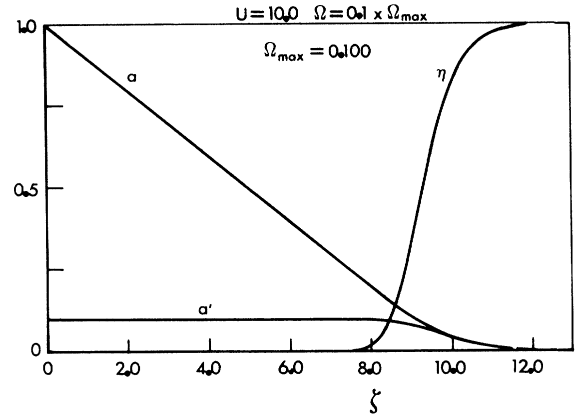

FIG. 2. Relative values of electric field (a), magnetic field (a'), and density η vs $\zeta = \omega_p/cz$ for a truncated Maxwellian plasma. The parameters on the curves are $\Omega = \omega/\omega_p$ and $U = u_M/u_T$.

To make further progress, the distribution function of the incident particles must be specified. We shall choose a Maxwellian, truncated at a maximum velocity u_m for reasons which will shortly become apparent.

$$F(u) = \frac{2}{u_T(\pi)^{1/2}\operatorname{erf}(u_m/u_T)}\, e^{-(u/u_T)^2};\quad 0 \le u \le u_m\ . \tag{53}$$

Here, u_T represents a thermal speed, and erf( ) denotes the error function

$$\frac{2}{\sqrt{\pi}}\int_0^{(\ )} e^{-\tau^2} d\tau\ .$$

With this choice of $F(u)$, the integral that appears in Eqs. (48), (51), and (52) becomes

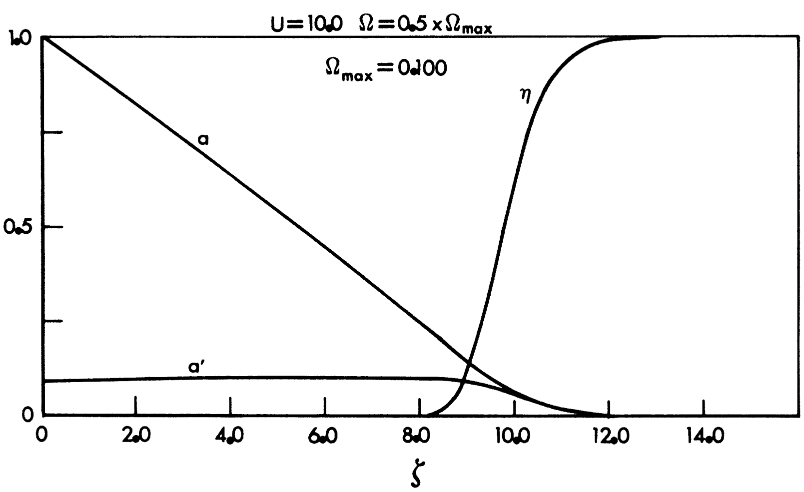

FIG. 3. Relative values of electric field (a), magnetic field (a'), and density η vs $\zeta = \omega_p/cz$ for a truncated Maxwellian plasma. The parameters on the curves are $\Omega = \omega/\omega_p$ and $U = u_M/u_T$.

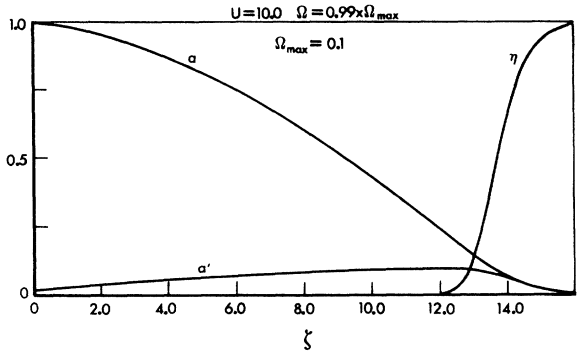

FIG. 4. Relative values of electric field (a), magnetic field (a'), and density η vs $\zeta = \omega_p/cz$ for a truncated Maxwellian plasma. The parameters on the curves are $\Omega = \omega/\omega_p$ and $U = u_M/u_T$.

