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High-Frequency Sound Attenuation and Dispersion in the Critical Region*

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A general theory is presented for sound attenuation and dispersion near the critical point in the region of frequencies which are much greater than the characteristic frequency of order-parameter dynamics, but are much smaller than the frequency of sound with the wavelength equal to the range of correlation of local order-parameter fluctuations. The method used is the most general version of the mode-mode coupling theory plus the static and dynamic scaling hypotheses and some general properties of time-correlation functions. If we ignore the small critical exponent α associated with the heat-capacity singularity, we find that the attenuation behaves as $f\epsilon^0$ and the dispersion (relative sound-velocity change with frequency) as $f^0\epsilon^0$, where f is the dimensionless frequency and ϵ the dimensionless temperature distance from the critical point.

I. INTRODUCTION

Owing to the coupling of sound waves with the order parameter of the phase transition, sound attenuation and dispersion have become important probes to gain information on the order-parameter dynamics of phase transitions.[1,2] Since the motions of the order parameter slow down enormously near the critical point, strong attenuation and dispersion of sound are expected near that point as are often observed experimentally.

On the other hand, many of the recent theoretical efforts which incorporate the scaling law ideas have been directed to low-frequency sound attenuation which can be described in terms of ordinary transport coefficients.[3] The efforts met only partial successes in accounting for available experiments.[1,2,4] The causes for failure may be found in our ignorance of important mechanisms contributing to low-frequency attenuation[5] as much as in possible inadequacy of the theory itself.

Under this circumstance, I have undertaken to extend the theory to higher frequencies with the purpose of providing additional areas of confrontation between theory and experiments. Recently, I presented a theory of sound attenuation and dispersion near the liquid-gas transition[6] which compared favorably with experiment.[7] I have also considered magnetic transitions[8] and the λ transition of liquid helium.[9] Quite unexpectedly, a rather striking similarity was found among these systems in the following aspect. There exists a region of sound-wave frequency which is much greater than the characteristic frequency of order-parameter dynamics Ω_κ but is much smaller than $c\kappa$, where c is the sound velocity and κ^{-1} the correlation range of local order-parameter fluctuations, because $\Omega_\kappa \propto \kappa^\theta$, with $\theta > 1$ and c, at most, approaching zero very slowly.[8] In this high-frequency region the sound attenuation and dispersion behave roughly in the same way as for the systems considered above. The method used to reach this conclusion is the simplest version of the mode-mode coupling theory and uses the specific forms for the short-time behavior of order-parameter time-correlation functions.

In view of the apparent universality of the high-frequency behavior mentioned above, the problem was again considered in a general way. Here this problem will be studied using the most general version of the mode-mode coupling theory plus static[10] and dynamic scaling ideas.[11] Otherwise, only some general properties of order-parameter time-correlation functions will be used, and a somewhat refined version of the conclusion given above will be reached regarding high-frequency sound attenuation and dispersion.

Section II is devoted to analysis of the complex sound-attenuation coefficient, followed by Secs. III and IV, which deal with high-frequency sound attenuation and dispersion, respectively.

II. COMPLEX SOUND-ATTENUATION COEFFICIENT

We start from the following general correlation-function expression for the complex sound-attenuation coefficient[12] $\hat{\alpha}(\omega)$ when the sound wavelength c/ω is much longer than the range of correlation of critical fluctuations κ^{-1}:

$$\hat{\alpha}(\omega) = \omega^2 (1/V) \int_0^\infty dt\, (\delta X(t), \delta X^\dagger(0)) e^{i\omega t}, \qquad (2.1)$$

where V is the volume of the system; $\delta X \equiv X - \langle X \rangle$, with X the appropriate flux that describes the coupling of the sound wave with the internal degrees of freedom that are responsible for the attenuation and dispersion; and the bracket notation stands for Kubo's canonical correlation

$$(A, B) = \beta^{-1} \int_0^\beta \langle e^{\lambda H} A e^{-\lambda H} B \rangle\, d\lambda\ , \qquad (2.2)$$

with $\beta = 1/k_B T$ and H the system Hamiltonian.[1,2]

For example,

$$X=(1-\mathcal{P})I^{xx}/(k_BTc^3\rho)^{1/2}$$

for fluids where I^{xx} is the molecular expression for the xx component of the local stress tensor integrated over V. $\mathcal{P}$ is Mori's projection operator onto the space of hydrodynamical variables,[13] and ρ is the mass density where we ignored unimportant contribution of heat conduction to $\hat{\alpha}(\omega)$.[14] Also for Heisenberg magnets,

$$X=(1-\mathcal{P}_H)\sum_{ij} U_{ij}\vec{S}_i\cdot\vec{S}_j ,$$

where $\vec{S}_i$ is the spin operator on the ith lattice site, and $\mathcal{P}_H$ designates the projection operator onto the magnetic Hamiltonian.[2,5]

According to the mode-mode coupling idea,[4,6,14] the critical anomaly in $\hat{\alpha}(\omega)$ is caused by the break-up of a sound wave into a multiplicity of critical modes. Here the critical modes are understood to be made up of the local order parameter and other hydrodynamic modes that couple with the order parameter in such a way as to determine the order-parameter dynamics self-consistently.[15] We denote the set of Fourier-transformed density operators of critical modes by $a_{\vec{k}}^j$ which are arranged in a column matrix $a_{\vec{k}}$, where $\langle a_{\vec{k}}\rangle=0$. The multiplicity of critical modes into which a sound wave breaks up is then given by a set of products like

$$a_{\vec{k}_1}^{j_1}a_{\vec{k}_2}^{j_2}\cdots a_{\vec{k}_n}^{j_n}-\langle a_{\vec{k}_1}^{j_1}a_{\vec{k}_2}^{j_2}\cdots a_{\vec{k}_n}^{j_n}\rangle , \tag{2.3}$$

which we denote by $A_{\vec{k}_1\cdots\vec{k}_n}^{j_1\cdots j_n}$. We also use renormalized version of A's:

$$\mathfrak{a}_{\vec{k}_1\cdots\vec{k}_n}^{j_1\cdots j_n}\equiv A_{\vec{k}_1\cdots\vec{k}_n}^{j_1\cdots j_n}/(A_{\vec{k}_1\cdots\vec{k}_n}^{j_1\cdots j_n}, A_{\vec{k}_1\cdots\vec{k}_n}^{j_1\cdots j_n\dagger})^{1/2}, \tag{2.4}$$

which are arranged as a column matrix denoted by $\mathfrak{a}$. These variables are essentially identical to the critical dynamical variables introduced previously.[15]

