Statistical Properties of Laser Radiation During a Transient Buildup*

F. T. Arecchi and V. Degiorgio
Centro Informazioni Studi Esperienze, Milano, Italy
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The transient statistical properties of a single-mode laser field are investigated both experimentally and theoretically. The physical system is a Q-switched He-Ne gas laser oscillating on the 6328-Å transition. The measurements are performed by photon-counting techniques, using the so-called linear method which enables resolving times to be in the nsec region. The effects of the photomultiplier and of the attenuators on the statistics are taken into account. The measurements are interpreted in terms of the available laser theories as well as in terms of a simple phenomenological model which attributes the statistical spread to the initial photon distribution, neglecting the stochastic forces in the course of the buildup. Finally, some of the approximations of the theories are discussed, and the sensitivity of a transient measurement is compared with that of a stationary-intensity-correlation measurement.

I. INTRODUCTION

In the past few years, much attention has been focused on the statistical properties of a single-mode laser oscillator. The problem is very important for laser applications, since knowledge of the laser statistics allows one to set an ultimate limit to the coherence of the laser field, as well as for fundamental physics, since the very central questions of nonequilibrium statistical mechanics are involved here. The stationary-laser statistics have already been fairly thoroughly investigated both experimentally[1] and theoretically.[2] Little attention, however, has been paid so far to the time-dependent statistical properties of a laser. A direct investigation of these properties is of particular interest, because they cannot be inferred in a simple way from the steady-state behavior of the laser.[3] On the other hand, pulsed lasers are very commonly used and a statistical interpretation of the experiments can be essential in some cases.

In this paper we make a systematic investigation of the transient statistical properties of a single-mode gas laser. Recently these properties have been the object of some experiments[4,5] which have been interpreted only on a semiqualitative basis. Here we present new sets of experimental data in a working region closer to threshold than in the case of Ref. 4, thus making possible a quantitative comparison with the available theories on the transient-laser statistics.[6,7] Such a comparison, however, is not straightforward. The measuring procedures will therefore be carefully analyzed in order to compare the outcome of the experiments and the theoretical approaches.

The experiments reported in Ref. 4, as well as in this paper, have been performed as follows. A single-mode laser cavity is switched from a situation with large losses (low quality factor "Q") to one with low losses. The switching operation is fast enough so that it can be considered as "instantaneous," and therefore all times can be referred to a time origin corresponding to the switching time. The gain in the laser medium is stabilized at such a value that in the low-Q case the laser is below the threshold of oscillation (losses larger than gain) and in the high-Q case the laser is above threshold (gain larger than losses). Starting at the switching time, the laser field undergoes a transient, eventually reaching a steady-state situation.

The transient-laser field is a statistical process, and it can be characterized by the time-dependent probability distribution for the field inside the cavity. Photon-counting techniques[8] are the most appropriate means to investigate the statistical properties of an optical field. The experimental photon-counting distribution measured by a photodetector outside the laser cavity is affected by the detection as well as the attenuation process. However, these features can easily be taken into account in order to infer the photon distribution inside the cavity from the measured statistical distribution of photoelectrons.[9]

When the laser is driven from the initial to the final state adiabatically, i.e., through a series of stationary conditions with increasing gain-to-loss ratio, at each point one can associate a steady-state photon distribution which has been investigated both theoretically[2] and experimentally.[1,10-12] A completely different problem is that of tracing the instantaneous statistical distribution during the transient evolution of the laser from below to above threshold. First of all, such a process is no longer invariant under time translation, hence ensemble averages cannot be replaced by time averages. From an experimental point of view, this means that each sample in a statistical ensemble must be taken starting from the *same* macroscopic initial condition,[13] and therefore the laser must be set to that same initial condition between two successive measurements. Furthermore, in the case of a transient field, each sample must be measured over a time interval which is much smaller than the buildup time, so that we can really describe an "instantaneous" situation.

Besides switching the cavity losses at constant gain, the laser transient can be induced by switching the gain of the lasing mode. This can be done either by detuning the resonant frequency of the cavity with respect to the center of the atomic line, or by varying the population inversion. In the first case, one could change the cavity length, and hence the mode position, by an electro-optic device. As far as the second case is concerned, one should change the pump strength, but it would take a rather long time for the population inversion to adjust itself to the new condition.[14]

Since most of the steady-state statistical experiments have been performed on the helium-neon laser oscillating on the 6328-Å neon transition, we study the transient on the same type of laser. The choice of a gas laser, besides the experimental advantage of stability, makes possible the comparison with the available statistical theories, which are developed for the case of atomic relaxation times much shorter than the cavity damping time.[15]

Besides the experiments of Refs. 4 and 5, some nonstatistical measurements have been published on the buildup of the radiation field in a Q-switched gas laser. Single transients were recorded[16] and compared with the predictions of the semiclassical laser theory.[17] The interpretation of measurements of single events is not straightforward, because only an ensemble average of the output intensity can be compared with the semiclassical theory. However, in that case, the agreement was quite good, because the laser used was well above threshold. When the oscillator is far from threshold, the main statistical effect is given by the spread in the initial number of photons inside the cavity, whereas the effect of the noise appearing during the time evolution is relatively small. The result is that all single transients have approximately the same shape and the effect of the statistics is mainly a random-time jitter of the leading edge of the transient with respect to the switching time. The interpretation of our preliminary experimental results[4] was given following these phenomenological considerations. Baklanov *et al.*[5] used similar, even if more refined, phenomenological considerations to interpret their measurements. Their laser, however, was pump switched; this casts some doubt on whether the gain-switching process in that experiment was really much faster than the buildup time of the laser radiation.

This paper is organized as follows: In Sec. II we discuss the main points of the laser theory, giving the range of validity of the approximations used. Since our review of the theory is directed toward the quantitative interpretation of the experimental results, we found it useful to include some conversion rules between different theoretical approaches. In Sec. III we describe the experimental setup, display the experimental results, and discuss how they must be manipulated in order to take into account the effects of the measuring procedures. In Sec. IV we give a quantitative comparison between the experimental results and some computer calculations done with the Scully-Lamb-Sargent theory.[6] Furthermore, it is shown that a simple phenomenological model can give most of the quantitative features of the experiments. In Sec. V we discuss the range of the approximations used in the theory and compare the transient measurements with the stationary-intensity correlations from the point of view of experimental sensitivity.

II. THEORETICAL CONSIDERATIONS

A. Review of Available Theoretical Approaches

From a theoretical point of view, laser dynamics consist of the interaction of a system of identical atoms (A) and a radiation field (F) in a cavity. These two systems are strongly interacting through resonant coupling. Furthermore, there are many other degrees of freedom loosely coupled to the above systems and acting as a thermal reservoir (R). The reservoir summarizes the effects of spontaneous-decay processes, thermal noise in the electromagnetic cavity, cavity losses, and atomic collisions.

A complete characterization of the laser field is obtained by knowledge of the reduced-density operator ρ for the field. Calling ρ_t the total-density operator for the laser system, ρ is given by

$$\rho = \mathrm{Tr}_{\mathrm{R,A}}\, \rho_t \,, \tag{1}$$

where $\mathrm{Tr}_{\mathrm{R,A}}$ denotes the trace operation over the reservoir and atomic variables.

It is rather easy to trace over the reservoir variables under some general assumptions on the spacing and range of the reservoir energy levels. Assuming a dense distribution of reservoir levels over a frequency band much larger than any reciprocal characteristic time of the field-atom system (Markovian approximation), the coupling with the reservoir can be treated under the first Born approximation and is rather insensitive to the model chosen for the reservoir. The latter is usually modeled as either a collection of harmonic oscillators[18] or as a collection of two-level atoms.[19]

The joint atom-field operator $\rho_{AF} = \mathrm{Tr}_R\, \rho_t$ obeys an equation

$$\frac{d\rho_{AF}}{dt} = \left(\frac{d\rho_{AF}}{dt}\right)_{\text{coh}} + \left(\frac{d\rho_{AF}}{dt}\right)_{\text{incoh}} \quad (2)$$

Here, the coherent part has the structure of a Liouville equation

$$i\hbar\left(\frac{d\rho_{AF}}{dt}\right)_{\text{coh}} = [H, \rho_{AF}] , \quad (3)$$

where

$$H = H_A + H_F + H_{AF} \quad (4)$$

is the sum of atom and field Hamiltonian H_A and H_F, respectively, plus their interaction H_{AF}. The incoherent part has the structure of an irreversible Markovian master equation as already treated in the literature.[20]

Instead of studying the evolution of the operator ρ_{AF}, one can deal with phase-space density functions which weight projector operators on suitable complete sets of atomic and field states.[21] Such phase-space densities have the same content of information as ρ_{AF}, and should not be confused with classical probability functions to which they are equal only in the classical limit.[22]

Both the operator equation and the corresponding partial differential equation for a phase-space density lead to severe problems[21] not yet solved. For instance, the phase-space equation has the structure of a multidimensional Fokker-Planck equation with a matrix of diffusion coefficients which is not positive definite, hence leading to unphysical solutions.[21] Furthermore, an exact elimination of the atomic variables yields an equation of motion for the operator $\rho = \text{Tr}_A \rho_{AF}$, or for the equivalent phase-space density, which is nonlocal in time, and thence implies convolution terms with a memory.[23]

In most experimental cases it is not necessary to deal with all the above complications. A time-scale consideration drastically reduces the difficulties, destroying the memory effects. In the usual laser devices, the decay time T_d of the cavity field is much longer than the relaxation time T_2 of the induced atomic dipoles. Furthermore, in the He-Ne gas laser the decay time T_1 of the population inversion is of the order of T_2, and thus also much shorter than T_d.[24] One can therefore make an "adiabatic elimination" of the atomic variables. Limiting our considerations to a single-mode gas laser, this adiabatic elimination consists of assuming that the rather slow variations of the field occur in a local-equilibrium situation for the atoms. Under this approximation, both the equation for the field-density operator and that for the field phase-space density become "local" in time. To be precise, the former has the following structure:

$$\frac{d\rho}{dt} = L(a, a^\dagger)\rho(t) , \quad (5)$$

where $L(a, a^\dagger)$ is a time-independent operator that includes the effects of the active medium as well as the dissipation mechanism, and a and $a^\dagger$ are the annihilation and creation operators for the single-mode laser field. This corresponds to a Markovian behavior of the single-field system. As shown in Ref. 2, there are several possible approaches to the laser problem, but all of them eventually have to use the abovementioned approximation in order to arrive at an equation of motion which can be solved and compared with the experiments.