$$\int_{(qA/mc)}^{\infty} \frac{F(u)du}{(1-(qA/mcu)^2)^{1/2}} = e^{-a^2U^2}\,\frac{\operatorname{erf}(U(1-a^2)^{1/2})}{\operatorname{erf}(U)}\ , \tag{54}$$

where

$$a \equiv |q| A/mcu_m \quad \text{and } U \equiv u_m/u_T\ . \tag{55}$$

It is important to notice that one must have $a \leq 1$ in any region that can be reached by the particles [see Eq. (16)], i.e., anywhere in the plasma.

One now substitutes (54) into (52), multiplies through $|q|/mcu_m$ to change A to a, and works with a normalized distance variable $\zeta \equiv (\omega_p/c)z$. Then (52) becomes with $\Omega \equiv \omega/\omega_p$

$$a'' + \Omega^2 a = a e^{-a^2U^2}\operatorname{erf}(U(1-a^2)^{1/2})/\operatorname{erf}(U)\ . \tag{56}$$

This is the self-consistent equation that governs the vector potential within the plasma.

B. Solutions

First, it is clear that if $a \to 0$ as $\zeta \to \infty$, the equation becomes $a'' = (1-\Omega^2)a$, so that the rf field will not penetrate deep into the plasma provided $\Omega < 1$. We shall assume $\Omega < 1$, or $\omega < \omega_p$.

Equation (56) can be integrated once, starting at the plasma boundary, $\zeta = 0$, where $A(0) = mcu_m/|q|$, so that $a(0) = 1$. The result is

$$[a'(\zeta)]^2 = [a'(0)]^2 - \int_{a^2}^{1} e^{-xU^2}\,\frac{\operatorname{erf}(U(1-x)^{1/2})}{\operatorname{erf}(U)}\,dx + (1-a^2)\Omega^2\ . \tag{57}$$

The expressions for the fields given by Eq. (28) remain valid provided we interpret $A(0) = mcu_m/|q|$. In order that the electric and magnetic fields vanish deep within the plasma $\zeta \to \infty$, one must have $a \to 0$ and $a' \to 0$ there. In this limit, Eq. (57) reduces to

$$[a'(0)]^2 = I - \Omega^2\ , \tag{58}$$

where

$$I = \int_0^1 \frac{e^{-xU^2}\operatorname{erf}(U(1-x)^{1/2})}{\operatorname{erf}(U)}\,dx\ . \tag{59}$$

As before, for the monoenergetic beam, Eq. (58) is a normalized statement of pressure balance. To see this, multiply by $\omega_p^2 A^2(0)/c^2$ and use the expression for the fields given by Eq. (28). The result is

$$[E^2(0) + B^2(0)]/8\pi = (\tfrac{1}{2} m N_\infty u_m^2) I\ . \tag{60}$$

The right-hand side of Eq. (60) can be recognized as the plasma pressure by noting the identity

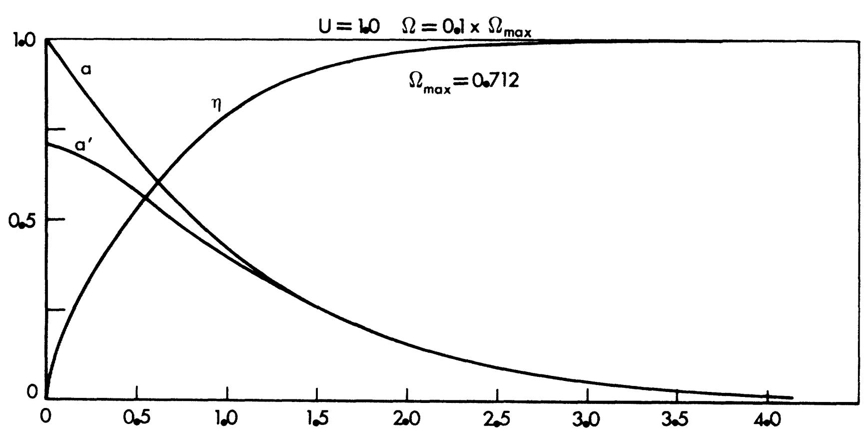

FIG. 5. Relative values of electric field (a), magnetic field (a'), and density η vs $\zeta = \omega_p/cz$ for a truncated Maxwellian plasma. The parameters on the curves are $\Omega = \omega/\omega_p$ and $U = u_M/u_T$.