The break-up of a sound wave is then described by projecting X onto the space of the variables $\mathfrak{a}$, which is formally expressed in matrix notation as[16]

$$\delta X\to(\delta X,\mathfrak{a}^\dagger)\cdot(\mathfrak{a},\mathfrak{a}^\dagger)^{-1}\cdot\mathfrak{a} . \tag{2.5}$$

Thus, the anomalous part of complex sound attenuation, which is again denoted as $\hat{\alpha}(\omega)$, becomes

$$\hat{\alpha}(\omega)=\omega^2\int_0^\infty e^{i\omega t}\Psi(t)\,dt, \tag{2.6}$$

$$\Psi(t)=(1/V)(\delta X,\mathfrak{a}^\dagger)\cdot(\mathfrak{a},\mathfrak{a}^\dagger)^{-1}\cdot(\mathfrak{a}(t),\mathfrak{a}^\dagger(0))\cdot(\mathfrak{a},\mathfrak{a}^\dagger)^{-1}\cdot(\mathfrak{a},\delta X^\dagger). \tag{2.7}$$

This result shows that $\hat{\alpha}(\omega)$ can be obtained from the knowledge of critical dynamics contained in $(\mathfrak{a}(t),\mathfrak{a}^\dagger(0))$ provided we know the coupling of X to $\mathfrak{a}$ which is represented by certain equal time correlations involving X and $\mathfrak{a}$.

We may also write

$$\Psi(t)=(I(t),I^\dagger(0)), \tag{2.8}$$

with

$$I\equiv V^{-1/2}(\delta X,\mathfrak{a}^\dagger)\cdot(\mathfrak{a},\mathfrak{a}^\dagger)^{-1}\cdot\mathfrak{a} . \tag{2.9}$$

The equal time correlations of the form $(\delta X, G)$, where G is an arbitrary operator, have been studied previously. One finds, for example, for fluids,[14]

$$(\delta X, G)=iT\rho^{-1/2}c^{-3/2}C_V^{-1}\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial\langle G\rangle}{\partial T}\right)_S, \tag{2.10}$$

and for magnets,[8,16]

$$(\delta X, G)=B'\left(\frac{\partial\langle G\rangle}{\partial T}\right)_S , \tag{2.11}$$

where C_V is the heat capacity at constant volume per unit volume, ρ the mass density, and B' remains finite at the critical point.[17] In general, one may write

$$(\delta X, G)=B'\epsilon^w\left(\frac{\partial\langle G\rangle}{\partial T}\right)_S , \tag{2.12}$$

where $\epsilon=|T-T_c|/T_c$ is the dimensionless temperature distance from the critical point, B is a finite number at the critical point and w is an exponent. For example, $w=\frac{1}{4}\alpha$ for liquid-gas transitions, and $w=0$ for magnets[17] and the λ transition of liquid helium.

In the Appendix it will be shown that the weak scaling law[18] yields for $(\delta X,\mathfrak{a}^\dagger)$

$$(\delta X,\mathfrak{a}_{\vec{k}_1\cdots\vec{k}_n}^{j_1\cdots j_n\dagger})=\epsilon^{w-1}G\left(\left\{\frac{\vec{k}'}{\kappa}\right\},\left\{\frac{\vec{k}_i}{k_j}\right\}, V\kappa^d, \{Vk_i^d\}\right), \quad \text{for } k_{\min}\lesssim\kappa \tag{2.13a}$$

$$(\delta X,\mathfrak{a}_{\vec{k}_1\cdots\vec{k}_n}^{j_1\cdots j_n\dagger})=\epsilon^{w-\alpha}k_{\min}^{-p}G\left(\left\{\frac{\vec{k}_i}{k_{\min}}\right\}, Vk_{\min}^d\right), \quad \text{for } k_{\min}\gg\kappa \tag{2.13b}$$

where $k_{\min}$ is the smallest of the wave numbers referring to the local order parameters contained in $\mathfrak{a}_{\vec{k}_1\cdots\vec{k}_n}^{j_1\cdots j_n}$. $\{\vec{k}'\}$ in (2.13a) is the set of $\vec{k}$'s among these wave numbers which are of the order of κ, whereas k_i and k_j are much greater than κ. Thus $(\delta X,\mathfrak{a}^\dagger)$ is a column matrix which again has the form (2.13) where G is now a column matrix whose elements are of the form of the functions G appearing in (2.13).

In the following we shall use the fact[15] that the product of two functions or matrices of type G or the inverse of such a matrix are again of type G. Here type G means that it contains only quantities like Vk_i^d, $V\kappa^d$ or the dimensionless ratios involving various wave numbers as well as κ (plus directions of various wave vectors). Then, in the end when the thermodynamic limit $V\to\infty$ is taken, quantities like Vk_i^d and $V\kappa^d$ must disappear. Thus the dependences of G upon such quanities may be ignored from the outset.