Using the photon-number representation, Scully and Lamb[2,19] give the following equation of motion for the diagonal-matrix elements of the density operator:

$$\frac{d\rho_{n,n}}{dt} = - [A - B(n+1)](n+1)\rho_{n,n} + [A - Bn]\, n\rho_{n-1,n-1} - C_n\rho_{n,n} + C(n+1)\rho_{n+1,n+1} , \quad (6)$$

where $\rho_{n,n} = \langle n|\rho|n\rangle$ is the probability of having n photons at time t, A is the unsaturated gain in the active medium and is proportional to the population inversion, B is the saturation parameter, and C represents the cavity loss factor. The relations between these parameters and measurable quantities are given in Eqs. (A1)–(A3) of Appendix A. The first two terms on the right-hand side of Eq. (6) account for the flow of probability from states $|n\rangle$ to $|n+1\rangle$ and from $|n-1\rangle$ to $|n\rangle$. The other terms proportional to the loss rate C describe the flow of probability from $|n\rangle$ to $|n-1\rangle$ and from $|n+1\rangle$ to $|n\rangle$. Equation (6) is valid not too far from threshold because the further approximation $(B/A)\langle n\rangle \ll 1$ has been used in the derivation.[25]

Considerations completely equivalent to those stemming from Eq. (6) can be made in the coherent-state representation.[26] In such a case, we make use of the diagonal phase-space density $P(\alpha)$ through the definition

$$\rho = \int |\alpha\rangle\langle\alpha| P(\alpha)\, d^2\alpha . \quad (7)$$

The function $P(\alpha)$ can be formally obtained by writing ρ in terms of the field operators a, $a^\dagger$ in antinormal order and then replacing the operators by the corresponding c numbers α, α^*.[27] An equation for $P(\alpha)$ can be derived from Eq. (6). The equation for $P(\alpha)$ can also be obtained by directly performing the adiabatic elimination on the generalized Fokker-Planck equation for the atoms plus field phase-space density.[21] The equation can be written as

$$\frac{\partial P}{\partial t} + \beta\, \text{div}_\alpha (d - |\alpha|^2)\, \alpha P = q\nabla_\alpha^2 P . \quad (8)$$

Here the differential operators div_α and ∇_α^2 refer to the complex variable $\alpha = x + iy = re^{i\varphi}$ and can be performed either in Cartesian or polar coordinates. Equation (8) was first derived by Risken[28] by a heuristic approach, and it has been given here using his notation. In order to give a meaning to the coefficients appearing in the above equation, we compare it with the standard Fokker-Planck equation for a classical harmonic oscillator undergoing Brownian motion. If α is the complex amplitude of the harmonic oscillator, γ the damping constant, and $\Gamma(t)$ the stochastic force having a Gaussian statistical distribution with zero average and a stationary-correlation function given by

$$\langle \Gamma^*(t)\,\Gamma(0)\rangle = 4q\,\delta(t)\ , \tag{9}$$

then $P(\alpha_0 0 \mid \alpha t)$, the conditional probability that the amplitude α be measured at time t, given the amplitude α_0 at the time $t=0$, obeys the following equation[29]:

$$\frac{\partial P}{\partial t} - \gamma\,\mathrm{div}_\alpha\, \boldsymbol{\alpha} P = q\nabla_\alpha^2 P\ . \tag{10}$$

Comparing Eq. (8) with (10) we see that there is a modification only in the drift term, βd playing the role of a damping constant with a negative sign (difference between linear part of the atomic gain and cavity losses) and $-\beta|\alpha|^2$ that of a nonlinear correction to the previous linear term. The diffusion coefficient q can have a similar interpretation as in the linear case.[30]

Comparison of Eq. (6) with (8) gives the following relations between the sets of macroscopic parameters A, B, C, and β, q, d (see Appendix A):

$$A = 4q,\quad B = 2\beta,\quad (A-C)/B = d\ . \tag{11}$$

The parameters A and B, as well as q and β, are defined in terms of the population inversion in the active medium, the dipole moment of the laser transition, and the atomic relaxation times. Slightly different microscopic models have been used in Refs. 19 and 28. Relations (11) can be derived directly from the microscopic definitions of the parameters only if one takes into account these differences (see Appendix A). The parameters built from a microscopic model cannot be considered quantitatively satisfactory. For instance, gain and losses are not uniformly distributed inside the cavity as is assumed in the above theories. The main significance of Eq. (11), however, is that one can always describe a single-mode laser not too far from threshold with three "effective" parameters, which can be determined by measurements without relying on specific models.

B. Moment Equations

An explicit expression for the photon distribution has been derived from Eq. (6) only in the steady-state condition.[19] The transient statistics can be described by giving the time evolution of the moments of the photon distribution. The average photon number $\langle n\rangle = \sum_{n=0}^{\infty} n\,\rho_{n,n}$ is found, from Eq. (6), to obey the equation of motion,

$$\frac{d\langle n\rangle}{dt} = (A-C)\langle n\rangle - B\langle n^2\rangle + A\ . \tag{12}$$

The main features of Eq. (12) are that (i) the time derivative of $\langle n\rangle$ is different from zero in the vacuum state ($\langle n\rangle = 0$), owing to the source term A which describes spontaneous emission; and (ii) the motion of $\langle n\rangle$ is coupled to the second moment $\langle n^2\rangle$; similarly the motion of $\langle n^2\rangle$ will be coupled to the third moment, and so on. Therefore, one has to deal with a hierarchy of moment equations. Numerical results can be obtained by truncating this hierarchy, dealing in this way with a closed set of differential equations.

Equation (12) should be compared with the result of the semiclassical theory,[17]

$$\frac{d\langle n\rangle}{dt} = (A-C)\langle n\rangle - B\langle n\rangle^2\ . \tag{13}$$

This is a closed equation for $\langle n\rangle$, which neglects fluctuations and requires a nonzero initial condition. In Sec. IV we shall show that our experimental results can be well fitted by using Eq. (13) associated with a statistically defined initial condition. This procedure can be easily handled by an analytic approach. In Sec. IV we present, also, a more elaborate fitting using a truncating procedure[31] at the tenth moment. The resulting nine coupled equations are solved by a computer.

III. DESCRIPTION OF EXPERIMENT

A. Experimental Setup

As discussed in Sec. I, the switching system which induces a transient in the laser oscillator must be much faster than the transient duration itself. This duration can be as short as a fraction of μsec whenever the atomic-population inversion is much higher than the steady-state inversion at threshold. If we want to span a large range of inversion values, it is therefore inconvenient either to switch the pump, or to switch the losses by a mechanical chopper. We have found it convenient to use an electro-optic shutter by inserting in the cavity a Kerr cell. This can be used in two ways, either to switch the losses as a regular shutter, or to switch the gain by changing the mode position with respect to the center of the gain line. In the first case, the axis of the Kerr cell (line normal to the two plane electrodes) must be oriented at 45° with respect to the polarization vector of the electric field in the cavity. In the second case the axis must be parallel to the polarization vector. A comparison between gain and loss switching has been carried out in Appendix B, where we show that the second method is better for two reasons: (a) because it requires a smaller voltage pulse applied to the cell electrodes; and (b) because it is less critical with respect to the onset of a second mode.

A scheme of the measuring apparatus is shown in Fig. 1. The laser cavity is 45 cm long with a nearly confocal configuration. The active medium is a helium-neon mixture in the ratio 7 : 1 with a total pressure of 1.2 Torr. The capillary discharge has a length of 25 cm and a diam of 0.2 cm. The laser oscillates on the 6328-Å transition with a single transverse mode in the TEM_{00} configuration. The measurements were performed with a single axial mode; that is, the pump power was kept below the threshold for the onset of a second mode. The position of the mode with respect to the gain line is controlled by a piezoelectric tuner as described in Ref. 32.

In order to minimize the insertion losses, the Kerr cell has been built with end windows at the Brewster angle. The Kerr cell is filled with orthodichlorobenzene and the glass windows chosen to match its refraction index. Use of the more common nitrobenzene was avoided because this liquid has about 2% losses over 1 cm at the laser wavelength against less than 1% for orthodichlorobenzene. The Kerr constant of the orthodichlorobenzene is 42.6 esu against 325 esu for nitrobenzene.[33] However, at the pump levels used, a phase retardation of only a few degrees between the two orthogonal components of the light field in the cell is sufficient to cut off the laser oscillation. This retardation is obtained with a field of 10 kV/cm. The geometry of the electrodes and the voltage pulser are designed in such a way that the rise time of the cell to full transparency is less than 4 sec.