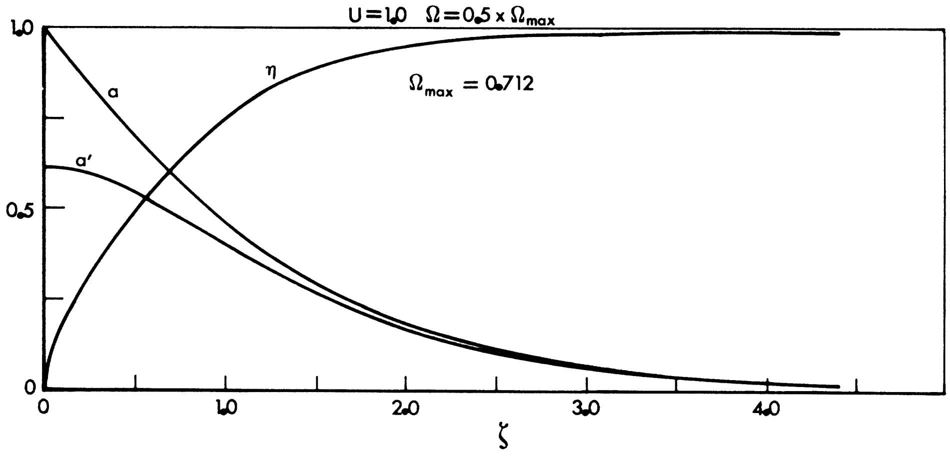

FIG. 6. Relative values of electric field (a), magnetic field (a'), and density η vs $\zeta = \omega_p/cz$ for a truncated Maxwellian plasma. The parameters on the curves are $\Omega = \omega/\omega_p$ and $U = u_M/u_T$.

$$\frac{4}{\sqrt{\pi}}\int_0^U x^2 e^{-x^2} dx = \mathrm{erf}(U)(I) \tag{61}$$

and by recognizing that the pressure is proportional to $\int u^2 f(u)du$.

Since I decreases with increasing U, it is clear from Eq. (58) that pressure balance cannot hold for given $\Omega = \omega/\omega_p$ if u_m is too large. The physical meaning of this seems to be as follows. According to Eq. (16), it is the presence of the vector potential itself that is directly necessary to stop the fastest particles. If the vector potential is indeed made large enough to stop particles of maximum initial velocity u_m, corresponding to the condition $a(0) = 1$, then it can happen that the resulting electromagnetic fields become too large to maintain pressure balance with the average thermal particle, as indicated by Eq. (58). Thus, if u_m is too large, the steady-state situation can no longer prevail, and the fields must push the plasma towards increasing values of z.

Substitution of Eq. (58) into (57) produces the following differential equation for the vector potential

$$(a')^2 = \int_0^{a^2} e^{-xU^2} \frac{\mathrm{erf}\left(U(1-x)^{1/2}\right)}{\mathrm{erf}(U)} dx - \Omega^2 a^2 . \tag{62}$$

From this equation, we numerically find solutions such that $a'(\zeta) < 0$.

In Figs. 2–10, plots are shown of the numerically obtained profile $a(\zeta)$ (proportional to electric field), $|a'(\zeta)|$ obtained from the square root of (62) (proportional to the magnetic field), and $\eta(\zeta) = N(\zeta)/N_\infty$, the normalized density profile obtained from Eqs. (48) and (54). According to the foregoing discussion, the plots are restricted to values of Ω and U such that the right-hand side of (58) is positive. The maximum allowable value of Ω is plotted vs U in Fig. 11.