Thus we split up I defined by (2.9) as $I=\sum_{\vec{k}} I_{\vec{k}}$, where $\mathfrak{a}_{\vec{k}_1\cdots\vec{k}_n}^{j_1\cdots j_n\dagger}$ of $(\delta X, \mathfrak{a}^\dagger)$ appearing in $I_{\vec{k}}$ contains wave numbers of the local order parameters whose minimum is k. [$\mathfrak{a}$'s contain the local order parameters as well as the densities of other conserved quantities which together determine the characteristic frequency (2.22) below.] Here it is convenient to write

$$I=I^{<}+I^{>}, \tag{2.14}$$

with

$$I^{<}\equiv\sum_{\vec{k}}^{k<\kappa} I_{\vec{k}},\quad I^{>}\equiv\sum_{\vec{k}}^{k>\kappa} I_{\vec{k}}. \tag{2.15}$$

Then, one has

$$\Psi(t)=\Psi^{\ll}(t)+\Psi^{<>}(t)+\Psi^{><}(t)+\Psi^{\ll}(t), \tag{2.16}$$

where

$$\Psi^{<>}(t)=(I^{<}(t), I^{>}(0)^\dagger),\quad \text{etc.} \tag{2.17}$$

Let us now investigate the properties of each term in (2.16). First consider $\Psi^{\ll}(t)$. Here (2.13a) is written in the matrix form:

$$(\delta X, \mathfrak{a}^{<\dagger})=\epsilon^{W-1}G^{<}, \tag{2.18}$$

where $\mathfrak{a}^{<}$ is a part of $\mathfrak{a}$ where $k_{\min}\le\kappa$, and an element of the row matrix $G^{<}$ has the form (2.13a). Then one has

$$\Psi^{\ll}(t)=\frac{1}{V}\epsilon^{2(w-1)}\sum_{\vec{k}}^{<}\sum_{\vec{l}}^{<}G_{\vec{k}}^{<}\cdot(\mathfrak{a},\mathfrak{a}^\dagger)^{-1}\cdot\Xi(t)\cdot G_{\vec{l}}^{<\dagger}, \tag{2.19}$$

where $\vec{G}_{\vec{k}}^{<}$ is a part of $\vec{G}^{<}$ where $\vec{k}_{\min}$ equals $\vec{k}$, and $\sum_{\vec{k}}^{<}$ is the sum over the $\vec{k}$ such that $k<\kappa$, and

$$\Xi(t)\equiv(\mathfrak{a}(t),\mathfrak{a}^\dagger(0))\cdot(\mathfrak{a},\mathfrak{a}^\dagger)^{-1} \tag{2.20}$$

describes time development of the variable $\mathfrak{a}$ such that $\Xi(0)\equiv\overleftrightarrow{1}$, a unit matrix. Now, we have

$$\sum_{\vec{k}}\cdots=(V\kappa^3)\int\frac{d\vec{k}}{(2\pi)^3}\kappa^{-3}\cdots,$$

and $(\mathfrak{a},\mathfrak{a}^\dagger)$ and its inverse matrix are of the type G. Therefore, assuming that (2.19) reduces to a finite value in the thermodynamic limit in which $V\kappa^3$ drops out, we obtain

$$\Psi^{\ll}(t)=L_1\epsilon^{2(w-1)-3\nu}\int^{<}d\vec{k}\int^{<}d\vec{l}\,g_{\vec{k}\vec{l}}^{\ll}(t), \tag{2.21}$$

where $g_{\vec{k}\vec{l}}^{\ll}(t)$ is a dimensionless function of time; L_1 is a finite constant; and $\int^{<}d\vec{k}$ is the integral over the $\vec{k}$ such that $k<\kappa$. Upon integration over $\vec{l}$, (2.21) further reduces to

$$\Psi^{\ll}(t)=\epsilon^{2(w-1)}\int^{<}d\vec{k}\,\tilde{g}_{\vec{k}}^{<}(t), \tag{2.22}$$

where $\tilde{g}_{\vec{k}}^{<}(t)$ is a function of time that remains finite at $t=0$.

Next, we consider $\Psi^{\gg}(t)$ which, using (2.13b), can be written in a form similar to (2.19):

$$\Psi^{\gg}(t)=\frac{1}{V}\epsilon^{2(w-\alpha)}\sum_{\vec{k}}^{>}\sum_{\vec{l}}^{>}k^{-p}l^{-p}G_{\vec{k}}^{>}\cdot(\vec{\mathfrak{a}},\vec{\mathfrak{a}}^\dagger)\cdot\Xi(t)\cdot G_{\vec{l}}^{>\dagger}, \tag{2.23}$$

where $G_{\vec{k}}^{>}$ is a part of the row matrix $G^{>}$ in which $k_{\min}$ equals k and the elements of $G^{>}$ have the form of G given by (2.13b). Then, using (2.13b) and noting that $(\mathfrak{a},\mathfrak{a}^\dagger)^{-1}$ is of type G and that

$$\sum_{\vec{k}}\cdots=\frac{V}{(2\pi)^3}\int d\vec{k}\cdots,$$

and assuming again that (2.23) reduces to a finite value in the thermodynamic limit, we find

$$\Psi^{\gg}(t)=L_2\epsilon^{2(w-\alpha)}\int^{>}d\vec{k}\int^{>}d\vec{l}\,k_{\min}^{-3}k^{-p}l^{-p}g_{\vec{k}\vec{l}}^{\gg}(t), \tag{2.24}$$

where $k_{\min}$ is the smaller of k and l, $g_{\vec{k}\vec{l}}^{\gg}(t)$ is a dimensionless function of time, and L_2 is a finite constant. Performing integration over $\vec{l}$, (2.24) becomes

$$\Psi^{\gg}(t)=\epsilon^{2(w-\alpha)}\int^{>}d\vec{k}\,k^{-2p}\tilde{g}_{\vec{k}}^{>}(t), \tag{2.25}$$

where $\tilde{g}_{\vec{k}}^{>}(t)$ is a function of t that remains finite at $t=0$.

Next, we take up $\Psi^{<>}(t)$, which we write in the same way as (2.19) and (2.23) as follows:

$$\Psi^{<>}(t)=\frac{1}{V}\epsilon^{2w-1-\alpha}\sum_{\vec{k}}^{<}\sum_{\vec{l}}^{>}l^{-p}G_{\vec{k}}^{<}\cdot(\mathfrak{a},\mathfrak{a}^\dagger)^{-1}\cdot\Xi(t)\cdot G_{\vec{l}}^{>\dagger}. \tag{2.26}$$

In a manner similar to that used before, we find that (2.26) has the following form:

$$\Psi^{<>}(t)=L_3\epsilon^{2w-1-\alpha}\int^{<}d\vec{k}\int^{>}d\vec{l}\,l^{-p}\kappa^{-3}(l/\kappa)^{-r}f(k/\kappa)g_{\vec{k}\vec{l}}^{<>}(t), \tag{2.27}$$

where $g_{\vec{k}\vec{l}}^{<>}(t)$ is a dimensionless function of time and L_3 is a finite constant. The factor $(l/\kappa)^{-r}$ arises from the fact that in the arguments that lead to (2.27) the factor κ^{-3} in the integrand can be l^{-3} or more generally $\kappa^{-3}(l/\kappa)^{-r}$. So far, r is arbitrary.