Insertion of a Kerr cell inside the He-Ne laser cavity raises some problems. An investigation of several interesting phenomena has been reported with cells containing polar as well as nonpolar liquids.[34] Transients with time constants of some seconds have been observed, and interpreted by taking into account very small changes in the refractive index of the liquid due to heating by absorption of laser light. This, however, plays a negligible role in our experiment for the following reason: We have used a duty cycle of 200 Hz, that is

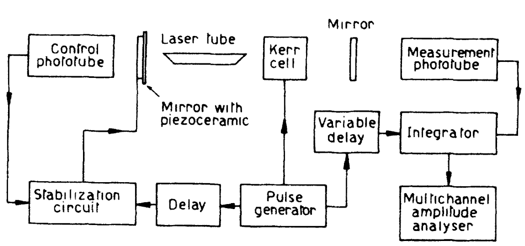


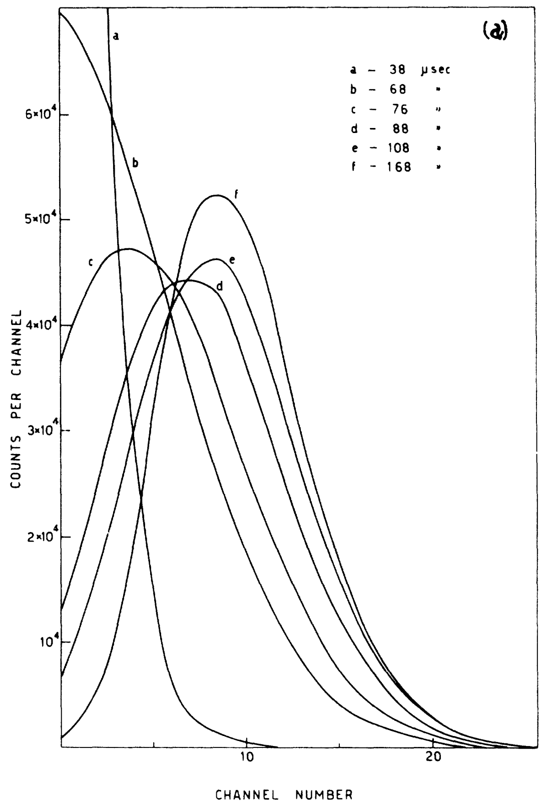
FIG. 1. Experimental setup.

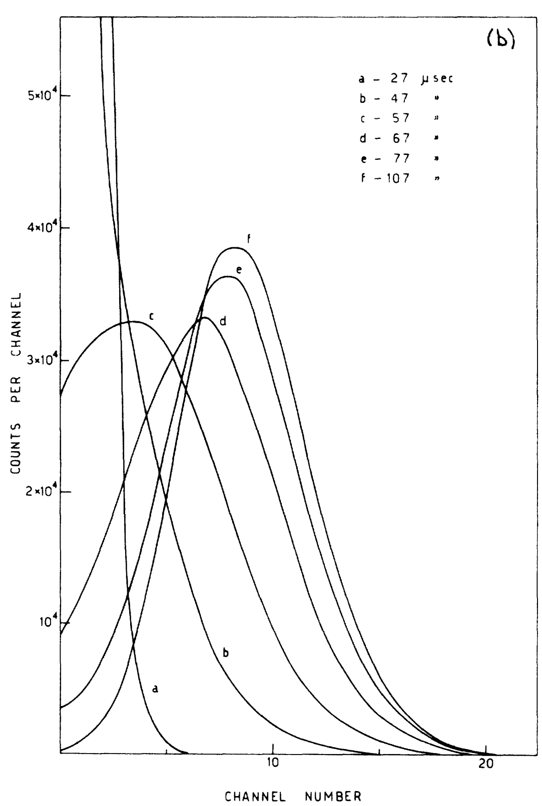
very fast with respect to the time constant of a thermal transient. Hence, the liquid in the cell is sensitive to the time average of the laser power. The over-all result can be explained by comparing the steady-state region of our transients (with no voltage on the cell) with a consistently dc situation (cell inserted without any voltage). In the two situations above, the average behavior of the laser (cavity losses, field spatial distribution) is different, because the average power is different. However, there are no further transient effects besides those occurring in the first few seconds after a series of measurements is started. Furthermore, it is very important to control the cleanness of the liquid. If dust particles remain in the liquid, they are strongly agitated by the application of the field, and give rise to spurious effects.[35]

As far as electrical disturbances are concerned, the fast discharge on the Kerr cell gives rise to a large induction pulse which triggers plasma oscillation in the laser tube. An accurate electrical shielding was accomplished by putting the Kerr cell in a copper box. With no shielding, large oscillating disturbances affect the transient intensity at a frequency of the order of 10^5 Hz. The effects mentioned above can appreciably influence measurements done very close to threshold. Indeed, experimental results taken at a pump parameter $a \leq 15$ (see Table I for the definition of a) were unreliable for lack of stability.

B. Conduction of Experiment

The experiment is carried out as follows. Pump and cavity parameters are chosen in such a way as to have the laser above threshold with the optical shutter open (no voltage on the Kerr cell). Starting with the optical shutter closed, the Kerr cell is switched on by a trigger pulse at the instant $t = 0$. The laser field undergoes a transient buildup from an initial statistical distribution, corresponding to the equilibrium between gain and losses far below threshold, up to an asymptotic condition above threshold. At the instant $t = \tau$ we perform a photocount measurement for a measuring interval T of 50 nsec, very small compared with the buildup time. Once a steady-state condition has been reached, an amplitude-stabilizing operation is performed by sampling the laser output and comparing this with a standard reference signal, following a procedure already described.[32] This is equivalent to "preparing" an identical initial state for a successive measuring cycle. After the sampling, the shutter is switched off for about 10 msec. This is long enough time to let the laser field decay completely, but short compared with the time scale of the slow drifts in the cavity length or in the atomic pumping. At the end of this interval the shutter is again switched on, and the cycle of opera-





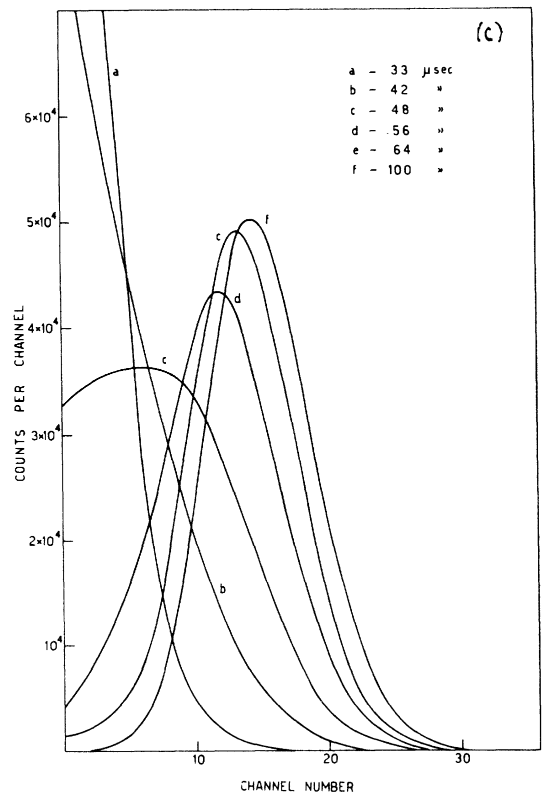


FIG. 2. Experimental statistical photocount distributions obtained with different delays with respect to the switching time. The solid lines join the experimental points which are not shown, to make the figure clearer. All distributions are normalized to the same area. The three figures (a)–(c) correspond to three different experimental situations (see Table I).

tions described above is repeated. This way an ensemble distribution of macroscopically identical events is collected.

By successively varying τ, the time evolution of the photocount distribution $p(m, T, \tau)$ is obtained. Experimental results for three different sets of laser parameters are shown in Fig. 2. The measured distributions must be corrected for some unavoidable background light through the mirrors of the cavity. This light is due to laser modes which are below threshold in the "on" operation. In the operating condition, it gives an average number of about 0.2 photoelectrons in 50 nsec. This background light depends on the mode spacing; therefore it becomes negligible with a shorter cavity (see Ref. 11).

Furthermore, to avoid dead-time problems on the small measuring interval T, we use the linear method[36,32]; that is, the current pulses from the photomultiplier are not standardized, but sent directly to an integrating capacitor. The statistical-charge distribution measured on the capacitor is the convolution of the photocount and amplitude distributions associated with the single-photoelectron response. Both the background light and the single-photoelectron amplitude distribution (see Fig. 3) are evaluated from independent measurements with the same apparatus. The moments of the photocount distributions are computed by assuming that the background light is uncorrelated with the laser light, and by using the formulas given in Ref. 32 for the linear method.

C. Experimental Results

The average photocount number $\langle m \rangle$ and the associated variance $\langle \Delta m^2 \rangle = \langle m^2 \rangle - \langle m \rangle^2$ are reported as functions of the time delay in Fig. 4 for three different pumping conditions a, b, and c. We may distinguish a first region, where $\langle m \rangle$ increases rapidly because of the stimulated-amplification process, and the variance grows much larger than if going slowly through a succession of stationary conditions.[37] This is explained as an enhancement of the initial spread in the photon distribution, due to linear amplification by stimulated emission.[38] Once a large amount of electromagnetic energy has been built inside the cavity, the field-atom interaction is no longer a linear process. In this region the curve of $\langle m \rangle$ has a point of inflection and eventually reaches a saturation value, while the variance goes through a maximum and then decreases to an asymptotic value appropriate to the stationary distribution.