The numerical results show that the fields always damp out in the plasma within a few (c/ω_p)'s. This is even true of the case $U = 10$ if one notes that the effective plasma boundary for this case occurs not at $\zeta = 0$ but around $\zeta = 10$. The position $\zeta = 0$ merely

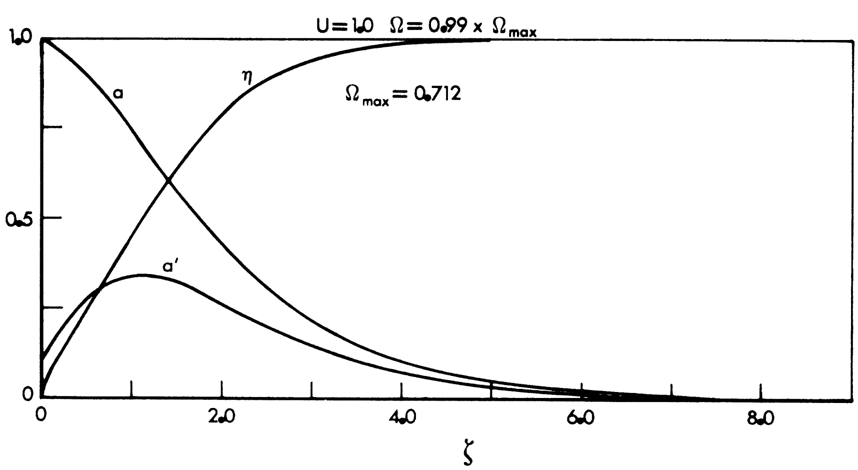

FIG. 7. Relative values of electric field (a), magnetic field (a'), and density η vs $\zeta = \omega_p/cz$ for a truncated Maxwellian plasma. The parameters on the curves are $\Omega = \omega/\omega_p$ and $U = u_M/u_T$.

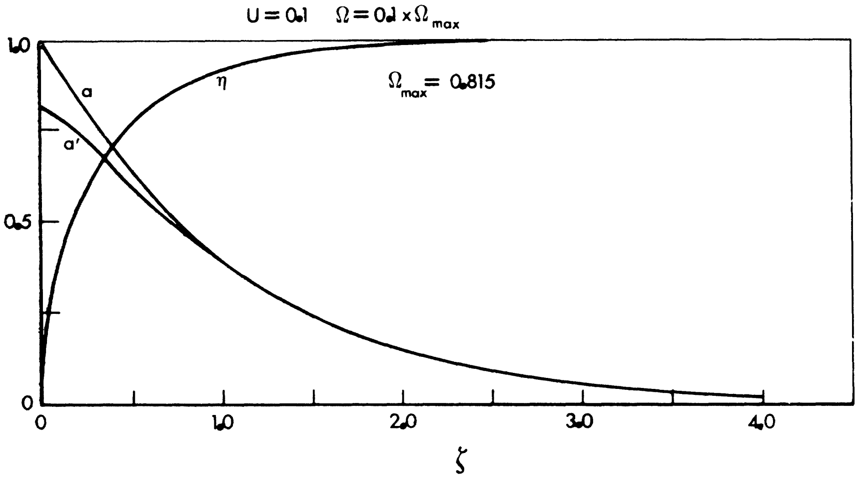

FIG. 8. Relative values of electric field (a), magnetic field (a'), and density η vs $\zeta = \omega_p/cz$ for a truncated Maxwellian plasma. The parameters on the curves are $\Omega = \omega/\omega_p$ and $U = u_M/u_T$.

represents the place where the highest-energy particles are stopped, and these have a very low density for $U = 10$.

C. Boundary Conditions

The procedure of matching the fields at the boundary is identical to that used in Sec. IIC, utilizing Eqs. (40) and (58).