To proceed further with the analysis we now introduce the dynamical scaling hypotheses for the functions g's that describe the order-parameter dynamics. Namely, for $k\le\kappa$ and $l>\kappa$ we assume

$$\tilde{g}_{\vec{k}}^{<}(t)=\tilde{g}(\Omega_k t, k/\kappa), \tag{2.28a}$$

$$\tilde{g}_{\vec{1}}^{>}(t) = \tilde{g}(\Omega_l^c t) \,, \qquad (2.28b)$$

$$g_{\vec{k}\vec{1}}^{<>}(t) = g^{<>}(\Omega_k t, \Omega_l^c t, k/\kappa), \qquad (2.28c)$$

where Ω_k is the characteristic frequency of the order-parameter dynamics given by

$$\Omega_k = k^\theta d(k/\kappa), \quad \theta > 0 \qquad (2.29)$$

and Ω_k^c is the asymptotic form in the critical regime $k \gg \kappa$, where

$$\Omega_k^c = Dk^\theta, \qquad (2.30)$$

with $D \equiv d(\infty)$.

Now, for $k \ll l$, $g^{<>}$ becomes $g^{<>}(\Omega_k t, \infty, k/\kappa)$ for $t \sim \Omega_k^{-1}$ and $g^{<>}(0, \Omega_l^c t, k/\kappa)$ for $t \sim (\Omega_l^c)^{-1}$. Thus, effectively $g^{<>}$ takes the form

$$g^{<>}(\Omega_k t, \Omega_l^c t, k/\kappa) = g_1(\Omega_k t, k/\kappa) + g_2(\Omega_l^c t, k/\kappa). \qquad (2.31)$$

First, consider the contribution from g_1 to (2.27). The integration over $\vec{1}$ converges at large l when

$$p + r > 3. \qquad (2.32)$$

We require (2.32), since otherwise fluctuations with microscopic wavelengths give important contributions to Ψ and hence to the sound attenuation; in this work we are solely concerned with the cases where the critical fluctuations of semimacroscopic wavelengths dominate the attenuation. Thus, in the integral over $\vec{1}$, major contributions arise from $l \sim \kappa$, where we should take for p the strong scaling value $p = (1-\alpha)/\nu$. Then we discover that this contribution to $\Psi^{<>}$ takes precisely the same form as (2.22) with $\tilde{g}_k^{<}$ replaced by a different function of time which still has the form (2.28a). Next, we consider a contribution from g_2. After integration over $\vec{k}$, it takes the following form (write $\vec{k}$ instead of $\vec{1}$):

$$\epsilon^{2(w-\alpha)} \int^{>} d\vec{k} \; k^{-2p} \frac{\kappa^{r-(1-\alpha)/\nu}}{k^{r-p}} \; \tilde{g}_2(\Omega_k^c t). \qquad (2.33)$$

This differs from $\Psi^{\gg}(t)$ [Eq. (2.25)] by the presence of a factor $\kappa^{r-(1-\alpha)/\nu}/k^{r-p}$. Using (2.32), we have

$$\kappa^{r-(1-\alpha)/\nu}/k^{r-p} < \kappa^{3-p-(1-\alpha)/\nu}/k^{3-2p}. \qquad (2.34)$$

If we use the strong scaling hypothesis where $p = (1-\alpha)/\nu$ and $3\nu = 2-\alpha$, (2.34) becomes $(\kappa/k)^\alpha$, which is smaller than unity for $\alpha > 0$. In this case, (2.33) can be ignored compared with $\Psi^{\gg}(t)$. Even when the strong scaling does not hold, we shall assume that at most $\kappa^{r-(1-\alpha)/\nu}/k^{r-p}$ does not exceed the order unity when k and κ are expressed in the unit of inverse microscopic distance.[19] $\Psi^{><}(t)$ can be taken care of by noting that $\Psi^{><}(t) = \Psi^{<>}(-t)^*$.

The results of the analysis of this section are now summarized by stating that $\Psi(t)$ has the following general form:

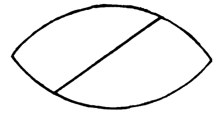

FIG. 1. A higher-order diagram.

$$\Psi(t) = \epsilon^{2(w-1)} \int_0^\kappa dk\, k^2 f(k/\kappa) g_k^{<}(t) + L\epsilon^{2(w-\alpha)} \times \int_\kappa^\infty dk\, k^{2-2p} g_k^{>}(t), \qquad (2.35)$$

where the dynamical scaling hypotheses give for $g_k^{\gtrless}(t)$

$$g_k^{<}(t) = g(\Omega_k t, k/\kappa), \quad g_k^{>}(t) = g(\Omega_k^c t) \,, \qquad (2.36)$$

with Ω_k and Ω_k^c given by (2.29) and (2.30), respectively, and L is a finite constant. We can further require that $g_k^{\gtrless}(0) = 1$, and $g_k^{\gtrless}(\infty) = 0$.

Corresponding to (2.35), $\hat{\alpha}(\omega)$ can be split as

$$\hat{\alpha}(\omega) = \hat{\alpha}^{<}(\omega) + \hat{\alpha}^{>}(\omega). \qquad (2.37)$$

Equations (2.35)–(2.37) are then sufficient for finding the high-frequency behavior of sound attenuation and dispersion in the critical region.