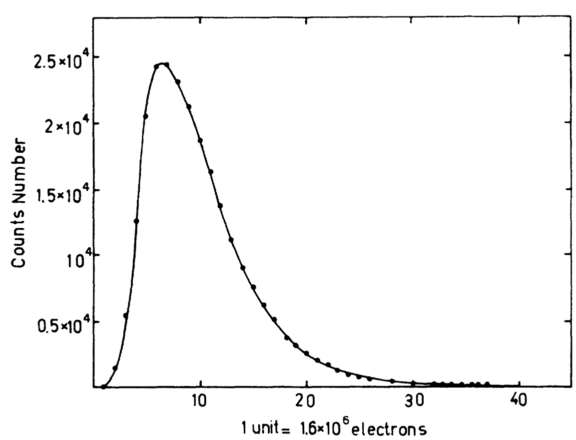


FIG. 3. Statistical distribution of the total electric charge collected on single-photoelectron pulses at the output of the photomultiplier. A RCA 7265 tube is used, with a total voltage of 2700 V; the voltage between the cathode and the first dynode is 300 V. The focusing-electrode potential has been adjusted to optimize the signal-to-noise ratio.

IV. DISCUSSION OF EXPERIMENTAL RESULTS

A. Comparison of Experiment and Theory

The photon distribution inside the laser cavity can be computed from the experimental photocount distribution, once the conversion factor η between photoelectrons emitted by the photocathode surface and photons in the cavity is known. η is given by

$$\eta = (cT/L)\,\theta_1\theta_2\theta_3 \,, \tag{14}$$

where c is the velocity of light, θ_1 the output-mirror transmittance, θ_2 the attenuating-filters transmittance, θ_3 the photocathode quantum efficiency, L the cavity length, and T the measuring time interval of a single sample.

The factorial moments F'_k of the photocount distribution $p(m, T, \tau)$ are related to the factorial moments F_k of the photon distribution $p(n, \tau)$ by the relation[9]

$$F'_k = \eta^k F_k \,. \tag{15}$$

From the above equation the following relations between first moments and variances are easily derived:

$$\langle n \rangle = \langle m \rangle / \eta \,, \tag{16}$$

$$\langle \Delta n^2 \rangle = (\langle \Delta m^2 \rangle - \langle m \rangle)/\eta^2 + \langle m \rangle / \eta \,. \tag{17}$$

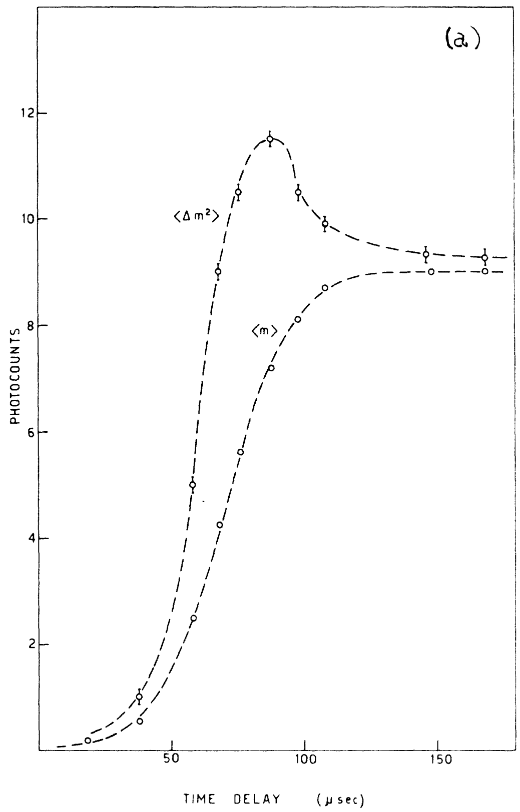
Equations (16) and (17) have been experimentally verified (see Ref. 9). These equations show that the relation between variance and first moment is conserved in the detection process only if $p(n, \tau)$ is a Poisson or a Bose-Einstein distribution.[39]

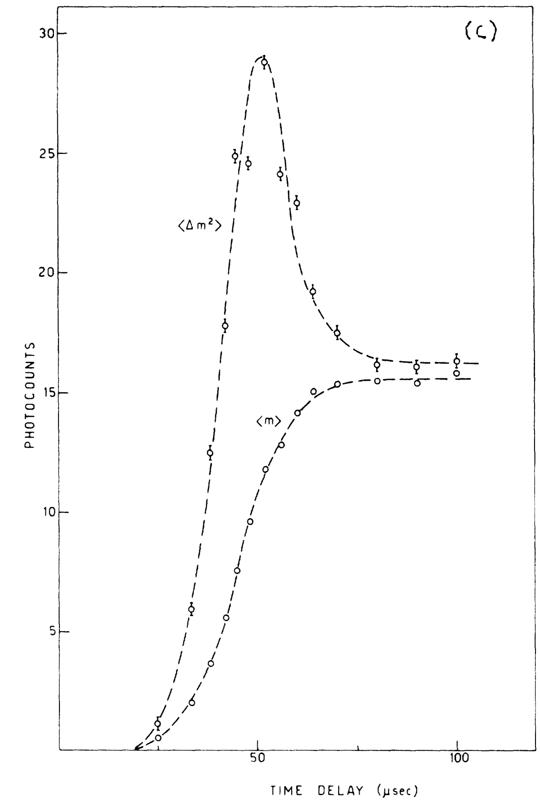
This fact can be given a physical meaning by the following considerations. In a problem where one pays attention only to the first and second moment of a statistical distribution, that distribution can be approximated by a Gaussian one having the same first two moments. In the case of an optical field, this can be considered as the linear superposition of a coherent field with an average photon number S and a Gaussian field with zero average and an average photon number N. Such a superposition corresponds to a displaced Gaussian distribution having an associated photon distribution with first moment and variance given, respectively, by[40]

$$\langle n \rangle = S + N \,, \tag{18}$$

$$\langle \Delta n^2 \rangle = S + N(1+N) + 2SN \,. \tag{19}$$

The variance is made of three terms. The first and second are, respectively, the variances associated with the two component fields. The third $2SN$ is an interference term that will be absent in the two-limit cases in which either S or N are zero. The disappearance of the interference term is suf-





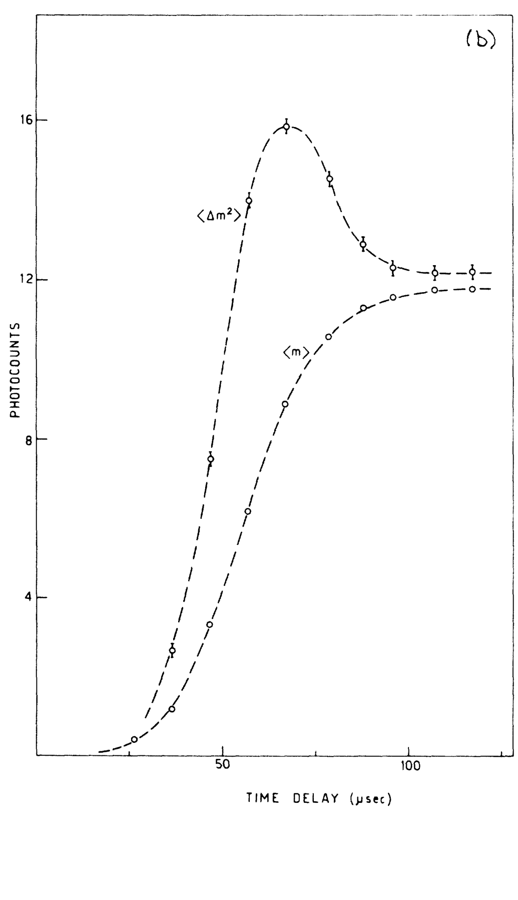


FIG. 4. Evolution of the mean value $\langle m \rangle$ and variance $\langle \Delta m^2 \rangle$ of the statistical distribution $p(m, T, \tau)$ as functions of the time delay τ. Dashed lines represent an interpolation of the experimental points. The three figures (a), (b), and (c) correspond to three different experimental situations (see Table I).

ficient to conserve the relation between first moment and variance in the detection process. Indeed, Eq. (17) is equivalent to the following relation between the variances of the photocount and photon distribution:

$$\langle \Delta m^2 \rangle = \eta^2 \langle \Delta n^2 \rangle + \eta (1 - \eta) \langle n \rangle . \qquad (20)$$

For a coherent field (Poisson photon distribution, $N = 0$), Eq. (20) becomes

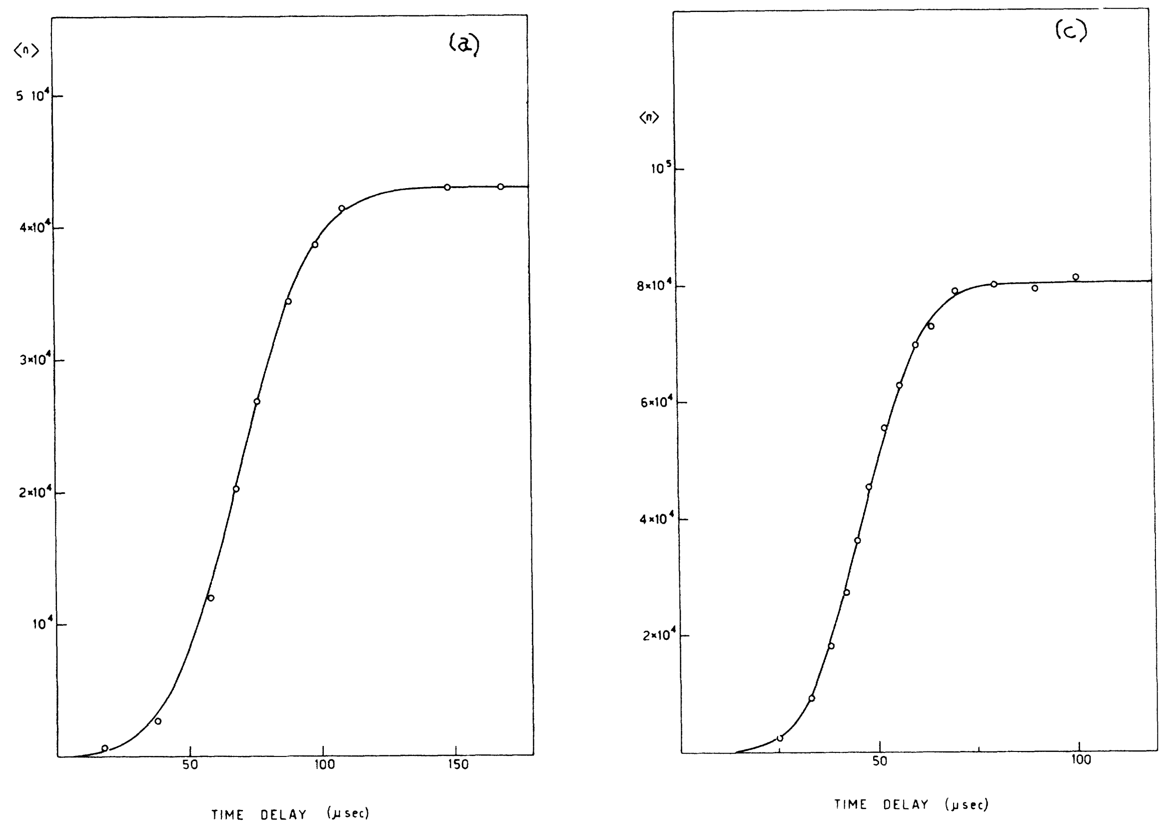
$$\langle \Delta m^2 \rangle = \eta S = \langle m \rangle , \qquad (21)$$