For the phase shift, one finds

$$\tan\phi = -(I/\Omega^2 - 1)^{1/2} \approx -\left(\frac{\omega_p^2 u_T^2}{\omega^2 u_M^2} - 1\right)^{1/2} \quad \text{for } u_M \gg u_T, \tag{63}$$

where, obviously, $(\omega_p u_T/\omega u_M) > 1$ for stable solutions, or

$$\cos\phi = \frac{\Omega}{\sqrt{I}} \approx \frac{\omega u_M}{\omega_p u_T} \quad \text{for } u_M \gg u_T\ . \tag{64}$$

One finds, with $A(0) = mcu_M/|q|$, and from Eqs. (40) and (64) that

$$A_i = \frac{A(0)\sqrt{I}}{2\Omega} = \frac{1}{2}\frac{mcu_T}{|q|}\frac{\omega\rho}{\omega} \quad \text{for } u_M \gg u_T\ . \tag{65}$$

For the reflected vector potential, one finds from Eqs. (37) and (40)

$$\vec{A}_r = A_i\,[\cos(k_0 z + \omega t - 2\phi),\ -\sin(k_0 z + \omega t - 2\phi)], \tag{66}$$

and for the total vector potential one has

$$\vec{A}(z<0) = 2A_i \cos(k_0 z - \phi) \times [\cos(\omega t - \phi),\ -\sin(\omega t - \phi)]\ . \tag{67}$$

Thus, if the position of the plasma boundary can be measured, perhaps with movable metal probes, the phase ϕ can be found by measurement of the position of the nearest node of $\vec{A}(z<0)$ or $\vec{E}(z<0)$.

The amplitudes of the incident electric and mag-

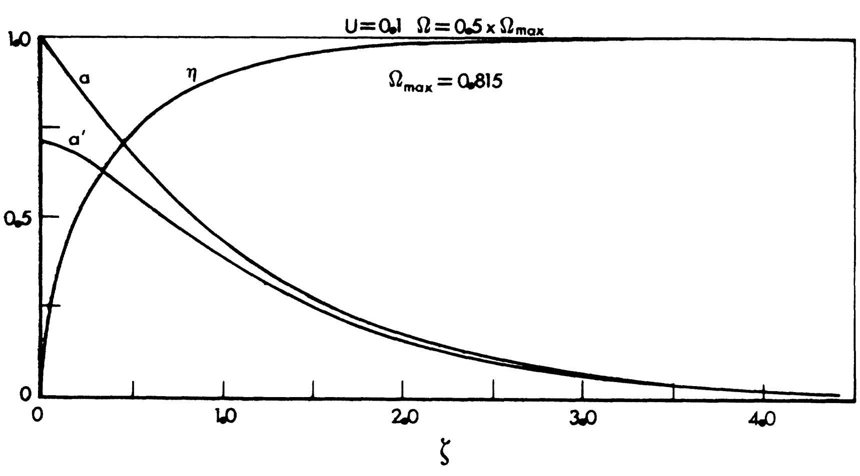

FIG. 9. Relative values of electric field (a), magnetic field (a'), and density η vs $\zeta = \omega_p/cz$ for a truncated Maxwellian plasma. The parameters on the curves are $\Omega = \omega/\omega_p$ and $U = u_M/u_T$.

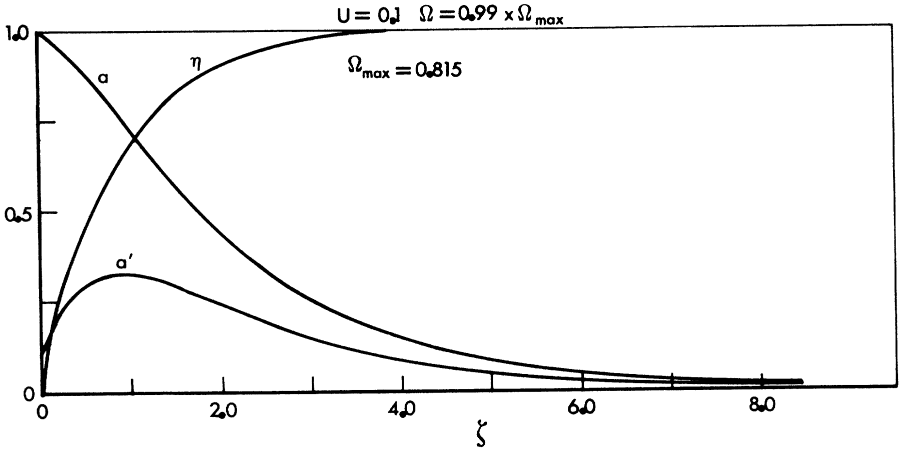

FIG. 10. Relative values of electric field (a), magnetic field (a'), and density η vs $\zeta = \omega_p/cz$ for a truncated Maxwellian plasma. The parameters on the curves are $\Omega = \omega/\omega_p$ and $U = u_M/u_T$.