III. SOUND ATTENUATION

By (2.36) we have

$$\mathrm{Re}\int_0^\infty e^{i\omega t} g_k^{<}(t)\omega \, dt = \mathrm{Re}\int_0^\infty d\tau\, e^{i\tau} g^{<}\left(\frac{\Omega_k}{\omega}\tau, \frac{k}{\kappa}\right) \equiv F^{<}\left(\frac{\omega}{\Omega_k}, \frac{k}{\kappa}\right), \qquad (3.1a)$$

$$\mathrm{Re}\int_0^\infty e^{i\omega t} g_k^{>}(t)\omega \, dt = \mathrm{Re}\int_0^\infty d\tau\, e^{i\tau} g^{>}\left(\frac{\Omega_k^c}{\omega}\tau\right), \equiv F^{>}\left(\frac{\omega}{\Omega_k^c}\right). \qquad (3.1b)$$

Thus the sound attenuation coefficient $\alpha(\omega) = \mathrm{Re}\hat{\alpha}(\omega)$ is

$$\alpha(\omega) = \alpha^{>}(\omega) + \alpha^{<}(\omega), \qquad (3.2)$$

$$\alpha^{<}(\omega) = \omega\epsilon^{2(w-1)} \int_0^\kappa dk\, k^2 f\left(\frac{k}{\kappa}\right) F^{<}\left(\frac{\omega}{\Omega_k}, \frac{k}{\kappa}\right), \qquad (3.3a)$$

$$\alpha^{>}(\omega) = L\omega\epsilon^{2(w-\alpha)} \int_\kappa^\infty dk\, k^{2-2p} F^{>}\left(\frac{\omega}{\Omega_k^c}\right). \qquad (3.3b)$$

Now, in the high-frequency region $\omega \gg \Omega_k$, in the integrand of (3.3a), the condition $\omega \gg \Omega_k$ is always satisfied. Thus we must examine $F^{<}(y, x)$ as $y \to \infty$, where

$$F^{<}(y. x) = \mathrm{Re}\int_0^\infty d\tau\, e^{i\tau} g^{<}\left(\frac{\tau}{y}, x\right). \qquad (3.4)$$

The functions $g^{<}$ and $g^{>}$ are determined by the dynamics of critical fluctuations in a complex way, and, in general, explicit expressions for them cannot be obtained. However, we know certain general properties of these functions. For example, for large y, we can expand $g^{<}(\tau/y, x)$ as

$$g^{<}(\tau/y, x) = 1 - Q'(x)(\tau/y)^{\mu} + \cdots, \quad \mu > 0 \quad (3.5)$$

where $Q'(x)$ is some function, μ is some positive exponent, the normalization condition $g^{<}(0, x) = 1$ has been used, and $\cdots$ stands for higher powers in τ/y. Thus we find for $y \gg 1$

$$F^{<}(y, x) = Q(x)\, y^{-\mu} + \cdots, \quad (3.6)$$

where $Q(x)$ is the same as $Q'(x)$ apart from a numerical factor.[20] Thus we find for $\omega \gg \Omega_\kappa$

$$\alpha^{<}(\omega) = \epsilon^{2(w-1)+3\nu+\mu\nu\theta}\omega^{1-\mu}\kappa_0^{3+\mu\theta} \times \int_0^1 dx\, x^{2+\mu\theta} f(x) Q(x)[d(x)]^{\mu}, \quad (3.7a)$$

where $\kappa = \kappa_0 \epsilon^{\nu}$.

Turning now to $\alpha^{>}(\omega)$, we find by (2.30) and (3.3b) that

$$\alpha^{>}(\omega) = \epsilon^{2(w-\alpha)}\omega^{1+(3-2p)/\theta} L D^{(2p-3)/\theta} \times \int_0^{\infty} dx\, x^{2-2p} F^{>}(x^{-\theta}), \quad \text{as } \kappa \to 0. \quad (3.7b)$$

Assuming the integrals in (3.7) are finite, these results are summarized by introducing the dimensionless frequency $f = \omega/\omega_0$, where ω_0 is some microscopic frequency such as

$$\alpha^{<}(\omega) \propto f^{1-\mu}\epsilon^{2(w-1)+(3+\mu\theta)\nu}, \quad (3.8a)$$

$$\alpha^{>}(\omega) \propto f^{1+(3-2p)/\theta}\epsilon^{2(w-\alpha)}. \quad (3.8b)$$

If we use the strong scaling $p = (1-\alpha)/\nu$, the ratio becomes

$$\frac{\alpha^{<}(\omega)}{\alpha^{>}(\omega)} \sim \left(\frac{\epsilon^{\nu\theta}}{f}\right)^{\mu+\alpha/\nu\theta},$$

which is much smaller than unity since $f \gg \epsilon^{\nu\theta}$ (the high-frequency condition).

Thus the high-frequency sound attenuation is now given by

$$\alpha(\omega) \propto \epsilon^{2(w-\alpha)} f^{1+(3-2p)/\theta}; \quad (3.9)$$

$$\alpha(\omega) \propto \epsilon^{2(w-\alpha)} f^{1+\alpha/\nu}, \quad \text{if} \quad p = (1-\alpha)/\nu, \quad 3\nu = 2 - \alpha. \quad (3.10)$$

If we use the fact that the low-frequency sound attenuation is

$$\alpha(\omega) = A(\epsilon)\omega^2, \quad A(\epsilon) \propto \epsilon^{2w-\alpha-\theta\nu}, \quad (3.11)$$

we can eliminate w and write (3.10) as

$$\alpha(\omega) \propto A(\epsilon)\epsilon^{\theta\nu-\alpha} f^{1+\alpha/\nu}. \quad (3.12)$$

Furthermore, if $\alpha = 0$, then $w = 0$ for liquid-gas transitions, or magnets, or the λ transition of helium. In this case, we find

$$\alpha(\omega) \propto f\epsilon^0. \quad (3.13)$$

The results obtained here agree with those obtained for the cases of liquid-gas transitions,[6] magnets,[8] and the λ transition in helium.[9] The method used here is more general than the previous calculations and does not depend upon the particular approximations or the specific forms for time-correlation functions of critical fluctuations.