and for a zero-average Gaussian field (Bose-Einstein photon distribution, $S = 0$), it becomes

$$\langle \Delta m^2 \rangle = \eta N (1 + \eta N) = \langle m \rangle (1 + \langle m \rangle) . \qquad (22)$$

Both relations (21) and (22) are independent of η.

In our case $\theta_1 = 0.5 \times 10^{-2}$, $\theta_2 = 2.5 \times 10^{-2}$, $\theta_3 = 5 \times 10^{-2}$, $\eta = 2.1 \times 10^{-4}$. The estimated relative error on η is $\pm 10\%$.



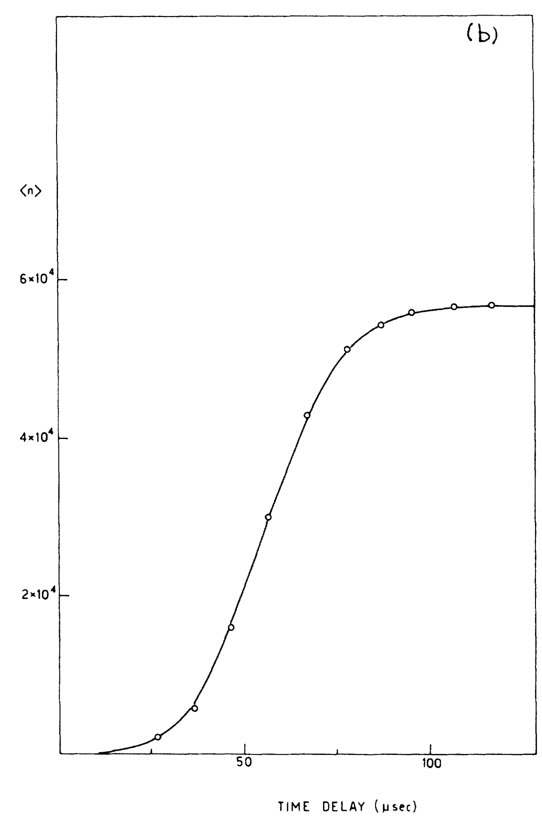
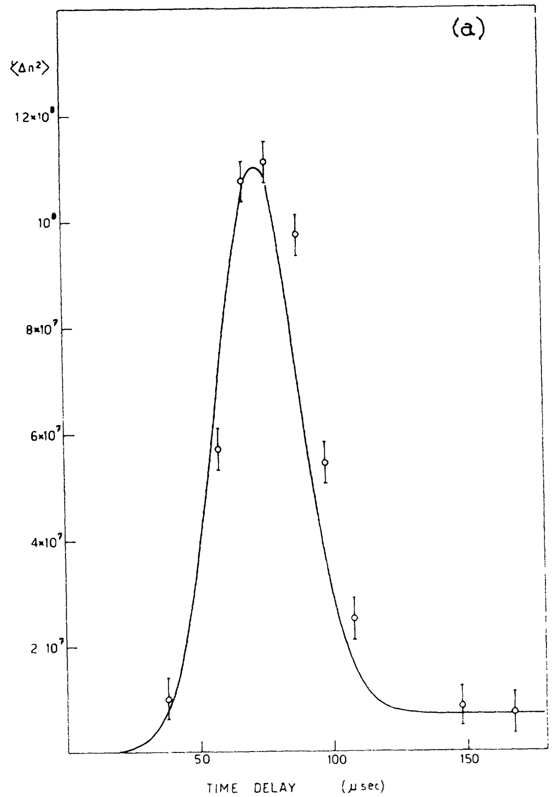


FIG. 5. Evolution of the average photon number $\langle n\rangle$ inside the cavity as a function of the time delay τ. Solid lines represent best-fit results computed from the theory of Scully, Lamb, and Sargent. Best-fit parameters for the three cases (a)–(c) are reported in Table I.

The points presented in Figs. 5 and 6 have been computed from the experimental data of Fig. 4, using Eqs. (16) and (17). The main differences between the plots of $\langle \Delta n^2 \rangle$ and $\langle \Delta m^2 \rangle$ versus time delay are (i) The ratio between the peak and asymptotic value is much larger for photon than for photocount variances, and (ii) the peak of the photocount variance is shifted toward a larger time value with respect to the peak of the photon variance. The solid lines in Figs. 5 and 6 represent theoretical results computed from the theory of Scully, Lamb, and Sargent[6] with a proper choice of the parameters in order to best fit the experimental points. The approximation used in deriving Eq. (6) implies the condition $(B/A)\langle n\rangle \ll 1$. This condition is fulfilled in the three experimental conditions studied in this paper.

Best-fit parameters $A-C$, B, and C are presented in Table I, together with Risken's parameters β, q, d computed from Eqs. (11). The difference $A-C=2\beta d$ is the reciprocal of the time constant related to the exponential buildup of $\langle n \rangle$



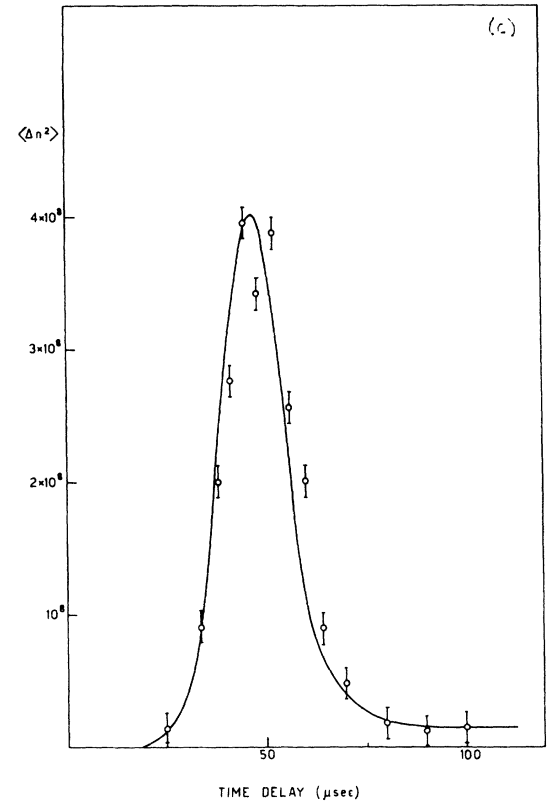
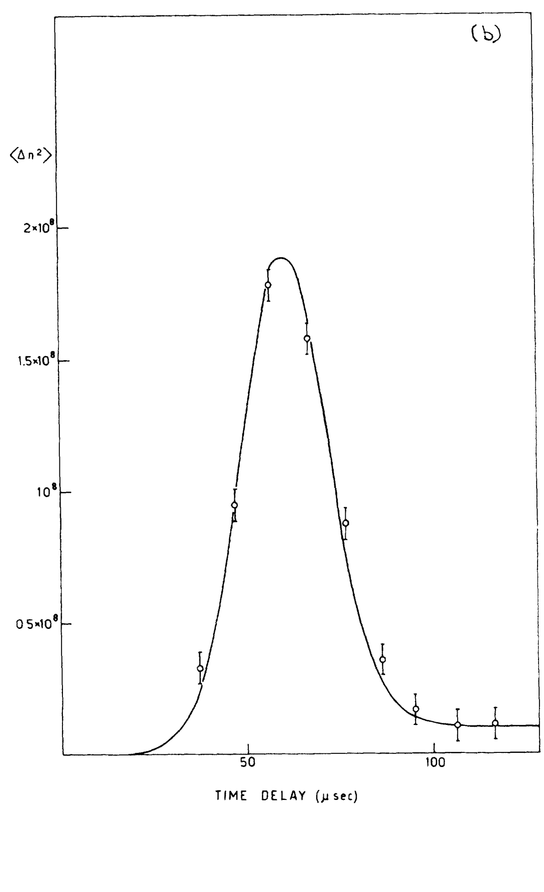


FIG. 6. Evolution of the variance $\langle \Delta n^2 \rangle$ of the statistical distribution of photons inside the cavity, as a function of the time delay. Solid lines represent theoretical results.



for short times (linear region). The shape of the theoretical curves is very sensitive to variations in $A - C$; therefore this quantity has been determined with a small relative error (about 3%). The relative error on B is substantially the same as for η, that is, about 10%. The relative error on C is also about 10%, because the theoretical curves are not very sensitive (at least in the region we have studied) to variations on C. The best-fit value of C is consistent with a rough evaluation of the cavity losses, which gives a round-trip relative loss of 5%.