netic fields are found from Eq. (64) to be

$$|E_i| = |B_i| = \frac{\omega}{c}\,\frac{A(0)\sqrt{(I)}}{2\Omega} \approx \frac{1}{2}\,\frac{mu_T\omega_p}{|q|} \quad \text{for } u_M \gg u_T \,. \tag{68}$$

Note that the incident intensity required for a steady state is independent of frequency.

The amplitudes of the total electric and magnetic fields at the boundary can be obtained from Eqs. (28) and (67) as

$$|E(0)| = \omega\,\frac{mu_M}{|q|}\,,$$

$$|B(0)| = E(0)\left(\frac{I-1}{\Omega^2}\right)^{1/2} \approx E(0)\left(\frac{\omega_p^2 u_T^2}{\omega^2 u_M^2} - 1\right)^{1/2} \quad \text{for } u_M \gg u_T \,. \tag{69}$$

Comparison of Eq. (69) with Eq. (44) shows the following. For a monoenergetic beam, one has $|\vec{E}| \ll |\vec{B}|$ at the plasma boundary when $\omega \ll \omega_p$. However, for an incident Maxwellian velocity distribution, one can have any ratio between $|\vec{E}|$ and $|\vec{B}|$ at the boundary, even when $\omega \ll \omega_p$. The ratio of the fields depends on the ratio u_T/u_m.

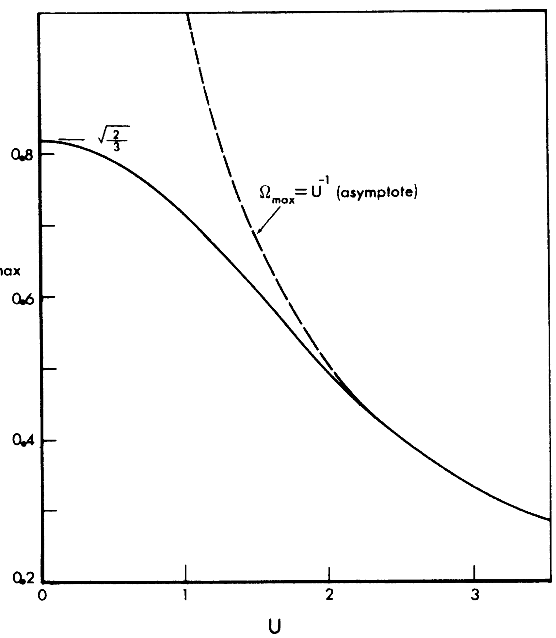

FIG. 11. The maximum value of Ω vs U:

$$\Omega_{max}^2 = I = \int_0^1 e^{-xU^2}\,\frac{\mathrm{Erf}(U(1-x)^{1/2})}{\mathrm{Erf}(U)}\,dx \,.$$

No stable solutions exist for $\Omega > \Omega_{max}$ because pressure balance fails.

IV. SUMMARY AND DISCUSSION

The standard treatment of rf confinement of plasmas requires the quantity $(v/c)\,(\omega_p/\omega)$ to be very small compared to unity. We have shown that, at least for circular polarization, rf confinement of plasmas should be possible even when this quantity is very large compared to unity. This regime is of thermonuclear interest.

We have computed electromagnetic field and plasma density profiles for the case of an electron plasma with no transverse temperature. Our method of computing the currents by means of the initial or incident distribution function seems to be essentially the same as Levin's.[1] We note from the numerical results that the fields generally decay significantly from their surface values within a few (c/ω_p)'s of the edge of the bulk plasma.

Also, we note that one can obtain some information about the plasma interior from measurements made only outside the plasma. First, one measures the phase shift ϕ which then gives the product $(I)^{1/2}/\Omega$ from Eq. (64). Knowing this product, one then measures the incident intensity to find $A(0)$ or u_m from Eq. (65). Knowing u_m, one then returns to the product $(I)^{1/2}/\Omega = \omega_p/\omega \times$ [known function of (u_m/u_T)] to obtain a relation between N and u_T. Thus, the density and temperature (of the full Maxwellian) deep in the plasma must correspond to some point on the curve of N_∞ vs u_T.