IV. SOUND DISPERSION

Sound dispersion can be studied in essentially the same way as attenuation was. If $c(\omega)$ is the sound velocity at the frequency ω, the sound dispersion is given in terms of $\Delta c \equiv c(\omega) - c$ as

$$\frac{\Delta c}{c} = \left(\frac{\Delta c}{c}\right)_{<} + \left(\frac{\Delta c}{c}\right)_{>} = (c/2\omega)\, \mathrm{Im}(\hat{\alpha}_{<}(\omega) + \hat{\alpha}_{>}(\omega)). \quad (4.1)$$

Equations (2.6), (2.29), (2.30), and (2.35)–(2.37), then yield

$$\left(\frac{\Delta c}{c}\right)_{<} = \tfrac{1}{2}c\epsilon^{2(w-1)}\int_0^{\kappa} dk\, k^2 f\left(\frac{k}{\kappa}\right) H^{<}\left(\frac{\omega}{\Omega_k}, \frac{k}{\kappa}\right), \quad (4.2a)$$

$$\left(\frac{\Delta c}{c}\right)_{>} = \tfrac{1}{2}c\epsilon^{2(w-\alpha)}\int_{\kappa}^{\infty} dk\, k^{2-2p} L H^{>}\left(\frac{\omega}{\Omega_k^c}\right), \quad (4.2b)$$

where

$$H^{<}(y, x) \equiv \mathrm{Im}\int_0^{\infty} d\tau\, e^{i\tau} g^{<}\left(\frac{\tau}{y}\, x\right), \quad x \lesssim 1 \quad (4.3a)$$

$$H^{>}(y) \equiv \mathrm{Im}\int_0^{\infty} d\tau\, e^{i\tau} g^{>}\left(\frac{\tau}{y}\right). \quad (4.3b)$$

For $\tau \ll y$, $g^{<}(\tau/y, x) = 1 + \cdots$, and therefore,

$$H^{<}(y, x) = 1 + \cdots, \quad y \gg 1. \quad (4.4)$$

Thus, for high frequencies $\omega \gg \Omega_\kappa$, we find

$$(\Delta c/c)_{<} = \tfrac{1}{2}c\epsilon^{2(w-1)+3\nu}\kappa_0^{3\nu}\int_0^1 dx\, x^2 f(x), \quad (4.5a)$$

$$(\Delta c/c)_{>} = \tfrac{1}{2}c\epsilon^{2(w-\alpha)}\omega^{(3-2p)/\theta} L D^{(2p-3)/\theta} \times \int_{\kappa(D/\omega)^{1/\theta}}^{\infty} dx\, x^{2-2p} H^{>}(x^{-\theta}). \quad (4.5b)$$

If we consider the fact that for $3 - 2p \le 0$, the lower limit of the integral of (4.5b) cannot be replaced by zero as $\kappa \to 0$ since $H^{>}(\infty) = 1$, we obtain

$$(\Delta c/c)_{<} \propto c\epsilon^{2(w-1)+3\nu} f^0, \quad (4.6a)$$

$$(\Delta c/c)_> \propto c\epsilon^{2(w-\alpha)}f^{(3-2p)/\theta}, \quad 3-2p \geq 0$$
$$\propto c\epsilon^{2(w-1)+3\nu}f^0, \quad 3-2p<0. \quad (4.6b)$$

Here we have used the fact that for $3-2p<0$ the fluctuations with $k \gg \kappa$ are unimportant, and hence (4.5b) cannot be used, and $(\Delta c/c)_> \sim (\Delta c/c)_<$. For $3-2p=0$, (4.5b) gives an additional factor $\ln[\kappa(D/\omega)^{1/\theta}]$, which we have dropped since we consider the theory not refined enough to justify including it. The ratio of the two contributions is

$$(\Delta c/c)_< /(\Delta c/c)_> \sim \epsilon^{2(\alpha-1)+3\nu}f^{(2p-3)/\theta}, \quad 3-2p \geq 0$$
$$\sim \epsilon^0 f^0, \quad 3-2p<0. \quad (4.7)$$

When the strong scaling $p=(1-\alpha)/\nu$ and $2=3\nu+\alpha$ hold and further $\alpha=0$, we obtain

$$\Delta c/c \propto \epsilon^0 f^0. \quad (4.8)$$

For the case of liquid-gas transitions where $\theta=3$ and $w=\frac{1}{4}\alpha$, the results (4.6) agree with our previous calculation[6] except that in Ref. 6 the scaling law relation $2=3\nu+\alpha$ was always assumed, which is expected to break down if the strong scaling does not hold.

It is interesting to note that (4.6) can be also derived using the Kramers-Kronig dispersion relations, which are

$$\frac{\alpha(\omega)}{\omega^2}=\frac{2}{c^2}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\frac{\Delta c(\omega')/c}{(\omega'-\omega)\omega'}, \quad (4.9)$$

$$\frac{\Delta c(\omega)}{c}=-\frac{c\omega}{2}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\frac{\alpha(\omega')}{(\omega'-\omega)\omega'^2}. \quad (4.10)$$

In view of Eqs. (3.8a) and (3.8b), the integral in (4.10) converges at large $|\omega'|$. Thus for high frequencies ω, we may take $|\omega'| \lesssim |\omega|$ in (4.10). Now, contributions to (4.10) from small $\omega' \ll \Omega_\kappa$ are proportional to ω since $\alpha(\omega') \propto \omega'^2$ for small ω', and turn out to be small. Thus the major contribution to the integral in (4.10), where $\alpha(\omega')$ is replaced by $\alpha^<(\omega')$, comes from $\omega' \sim \Omega_\kappa$, where we can use (3.8a) for $\alpha^<(\omega')$ because the low-frequency and high-frequency forms for $\alpha^<(\omega')$ should match at $\omega'=\Omega_\kappa$. Therefore, we find

$$\left(\frac{\Delta c(\omega)}{c}\right)_< \cong \frac{c}{2}\int_{\omega'\sim\Omega_\kappa}\frac{d\omega'}{\pi}\frac{\alpha^<(\omega')}{\omega'^2} \sim c\frac{\alpha^<(\Omega_\kappa)}{\Omega_\kappa}$$
$$\propto c\epsilon^{2(w-1)+(3+\mu\theta)\nu}\Omega_\kappa^{-\mu},$$

which is identical to (4.6a) since $\Omega_\kappa \propto \epsilon^{\nu\theta}$. Next, we consider the integral (4.10) where $\alpha(\omega')$ is replaced by $\alpha^>(\omega')$. If $3-2p \geqq 0$, then major contributions to the integral come from $|\omega'| \sim |\omega|$ and Eq. (4.10) $\sim c\alpha^>(\omega)/\omega \sim c\epsilon^{2(w-\alpha)}f^{(3-2p)/\theta}$. If $3-2p<0$, major contributions are from $\omega \sim \Omega_\kappa$; $\alpha^>$ takes the same form as $\alpha^<$ in this case, and Eq. (4.10) $\sim c\alpha^<(\omega')/\omega' \sim c\epsilon^{2(w-1)+3\nu}$. These results agree with (4.6b).

Finally, we note that for $3-2p<0$, Δc is very simply related to $\Psi(0)$. In this case, main contributions to the integral (4.5b) come from $x \sim \kappa(D/\omega)^{1/\theta}$ or critical fluctuations with $k \sim \kappa$, and $(\Delta c/c)_<$ come from critical fluctuations with $k<\kappa$. That is, the critical fluctuations with k of the order of κ or smaller whose characteristic frequencies are much smaller than ω give major contributions to

$$\Delta c/c=\tfrac{1}{2}c\omega\int_0^\infty \Psi(t)\sin\omega t\,dt. \quad (4.11)$$

Furthermore, in $\Psi(0)$, which is given by (2.35) as

$$\Psi(0)=\epsilon^{2(w-1)}\int_0^\kappa dk\,k^2 f(k/\kappa)+L\epsilon^{2(w-\alpha)}\int_\kappa^\infty dk\,k^{2-2p}, \quad (4.12)$$

the fluctuations with $k \gg \kappa$ are unimportant because the integral in the second term converges at $k=\infty$. Thus we can totally ignore high-frequency components in $\Psi(t)$ in (4.11) and in $\Psi(0)$, which allows us to replace $\Psi(t)$ in (4.11) by $\Psi(0)$ to obtain

$$\frac{c_\infty-c}{c}=\frac{\Delta c}{c}=\frac{c}{2}\Psi(0)$$
$$=\frac{c}{2V}(\delta X,\mathfrak{a}^\dagger)\cdot(\mathfrak{a},\mathfrak{a}^\dagger)^{-1}\cdot(\mathfrak{a},\delta X^\dagger), \quad (4.13)$$

where c_∞ is the sound velocity at high frequency.

In this manner, when $3-2p<0$, high-frequency dispersion yields information on the strength of coupling between the sound wave and critical fluctuations which do not involve relaxation times. Furthermore, if we combine this with low-frequency attenuation divided by ω^2,

$$\alpha(\omega)/\omega^2=A=\int_0^\infty \Psi(t)\,dt=\Psi(0)\tau, \quad (4.14)$$

we find for the relaxation time τ defined by (4.14):

$$\tau=\frac{A}{2\Delta c/c^2}. \quad (4.15)$$

This relation looks the same as that pointed out by Bennett.[21] However, in Bennett's case Δc was the anomalous change in the *zero-frequency* sound velocity near the transition which involves the full interaction of the sound wave with order-parameter fluctuations *without* the projection operator $(1-\mathcal{P})$ or $(1-\mathcal{P}_H)$ (see Sec. II). Since in our case Δc and A both involve the same δX, we believe that our relation (4.15) is theoretically more satisfactory, although it is less general and may not be as convenient in practice.

V. DISCUSSION AND CONCLUDING REMARKS

In Secs. II–IV we considered in a rather general way the problem of sound attenuation and dispersion in the frequency region, which is much greater than the characteristic frequency of critical fluctuations.

The method used was a combination of scaling law ideas[22] (both static and dynamic) and mode-mode coupling in its most general form, where, however, we assumed that there exists only one kind of characteristic frequency spectrum $k^\theta d(k/\kappa)$ that dominates the problem.

The most interesting result is that high-frequency sound attenuation and dispersion behave roughly in the same way for various different systems [see (3.13) and (4.8)]. This may be understood very crudely as follows.[23] If there is only one order-parameter relaxation time τ, sound attenuation and dispersion are written as[1]

$$\alpha(\omega)=\frac{(c_\infty^2-c^2)}{2c^3}\,\frac{\omega^2\tau}{1+(\omega\tau)^2}\ , \tag{5.1}$$

$$c(\omega)^2-c^2=\frac{(c_\infty^2-c^2)(\omega\tau)^2}{1+(\omega\tau)^2}\ , \tag{5.2}$$

where c_∞ is the infinite-frequency ("frozen") velocity. However, this picture is oversimplified, and, in fact, we have a whole spectrum of τ_k^{-1} which goes as $k^\theta d(k/\kappa)$ instead of a single τ^{-1}. As long as we are in the region $\omega \ll \tau_k^{-1}=\kappa^\theta d(1)$, most τ_k^{-1} are greater than ω, with the average at τ_κ^{-1}, and attenuation and dispersion behave as though there were a single relaxation time $\tau=\tau_\kappa$. On the other hand, for $\omega\gg\tau_\kappa^{-1}$, most of the relaxation times are much greater than ω^{-1}, and the corresponding relaxation processes cannot effectively contribute to attenuation and dispersion ("frozen"). Thus, there remains only a small contributing part of the spectrum which satisfies $\tau_k^{-1}\gtrsim\omega$. Thus, we may estimate high-frequency attenuation and dispersion by replacing τ by ω^{-1} in (5.1) and (5.2) to obtain

$$\alpha(\omega)\sim\omega(c_\infty^2-c^2)/c^3,\quad \Delta c(\omega)/c\cong(c_\infty^2-c^2)/2c^2,$$

which agree with (3.13) and (4.8).

There appear to be some experimental indications to support this sort of behavior,[24] but further extensive experimental study is clearly desirable.

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APPENDIX

Here let us discuss properties of equal-time correlation functions of long-wavelength fluctuations, and in particular derive (2.13). In all the cases studied, the critical dynamical variables[15] consist of the products of Fourier components of the local order parameters denoted by $\sigma_{\vec{k}}$, which in certain cases also include Fourier components of noncritical conserved variables denoted by $\mu_{\vec{k}}$, where we always take $\langle\sigma_{\vec{k}}\rangle=\langle\mu_{\vec{k}}\rangle=0$. Thus, it is enough to consider a correlation like

$$\langle\sigma_{\vec{q}_1}\cdots\sigma_{\vec{q}_m}\,\mu_{\vec{k}_1}\cdots\mu_{\vec{k}_n}\rangle\ , \tag{A1}$$

which is abbreviated as $\langle\sigma^m\mu^n\rangle$. Elsewhere[15] it has been shown that use of the scaling law ideas[10] gives (A1) the following effective form[25]:

$$\langle\sigma^m\mu^n\rangle=\kappa^{-mx}\kappa^{-[n/2]d}G, \tag{A2}$$

where d is the dimensionality of space; and $x=\frac{1}{2}(d+2-\eta)$, $[n/2]=n/2$ for even n and $n/2-1$ for odd n; and G is some function of $V\kappa^d$, q_i/κ, and k_i/κ. (We always use the same symbol G whenever such a function appears.)

First, since $\mathfrak{a}$ consists of the variables like $\sigma^m\mu^n-\langle\sigma^m\mu^n\rangle$ which are properly normalized so that the canonical correlation (2.2) of a variable with itself reduces to unity, we see that[15]

$$(\mathfrak{a},\mathfrak{a}^\dagger)=\tilde{G}_0+\kappa^{d/2}\tilde{G}_1\ , \tag{A3}$$

where the $\tilde{G}$'s are infinite matrices whose elements have the forms of G.

Next, let us turn to $(\partial\langle\sigma^m\mu^n\rangle/\partial T)_S$. In general, it is expected from (A2) that

$$\left(\frac{\partial\langle\sigma^m\mu^n\rangle}{\partial T}\right)_s=\epsilon^{-1}\kappa^{-mx}\kappa^{-[n/2]d}G. \tag{A4a}$$

However, the situation is different if all $\vec{q}$'s are much greater than κ. Here $\langle\sigma^m\mu^n\rangle$ approaches its asymptotic form at T_c, which no longer depends upon κ, as follows:

$$\langle\sigma^m\mu^n\rangle\to q_1^{-mx}k_1^{-[n/2]d}\times G(Vq_1^d,\,Vk_1^d,\,\{\vec{k}_i/k_j\},\,\{\vec{q}_i/q_j\},\,\{\vec{k}_i/q_j\}), \tag{A4b}$$

which is analogous to the first term of the asymptotic behavior of $\langle\sigma_{\vec{q}}\sigma_{-\vec{q}}\rangle$ as $\kappa\to0$, that is,

$$\langle\sigma_{\vec{q}}\,\sigma_{-\vec{q}}\rangle=\frac{A_1}{q^{2-\eta}}+A_2\,\frac{\kappa^{(1-\alpha)/\nu}}{q^{2-\eta+p}}+\cdots, \tag{A5}$$

with A_1 and A_2 independent of κ and $\vec{q}$. Here p is a new exponent which is equal to $(1-\alpha)/\nu$ if the strong scaling holds, but it may differ from this strong scaling value.[19] An analogous expansion of $\langle\sigma^m\mu^n\rangle$ then will be found by assuming the correction to (A4) corresponding to the second term of (A5) of the following form:

$$q_1^{-mx}k_1^{-[n/2]d}\,(\kappa^{(1-\alpha)/\nu}/q_1^p)G, \tag{A6}$$

where G is of the same type as that appearing in (A4), which no longer contains κ. Namely, we have

$$(\partial\langle\sigma^m\mu^n\rangle/\partial T)_S = \epsilon^{-\alpha} q_1^{-mx-p} k_1^{-[n/2]d} G. \qquad \text{(A7)}$$

We also note that

$$([\sigma^m\mu^n - \langle\sigma^m\mu^n\rangle], [\sigma^m\mu^n - \langle\sigma^m\mu^n\rangle]^\dagger)^{1/2}$$

$$= \kappa^{-mx}\kappa^{-nd/2} G \text{ for all } q_i \sim \kappa,$$

$$= q_1^{-mx} k_1^{-nd/2} G \text{ for all } q_i \gg \kappa. \qquad \text{(A8)}$$

Therefore, we finally find that

$$(\delta X, \mathfrak{a}^\dagger) = \epsilon^{w-1}\kappa^{\{n/2-[n/2]\}d} G \text{ for all } q_i \sim \kappa,$$

$$= \epsilon^{w-\alpha} q_1^{-p} k_1^{\{n/2-[n/2]\}d} G \text{ for all } q_i \gg \kappa. \qquad \text{(A9)}$$

If we ignore the terms with odd n which are small, we finally obtain (2.13).

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Bose-Einstein Condensation in Two Dimensions

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It is proven that for Bose-Einstein condensation, in the sense of quasiaverages, condensation into *any* one-particle state is forbidden at nonzero temperatures in any two-dimensional system, whether under rotation or not, regardless of the external potential, so long as the particle density is bounded everywhere. The proof is based on Bogolyubov's inequality.

I. INTRODUCTION

Because of the work of London,[1] superfluidity has been associated with Bose-Einstein condensation, which, for homogeneous systems, means macroscopic occupation of the zero-momentum single-particle state. A theory of superfluid flow,[2] as well as of the λ transition,[3] has been constructed based on London's[3] prediction.

It has generally been assumed that the fraction of particles in the zero-momentum state is equal to the quasiaverage[4] $|\langle\psi(\vec{r})\rangle|^2$ (superfluid order