Our results are also in qualitative agreement with the transient statistical theory of Risken and Vollmer.[7] A numerical comparison between this theory and the one by Scully and Lamb has not yet been tried.

There is another point to discuss when experimental results are compared with theory. Equation (12), which describes the motion of $\langle n \rangle$, can be solved for short times by neglecting the nonlinear term, obtaining

$$\langle n \rangle = [A/(A - C)]\,[e^{(A-C)\tau} - 1]\ . \tag{23}$$

TABLE I. Numerical values of the laser parameters corresponding to the three experimental cases a–c. A, B, and C are the parameters used by Scully and Lamb (Ref. 19); β, q, and d are Risken's parameters (Ref. 28) and have been computed by Eqs. (11). $a=(\beta/q)^{1/2}d$, Risken's pump parameter.

	(10^5 sec^{-1})	C (10^7 sec^{-1})	B (sec^{-1})	q (10^6 sec^{-1})	β (sec^{-1})	d (10^5)	a
a	0.95	1.67	2.20	4.199	1.10	0.43	22
b	1.25	1.67	2.24	4.206	1.12	0.56	29
c	1.75	1.67	2.13	4.219	1.065	0.82	42

Here it has been assumed that $\langle n\rangle=0$ at $\tau=0$. From an experimental point of view, the photon number in the oscillating mode is not zero at the switching time; precisely, it is $A/(C'-A)$, where C' are the cavity losses in the low-Q case. In our case, this number is of the order of 1. For time delays short with respect to the time constant $(A-C)^{-1}$, the photon number increases at the constant rate A. Therefore, provided that $C'-A \gg A-C$, the theoretical transient, which starts with zero photons, has a time lag of $(C'-A)^{-1}$ with respect to the experimental transient. This effect is too small to be detected in our experiment[41]; that is, we can assume with a good approximation that at $\tau=0$ the oscillating mode is in the vacuum state.

B. Phenomenological Considerations

We want to discuss here a simpler alternative approach to the statistical description of the laser transient. This approach is basically phenomenological and represents a development of the considerations used in Ref. 4 to interpret our earlier results.

When the laser is not very close to threshold, all single transients are observed to have the same shape with small amplitude fluctuations. The statistical character of the process is mainly revealed by a random jitter in time, as was already said in Sec. I. Therefore, it seems a good approximation to assume that a single transient can be represented as a "deterministic" evolution from a statistically defined initial condition. The evolution will be described by the semiclassical equation (13), whose integration gives

$$n(\tau, n_0)=\frac{n_0 d}{d\,e^{-(A-C)\tau}+n_0[1-e^{-(A-C)\tau}]} \;. \tag{24}$$

Here $d=(A-C)/B$ is the asymptotic value of n for large τ and n_0 is photon number inside the cavity at the switching time $t=0$. We assume that n_0 is distributed around an average value $\bar{n}_0$ with a Bose-Einstein distribution, which in the limit for $\bar{n}_0 \gg 1$, can be written as

$$p(n_0)=(1/\bar{n}_0)\,e^{-n_0/\bar{n}_0} \;. \tag{25}$$

The time-dependent kth moment of the photon distribution is given by

$$\langle n^k(\tau)\rangle=\int_0^{+\infty} n^k(\tau, n_0)\,p(n_0)\,dn_0 \;. \tag{26}$$

Using Eq. (26), one obtains for the first two moments

$$\langle n\rangle=\frac{d}{1-e^{-(A-C)\tau}}\,[1-H(z)] \;, \tag{27}$$

$$\langle n^2\rangle=\left[\frac{d}{1-e^{-(A-C)\tau}}\right]^2 [1+z-(2+z)H(z)] \;, \tag{28}$$

and hence,

$$\langle \Delta n^2\rangle=\left[\frac{d}{1-e^{-(A-C)\tau}}\right]^2 [z(1-H(z))-H^2(z)] \;. \tag{29}$$

In the above formulas we have called the time-dependent quantity z,

$$z=\frac{d\,e^{-(A-C)\tau}}{\bar{n}_0[1-e^{-(A-C)\tau}]} \;, \tag{30}$$

and mean by $H(z)$, the following function of z

$$H(z)=z\,e^z\int_z^{+\infty}\frac{e^{-y}}{y}\,dy \;. \tag{31}$$

We note that the ratio r between the variance and the square of the mean photon number is a function only of the variable z, that is,

$$r=\frac{\langle \Delta n^2\rangle}{\langle n\rangle^2}=\frac{z(1-H)-H^2}{(1-H)^2} \;. \tag{32}$$

Equation (32) gives, therefore, a universal function for the laser transient. The ratio r is plotted as a function of z in Fig. 7. As one can see, r goes from zero for small values of z (large times) to 1 for large values of z (short times). The behavior of the quantity r can be better understood if one recalls that, within the approximation $\langle n\rangle \gg 1$ used in our treatment, r coincides with the second-reduced normalized-factorial moment.[10,11]

After a few time constants, the term $e^{-(A-C)\tau}$ in the denominator at the right-hand side of Eqs. (27)–(29) can be neglected. Therefore, $\langle n\rangle/d$, $\langle n^2\rangle/d^2$, and $\langle \Delta n^2\rangle/d^2$ can be considered as functions of the variable z only. The maximum of the variance occurs at a value $z_m=0.4$. The average number of photons, variance, and ratio r corresponding to this value of z can be easily computed. The results are reported in Table II together with the experimental values. The agreement is good.

Using the definition of the variable z, one can also compute the effective delay τ_m corresponding to the maximum in the variance. From Eq. (30) one would obtain a value τ_m' given by

$$\tau_m'=(A-C)^{-1}\ln(d/\bar{n}_0 z_m) \;. \tag{33}$$

The model used here assumes an average num-

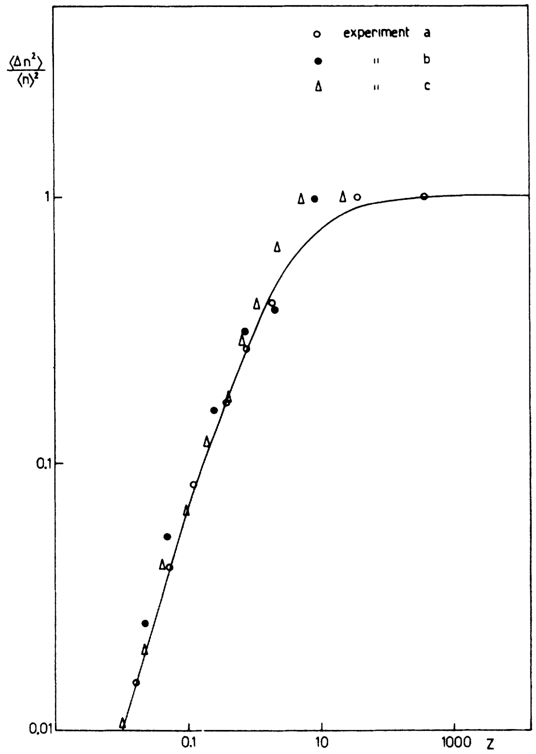


FIG. 7. Plot of the relative variance $\langle \Delta n^2 \rangle / \langle n \rangle^2$ vs $z = (d/\bar{n}_0)\, e^{-(A-C)T}\, [1 - e^{-(A-C)T}]^{-1}$. Large values of z correspond to small delays, and small values of z to large delays. The solid line represents the results of the phenomenological model discussed in Sec. IV.

ber $\bar{n}_0$ of photons at the initial time. Therefore, the delay τ_m differs from τ'_m by an amount given by the time interval Δt taken to build up $\bar{n}_0$ photons in the laser cavity starting from the vacuum state. This additional delay is computed from Eq. (23) which gives an exact description of the average transient for short times. A value $\Delta t = (A - C)^{-1} \ln 2$ has been derived by choosing $\bar{n}_0 = A/(A - C)$. There is some arbitrariness in this choice, but the final result is not critically dependent on the value of $\bar{n}_0$, provided that the choice is consistent with the assumptions made in the phenomenological model. Table II shows that the computed values of τ_m lie very close to the experimental ones.

The considerations made above have also been used to determine the points reported in Fig. 7 from the experimental results. The fit with the theoretical curve is satisfactory, except for the region of high z (short times), where the approximations involved in the phenomenological approach become too crude. One could improve the treatment of this region by introducing the following consideration: The "initial" photon distribution is effectively built up in the first few instants after the switching on; therefore, the highest values of n_0 are influenced by the saturation term. The distribution $p(n_0)$ shows deviations from that of Bose and Einstein for high values of n_0. In the region of high z, the agreement with experimental results and with the more refined Scully-Lamb theory could be greatly improved by putting a cutoff in the distribution $p(n_0)$. However, the resulting improvement would not add very much, because the main aim of the phenomenological approach is to describe the relevant region of the transient, and this is already achieved with a good accuracy, as shown by the data of Table II.

V. CRITICAL REMARKS ON RELATIONS BETWEEN EXPERIMENTS AND THEORY

In the previous sections we have presented a series of statistical measurements on a transient single-mode laser, and shown a good agreement with the laser theories. We have furthermore shown that the experiments can be rather accurately interpreted by a simple phenomenological theory, which consists of considering only the initial statistical spread and neglecting the fluctuations along the time evolution. The following two questions deserve further discussion: (a) The theories are fully developed for simple models. Is it possible to perform experiments in working conditions where some of the usual approximations fall down? (b) In a linear system the fluctuation-dissipation theorem relates the time-dependent response (first-order average of a given parameter of the system) with the stationary fluctuations (correlation function for the same parameter). Therefore, a transient experiment and a stationary-correlation experiment carry the same amount of information. Can we make a similar assertion for the laser, which is a nonlinear system?