Finally, we compute the incident rf power density that is required to confine a plasma that begins to be of thermonuclear interest. For the density, we choose $n = 10^{13}$ cm^{-3} and for the temperature, $T = 10$ keV $\approx 10^8$ K. The electron thermal speed then becomes $u_T \approx 6\times 10^9$ cm/sec, and the electron plasma frequency proves to be $\omega_p \approx 2\times 10^{11}$ sec^{-1}. Assuming that most of the tail of the Maxwellian distribution is confined ($u_M \gg u_T$), Eq. (68) then gives the magnitude of the incident electric field, namely, $|E_i| \approx \frac{1}{2} m u_T \omega_p/|q| \approx 1000$ statvolt/cm $= 3\times 10^5$ V/cm. The Poynting vector gives the incident power density as $S = (c/4\pi)E_i^2 \approx 25\times 10^{14}$ erg/cm^2/sec $= 250$ MW/cm^2. (We note that this calculation applies only to a hot-electron plasma, the cold heavy ions being confined mainly by the electron space charge.)

ACKNOWLEDGMENTS

Mrs. Fran Gasche and Steve Hansen of the Boeing Scientific Research Laboratories computer facility obtained the numerical solutions of the differential equations.

APPENDIX A: PROOF THAT THE PHASE IS SPACE INDEPENDENT

The arguments that led to Eq. (22) can easily be generalized to include a z-dependent phase. Instead of (22) one finds

$$\partial_z^2 A - A(\partial_z \phi)^2 + \frac{\omega^2}{c^2} A = \frac{\omega_p^2}{c^2} \frac{A}{[1 - A^2/A^2(0)]^{1/2}} , \quad \text{(A1)}$$

and, simultaneously,

$$2(\partial_z A)(\partial_z \phi) + A\partial_z^2 \phi = 0. \quad \text{(A2)}$$

Equation (A2) implies that $\partial_z (A^2 \partial_z \phi) = 0$, or

$$A^2 \partial_z \phi = \text{const} \qquad \text{independent of } z. \quad \text{(A3)}$$

But, since A is assumed to vanish deep within the plasma ($z \to \infty$), the constant must be zero. Hence

$$\partial_z \phi = 0 \qquad \text{so } \phi \text{ is independent of } z. \quad \text{(A4)}$$

Thus (A1) reduces to Eq. (22). The fact that ϕ is constant is important in this case since this is generally not true. For example, if one considers the classical skin problem for a conductor one finds a nonconstant phase.

APPENDIX B: DISTRIBUTION FUNCTION

Although not utilized specifically here, it is frequently necessary to have available an expression for the distribution function in order to compute such phenomena as wave stability.

The Vlasov equation states that the velocity distribution function $F(\vec{v})$ is constant along the orbits. For a plasma with a longitudinal temperature and zero transverse temperature, the incident distribution function deep in the plasma can be written

$$F_\infty(\vec{u}) = f_\infty(u_z)\,\delta(\vec{u}_\perp), \quad \text{(B1)}$$

where $\vec{u}$ is the velocity vector deep in the plasma, and we can, for example, set $f_\infty(u_z)$ equal to a truncated Maxwellian. Here $\delta(\vec{u}_\perp)$ is the two-dimensional Dirac δ function.

The velocity distribution at any other point, where the velocity is denoted by $\vec{v}$, can then immediately be written down as

$$F(\vec{v}) = F_\infty(\vec{u}) = f_\infty(u_z)\,\delta(\vec{u}_\perp) \ , \quad \text{(B2)}$$

where $\vec{v}$ is related to $\vec{u}$ via the orbit equations. Thus, we can immediately write

$$F(\vec{v}) = f_\infty \{[v_z^2 + (qA/mc)^2]^{1/2}\}\, \delta\,[\vec{v}_\perp + (q\vec{A}/mc)] \ . \quad \text{(B3)}$$

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