TABLE II. Comparison between experiments and the phenomenological model discussed in Sec. IV. τ_m is the delay corresponding to the peak value of the variance $\langle \Delta n^2 \rangle$ of the photon distribution inside the cavity. The values $\langle n \rangle_m$ and $\langle \Delta n^2 \rangle_m$ reported in the table are taken at the delay τ_m. d is the asymptotic value of $\langle n \rangle$ for large delays.

	$\langle n \rangle_m / d$		$\langle \Delta n^2 \rangle_m / d^2$		$\langle \Delta n^2 \rangle_m / \langle n \rangle_m^2$		τ_m(nsec)	
	Theor	Expt	Theor	Expt	Theor	Expt	Theor	Expt
a	0.581	0.568	0.0567	0.0596	0.168	0.185	74.5	73
b	0.581	0.588	0.0567	0.0585	0.168	0.169	61.0	60
c	0.581	0.557	0.0567	0.0618	0.168	0.199	47.6	47

A. Approximations and Theory

Both the master equation (6) and the Fokker-Planck equation (8) have veen developed for a resonant interaction of a single-mode field with a homogeneously broadened atomic line, and within the Markov approximation (adiabatic elimination of the atomic variables). In fact, the neon 6328-Å transition is Doppler broadened, hence there is a natural selection of a homogeneous packet around the frequency of oscillation of the lasing mode, and the resonance assumption is justified for a wide range of mode positions near the center of the Doppler line.

As far as the type of line broadening is concerned, the atomic gain to be introduced in the equation of evolution for the average photon number n (from here on for simplicity we leave out the angular brackets $\langle\,\rangle$) would be given by

$$g_h(n) = A/[1 + (B/A)n] \tag{34}$$

for a purely homogeneous line, and by

$$g_i(n) = A/[1 + (B/A)n]^{1/2} \tag{35}$$

for a purely inhomogeneous line.[42] Here A and B are the parameters already used in the master equation (6). In the cubic approximation,[43] the quantities above are given by

$$g_h = A - Bn \tag{36}$$

and

$$g_i = A - \tfrac{1}{2}(Bn)\ . \tag{37}$$

Since we do not assign A and B by their microscopic definitions, but rather evaluate them from the measurements, it is not possible to distinguish between Eqs. (36) and (37), the factor $\frac{1}{2}$ being included in the *a posteriori* definition of B. Therefore, the experiments reported in this paper, which have been performed on a laser not too far from threshold, cannot give information on the homogeneity of the atomic line broadening.

The situation is different for the measurements reported in Ref. 4. First of all, some of the laser parameters reported in Table I of Ref. 4 had not been computed correctly, because at that time we were not aware of the considerations later developed in Ref. 9, and used in Sec. IV A of this paper. Furthermore, in order to derive the parameters A, B, and C from the experiments, one must already assume a specific broadening, and hence make a choice among the gain relations (34)–(36). Precisely, in the measurements of Ref. 4, the reciprocal of the time constant of the initially exponential buildup and the asymptotic average number of photons in the cavity are higher than in the cases reported in this paper. Therefore, one is not sure *a priori* that the cubic approximation is still valid. We have performed a check of validity by using the theoretical results obtained here in Sec. IV B, and based on only scaled quantities (i.e., independent of the parameters). The experimental values of the ratios $\langle n\rangle_m/d$, $\langle\Delta n^2\rangle_m/d^2$, and $\langle\Delta n^2\rangle_m/\langle n\rangle_m^2$ are reported in Table III for the two cases denoted as I and II in Ref. 4. Deviations from the theoretical values given in Table II are within the estimated experimental errors. Therefore, the cubic approximation is still acceptable, and has been used to compute the laser parameters also reported in Table III. Of course, our interpretation relies on the fact that the above-mentioned ratios are no longer independent of the laser parameters, when the cubic approximation is not used. We have checked this point by inserting the gains g_h and g_i given in Eqs. (36) and (37) in Eq. (13) and applying the method described in Sec. IV B. One obtains a detectable difference in the values of the scaled parameters with respect to the cubic case for a pump parameter *twice* the maximum value reached in the experiments of Ref. 4. We do not have, however, experimental data for such a high pump value.

We conclude with two further comments on the approximations used. As far as the Markov approximation is concerned, deviations would appear for pump values so high that the transient buildup time is comparable to the atomic relaxation times. Our fastest buildup time is in the μsec region,[4] whereas the atomic relaxation times are in the nsec region; hence, the Markov approximation is valid in the present experiments as well as in the previous ones.

As far as the phenomenological model of Sec. IV B is concerned, one may wonder why the initial statistical spread is so much more important than the effect of a stochastic force in the equations of motion. A heuristic answer is that any further fluctuation introduced after the initial instant would take a

TABLE III. Correct values of the parameters A, B, C, and a for the measurements reported in Ref. 4. The quantities appearing in the first three columns are computed from the experimental results to check the validity of the cubic approximation and should be compared with the theoretical values appearing in Table II.

	$\langle n\rangle_m/d$	$\langle\Delta n^2\rangle_m/d^2$	$\langle\Delta n^2\rangle_m/\langle n\rangle_m^2$	$\langle n\rangle_\infty = d$ (10^5)	$A-C$ (10^6 sec^{-1})	C (10^7 sec^{-1})	B (sec^{-1})	a
I	0.533	0.0639	0.227	9.96	2.18	1.7	2.19	503
II	0.561	0.0533	0.170	5.76	1.17	1.7	2.03	280

time of the order of the buildup time before being influential, hence most of the transient can be correctly studied by taking into account only the initial fluctuations.

B. Transient- versus Stationary-Intensity Correlations

The photon number, or intensity, correlation function $R(\tau)=\langle n(\tau)n(0)\rangle$ of a stationary single-mode laser has an initial value $R(0)=\langle n(n-1)\rangle$, and an asymptotic value $R(\infty)=\langle n\rangle^2$, where averages are evaluated on the single-time photon distribution. In general, the decay from the initial to the asymptotic value can be described as a series of exponential terms, whose time constants and relative weights are dependent on the three laser parameters. Somewhat above threshold (pump parameter $a>10$) only one of the exponential terms is predominant, because in this case the laser equations can be linearized around the equilibrium position.[44] The decay time τ_c of the intensity-correlation function is given, within this approximation, by $\tau_c = (A-C)^{-1}=(2\beta d)^{-1}$. All the quantities $R(0)$, $R(\infty)$, and τ_c are measured in a transient statistical experiment, as has been shown in this paper. The drawback of an intensity-correlation measurement on a stationary laser[45] is that the time-dependent part of the correlation function becomes very small with respect to the constant part when the laser is very far away from threshold. To be precise, the ratio $[R(0)-R(\infty)]/R(\infty)$ is proportional to a^{-2}, where a is Risken's pump parameter (see Table I). Therefore, the signal-to-noise ratio becomes quite small, making the measurement more difficult. This is not the case in a transient experiment, where the precision in the evaluation of τ_c is not affected by an error in the evaluation of $R(0)-R(\infty)$. An open point is whether one can derive the exact intensity-correlation function from the transient behavior, and not simply its single-exponential approximation. This means taking the full nonlinearity of the problem into account.[2] We are still investigating this point.

ACKNOWLEDGMENTS

We are very grateful to M. O. Scully and W. E. Lamb for having suggested the method used in the computation of numerical results from their theory. We acknowledge also financial support from M. O. Scully for use of the MIT computer center and help in computer programming by D. Limbert. Thanks are due to F. Gardossi for valuable technical support in the conduct of the experiment and R. Bonifacio for helpful discussions.

APPENDIX A: RELATIONS BETWEEN COEFFICIENTS OF MASTER EQUATION AND FOKKER-PLANCK EQUATION

The relations (11) between the parameters used by Risken and those used by Scully and Lamb can be computed without deriving the Fokker-Planck equation (8) from the master equation (6) and successively comparing the coefficients. The alternative method presented here is based on the comparison of the expressions for three measurable quantities, which are derived from the known stationary solutions of Eqs. (6) and (8). We have chosen the average photon number $\langle n\rangle$, the ratio between the variance $\langle\Delta n^2\rangle$ and the average photon number, and the decay constant τ_c of the intensity-correlation function.[44] All these quantities are evaluated, for sake of simplicity, somewhat above threshold where a quasilinearization of the laser equations is a reasonable approximation. Since the same approximation is performed on both treatments, there is no loss of generality for the final result. The following expressions hold:

$$\langle n\rangle = (A-C)/B = d\ , \tag{A1}$$

$$\langle\Delta n^2\rangle/\langle n\rangle = A/(A-C) = 2q/\beta d\ , \tag{A2}$$

$$\tau_c = (A-C) = 2\beta d\ . \tag{A3}$$

Using the above equations, relations (11) are immediately derived.

The laser parameters A, B, C and β, q, d are defined in terms of microscopic quantities, respectively, in Eqs. (87)–(89) of Ref. 19 and in Eqs. (VI. 12.24), (VI. 12.25), and (VI. 12.31) of Ref. 46. These definitions are not exactly consistent with relations (11), because the two microscopic models are slightly different. They become consistent if the following assumptions are made: (i) The population inversion is coincident with the population of the upper level of the laser transition; and (ii) transverse atomic relaxation times are equal to longitudinal atomic relaxation times.

APPENDIX B: COMPARISON BETWEEN LOSS AND GAIN SWITCHING

Let us consider a Doppler-broadened gain line (gas laser). The linear gain $A(s^{-1})$ as a function of the frequency ν is given by

$$A(\nu)=A_0\exp\left[-\left(\frac{2(\nu-\nu_0)}{\Delta\nu_D}(\ln 2)^{1/2}\right)^2\right]\ , \tag{B1}$$

where A_0 is the gain at the center frequency ν_0, and $\Delta\nu_D$ is the width at half-height. Let C represent the losses in the laser cavity. The threshold condition $C=A_T$ defines the frequency range $\Delta\nu_T$ in which laser action is possible (see Fig. 8). If single-mode operation is required, a sufficient condition will be $\Delta\nu_T\leq\Delta\nu_0$, where $\Delta\nu_0=c/2L$ is the frequency difference between two adjacent axial modes of the laser cavity. In Fig. 8 we have denoted by ν_A and A, respectively, the frequency and gain of the lasing mode, and by ν_T the threshold frequency.

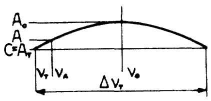


FIG. 8. Top region of the Gaussian gain line as a function of the frequency. $C = A_T$ represents the threshold gain, and A_0 is the gain at the center frequency.

The laser can be switched off by using an electro-optic device inside the cavity. Let z be the direction of propagation of the light inside the electro-optic material and n the index of refraction of the material. By applying an electric field on the material, one can change n_x (index of refraction for light polarized along the x axis) and n_y. If the x axis is chosen to be coincident with the direction of polarization of the laser light, one simply changes the length of the cavity. Therefore, the mode position is displaced with respect to the center of the atomic line. In order to bring the laser to threshold, the displacement has to be equal to $\nu_A - \nu_T$. If A is much closer to C than to A_0, one obtains

$$\nu_A - \nu_T = (A - C) \Big/ \left(\frac{dA}{d\nu}\right)_{\nu=\nu_T} = \frac{A-C}{C} \frac{\Delta\nu_D}{4\ln 2} \frac{\Delta\nu_D}{\Delta\nu_T} . \tag{B2}$$

Denoting the voltage required to change the optical length by $\frac{1}{2}\lambda$ $V_{\lambda/2}$, the voltage V_G required to bring the laser to threshold is

$$V_G = V_{\lambda/2} \frac{\nu_A - \nu_T}{\Delta\nu_0} = V_{\lambda/2} \frac{\Delta\nu_D}{\Delta\nu_0} \frac{\Delta\nu_D}{4\ln 2\, \Delta\nu_T} \frac{A-C}{C} . \tag{B3}$$

$V_{\lambda/2}$ is obviously a function of the electro-optic coefficients of the material and of the geometry of the arrangement.

Instead of switching the gain as described above, one can Q switch the cavity in the following way: The x axis is chosen to form a 45° angle with the direction of polarization of the optical field propagation along the z axis. In this case, by applying an electric field to the electro-optic material, a phase retardation is introduced between the two components of the optical field along x and y. This gives rise to a component of the optical field with a polarization perpendicular to that of the incident light beam. For this new component, losses are much larger because of the Brewster windows inside the cavity. Therefore, the total effect will be to increase the losses of the cavity. It is easy to show that the voltage V_Q required to increase the losses by $A - C$, that is to bring the laser to threshold, is

$$V_Q = V_{\lambda/2} (A - C)/\pi\, \Delta\nu_0 . \tag{B4}$$

Comparison of Eqs. (B3) and (B4) shows that V_Q is much smaller than V_G, taking into account that C is always much smaller than $\Delta\nu_D$. More precisely, the ratio between the two voltages is

$$V_Q/V_G \approx (C\, \Delta\nu_0)/(\Delta\nu_D)^2, \tag{B5}$$

where we have assumed $\Delta\nu_T = \Delta\nu_0$.

Furthermore, one should note that a gain switch is more critical than a Q switch. Indeed, if the mode spectrum is shifted by a rather large amount, the mode adjacent to the lasing one can be brought above threshold.

We must finally notice that in the case of Q switching we exploit the difference of the refractive index between two perpendicular directions of polarization, due to the induced orientation of anisotropic molecules. The main effect will be a rotation of the polarization without increase in the optical length. However, at higher orders, there might be a small shift in the mode position due to distortion, besides orientation, of the molecules.

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[3]Linear techniques such as the Onsager regression theorem or Kubo fluctuation-dissipation formula do not depend on the detailed structure of the physical system, whereas nonlinear problems must be solved case by case. In Secs. II–V we show that measurement of a set of three independent parameters specifies completely the laser dynamics. However, such specification will be strongly dependent on the approximations.

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Fluctuation Spectra and Quasithermodynamics of a Linearized Markov Process*

Robert H. G. Helleman
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033
(Received 27 July 1970)

An n-dimensional linear Markov process with parameters $\alpha_i (i=1, \ldots, n)$ and $\underline{\dot{\alpha}} = -\underline{M}\underline{\alpha}$ is considered. Criteria for stationarity of the process and spectral properties of the fluctuations around the stationary state $\underline{\alpha}=\underline{0}$ are derived. When the stationary state is a thermodynamic equilibrium state, the spectra are proven to be monotonic functions of the frequency. Positive-definiteness of the matrix $\underline{M}^3 \langle \underline{\alpha}\,\underline{\alpha}^T\rangle$ turns out to be the necessary and sufficient condition for the absence of local maxima in the spectra. A process with $\underline{M}^3 \langle \underline{\alpha}\,\underline{\alpha}^T\rangle$ positive-definite also has the property that $\underline{M}^2 \langle \underline{\alpha}\,\underline{\alpha}^T\rangle$ is positive-definite, which guarantees the absence of external driving forces, and corresponds to a pure relaxation process. $\underline{M} \langle \underline{\alpha}\,\underline{\alpha}^T\rangle$ positive-definite is a necessary condition for stationarity. Moreover, it is a necessary and sufficient condition for the existence of spectra and transport coefficients. Positive-definiteness of $\underline{M} \langle \underline{\alpha}\,\underline{\alpha}^T\rangle$, $\underline{M}^2 \langle \underline{\alpha}\,\underline{\alpha}^T\rangle$, and $\underline{M}^3 \langle \underline{\alpha}\,\underline{\alpha}^T\rangle$ is linked to properties of the excess-entropy production.

I. INTRODUCTION

We shall derive some properties of fluctuation spectra ("noise") and the excess-entropy production during those fluctuations from equilibrium, as well as from stationary nonequilibrium states of systems with n-coupled macroscopic variables.

As an example, one may think of an n-"level" semiconductor with the electron occupancy numbers of each level as the variables. In the equilibrium state there are only thermal transitions between the levels. A stationary nonequilibrium state may result when we continuously "pump" electrons from one level to another by means of a steady light source. Spectra can be obtained experimentally through a Fourier decomposition of the temporal history of the fluctuations in the conduction current: The conductivity is a linear combination of these occupancies, with the mobilities as coefficients.

The time dependence of the n macroscopic variables a_i $(i=1, \ldots, n)$ of this and many other kinetic processes is assumed to behave like a Markov process and thus to be governed by a first-order differential equation in time[1-4]:

$$\underline{\dot{a}} = \underline{f}(\underline{a}), \quad \underline{a} \equiv (a_1, \ldots, a_n)^T . \tag{1a}$$

For the semiconductor example discussed before, the explicit form of Eq. (1a) may be obtained[1,5] by assuming that the transition current p_{ij} from the ith level to the jth level is proportional to the occupation a_i of the ith level and to the number of vacancies in the jth level, the "mass-action" laws: $p_{ij} = \gamma_{ij} a_i (N_j - a_j)$, where N_j is the total number of states in the jth level. This gives

$$\dot{a}_i = -\sum_{j=1}^{n} (p_{ij} - p_{ji})$$
$$= -\sum_{j=1}^{n} [\gamma_{ij} N_j a_i - \gamma_{ji} N_i a_j + (\gamma_{ji} - \gamma_{ij}) a_i a_j] . \tag{1b}$$

In many cases the energy gap between the valence band, defined as $i=n$, and the higher levels, defined as $i=1$ or 2, is equal to the energy of photons in the optical spectrum. In the case of steady light absorption there results a current U_{nk}, for example, from the nth level to the kth level. One has $U_{nk} = a_n q E$ with q, the quantum efficiency, and E, the constant number of incident photons per second. For all practical cases, however, the net decrease in a_n is negligible compared to the large number a_n of electrons in the valence band. Hence, one takes $a_n \approx N_n$ and U_{nk} a constant as a result. This is incorporated in Eq. (1b) by adding U_{nk} to the equation for $\dot{a}_k$ and $-U_{nk}$ to the equation for $\dot{a}_n$.

We solve each model for its stationary state $\underline{a}_0$, i.e., $\underline{f}(\underline{a}_0) = \underline{0}$. We linearized Eq. (1a) defining $\underline{\alpha}(t) \equiv \underline{a}(t) - \underline{a}_0$ and find

$$\underline{\dot{\alpha}} = -\underline{\tilde{M}}\,\underline{\alpha}, \quad \underline{\tilde{M}}_{ij} \equiv -\left.\frac{\partial f_i(\underline{a})}{\partial a_j}\right|_{\underline{a}=\underline{a}_0} \quad (i, j = 1, \ldots, n) . \tag{2}$$

We now remove any dependent equations and variables. We therefore obtain a reduced matrix $\underline{M}$ with dimension $\leq n$, $\det\underline{M} \neq 0$, and $\underline{\alpha} = \underline{0}$ as the only point where $\underline{\dot{\alpha}} = \underline{0}$: