calculated values which are close to the measured values for $d_{33}(T, P)$.

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Two-Dimensional System of Hard-Core Bosons*

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As a model of a helium monolayer a system of hard-core bosons of mass m and diameter a constrained to motion in two dimensions is considered at absolute zero. In the low-density limit, the ground-state energy per particle and condensate depletion are found to be $E/N = -2\pi\hbar^2 n/m \ln na^2$ and $n_0 = n(1 + 1/\ln na^2)$, where n is the areal density of the system. The expansion parameter $-1/\ln na^2$ is approximately equal to unity for real helium monolayers. The variation of the above results with temperature is discussed for a system of finite size.

INTRODUCTION

The recent spate of experiments[1] on adsorbed helium monolayers has occasioned the investigation of several models of such systems.[2] One simple model which has received little attention consists of a system of hard-disk bosons constrained to motion in two dimensions. Except for the restriction on the dimensionality, this model ignores the effects of the substrate and the weak attractive interaction between particles, thereby displaying the consequences of the hard-core interaction. Once these are well understood, the remaining interactions can be introduced. As the first step in the above program, the ground-state energy, condensate density, and excitation spectrum of a

system of hard-disk bosons of diameter a are calculated below in the limit of low densities and at the absolute zero of temperature.

It might be objected to that such a calculation, which is predicated upon the existence of a Bose condensate, can say little of the properties of the system at finite temperatures in light of the fact that the infinite two-dimensional boson system possesses no condensate at any finite temperature.[3] However, we are concerned with a model of a real, and therefore finite, system of helium. In such a model, a finite fraction of the bosons occupies the lowest-energy single-particle state for temperatures less than and of the order of $T_0/\ln N$, where T_0 is the condensation temperature in three dimensions and N is the number of particles. For some recent experimental configurations[4] this temperature is typically of the order of 0.1 °K.

CHOICE OF DIAGRAMS

We first consider an interaction $U(r)$ with a well-behaved Fourier transform $U_{\vec{p}}$ and of range a. With $\hbar$ and m, the particle mass, set equal to unity, the dimensionless expansion parameters are

$$\xi = U_0 \quad , \quad \beta = na^2 \ ,$$

where n is the areal number density of particles. The parameter β is assumed to be small. We follow Beliaev[5] in classifying the diagrams and consider various contributions to the anomalous self-energy Σ_{02}. The first-order contribution is shown in Fig. 1(a) and has the magnitude

$$\Sigma_{02}^{a}(\vec{p}) = n_0 U_{\vec{p}} \quad .$$

The only nonzero diagram in second order is shown in Fig. 1(b). The magnitude of this contribution is

$$\Sigma_{02}^{b}(\vec{p}) \propto n_0 \int U_{\vec{q}}\, U_{\vec{p}-\vec{q}}\, G^0(q) G^0(-q) d\vec{q} dq^0 \ ,$$

where

$$G^0(q) = (q^0 + \mu - \epsilon_q + i\delta)^{-1} \ ,$$

$$\epsilon_q = \tfrac{1}{2} q^2 \ .$$

The value of the chemical potential μ may be taken to be the lowest-order contribution to μ which is $n_0 U_0$. Assuming that $U_{\vec{q}}$ is essentially constant for $qa < 1$ and negligible otherwise, the magnitude of the above contribution is found to be

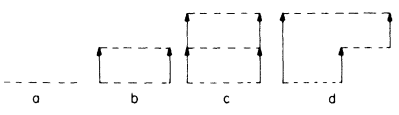

FIG. 1. Some diagrams for the anomalous self-energy Σ_{20}. The dotted lines represent interactions and the solid lines represent propagators G^0.

$$\Sigma_{02}^{b}(\vec{p}) \propto -\Sigma_{02}^{a}(\vec{p}) \xi \ln \xi\beta \ . \tag{1}$$

It is to be noted that this contribution depends on the density parameter β, which is not the case in three dimensions. In third order we may compare the contribution of the diagrams shown in Figs. 1(c) and 1(d). These are

$$\Sigma_{02}^{c}(\vec{p}) \propto n_0 \int d\vec{q}\, dq^0 U_{\vec{q}} G^0(q) G^0(-q) \times \int d\vec{t}\, dt_0\, U_{\vec{t}}\, U_{\vec{p}-\vec{t}}\, G^0(t) G^0(-t)$$

$$\propto \Sigma_{02}^{a}(\vec{p}) (\xi \ln \beta\xi)^2 \tag{2}$$

and

$$\Sigma_{02}^{d}(\vec{p}) \propto \int d\vec{q} dq^0 U_{\vec{q}}^2 U_{\vec{p}-\vec{q}} G^0(q) [G^0(-q)]^2$$

$$\propto \Sigma_{02}^{a}(\vec{p}) \xi^2 \ . \tag{3}$$

Again it should be noted that Σ_{02}^{c} depends on β. As this parameter is small, Σ_{02}^{c} is larger than Σ_{02}^{d} as expected in that the latter arises from the scattering of three particles, a less likely process, in the low-density limit, than repeated scattering of two particles.

From the above analysis, it can be seen that the choice of diagrams to be summed is not as straightforward in two dimensions as in three. In three dimensions, only the so-called ladder diagrams are independent of the small parameter β, while all other diagrams vanish with vanishing β. Therefore the sum of the ladder diagrams gives the first term in the expansions of all quantities in terms of the density. In the calculation of the properties of the two-dimensional system below, we choose to sum the ladder diagrams. That this is the correct choice is demonstrated as follows.

The sum of the repeated interaction of two particles yields an effective interaction which will be shown to be density dependent. In particular, in the limit of small-momentum transfers, less than a^{-1}, the effective interaction has a $(\ln\beta)^{-1}$ dependence. This is to be contrasted with the behavior of the effective interaction in three dimensions which, in the same limit, is equal to a nonzero constant. Returning to the evaluation of the diagrams of Fig. 1 and replacing the bare interaction by the effective interaction, which amounts to replacing ξ by $(\ln\beta)^{-1}$ in Eqs. (1)–(3), one finds the following results: First, diagrams a, b, and c, which are now the same diagram, as the effective interaction is the sum of all repeated interactions, are indeed identical to within any power of the small parameter $(\ln\beta)^{-1}$ which turns out to be the expansion parameter of this problem. Second, the diagram of Fig. 1(d), which involves a third particle, is smaller than the other contributions by the expansion parameter $(\ln\beta)^{-1}$. Thus it is seen that in

two dimensions, as in three, the ladder diagrams provide the largest term in the expansion of quantities in terms of density and that the corrections include terms generated by the scattering of several particles.

GROUND-STATE PROPERTIES

The ladder diagrams are summed in the usual way by means of the integral equation

$$\Gamma(\vec{p}', \vec{p}, P) = U_{\vec{p}'-\vec{p}} + i(2\pi)^{-3} \int d\vec{q}\, dq^0 U_{\vec{p}'-\vec{q}} G^0(\tfrac{1}{2}P + q) \times G^0(\tfrac{1}{2}P - q)\Gamma(\vec{q}, \vec{p}, P).$$

In a manner identical to that of Ref. 5, the solution of this equation can be brought to the following forms which do not depend on the existence of a Fourier transform for the bare interaction:

$$\Gamma(\vec{p}', \vec{p}, P) = f(\vec{p}', \vec{p}) + \int f(\vec{p}', \vec{K}) f^*(\vec{p}, \vec{K}) \left(\frac{1}{K_0^2 - K^2 + 2\mu + i\delta} + \frac{1}{K^2 - p^2 - i\delta} \right) \frac{d\vec{K}}{(2\pi)^2} ,$$

$$\Gamma(\vec{p}', \vec{p}, P) = f^*(\vec{p}, \vec{p}') + \int f(\vec{p}', \vec{K}) f^*(\vec{p}, \vec{K}) \left(\frac{1}{K_0^2 - K^2 + 2\mu + i\delta} + \frac{1}{K^2 - (p')^2 + i\delta} \right) \frac{d\vec{K}}{(2\pi)^2} ,$$

where

$$K_0^2 = P^0 - \tfrac{1}{4}P^2,$$

$$f(\vec{p}', \vec{p}) = \int U(r) \psi_{\vec{p}}(\vec{r}) e^{-i\vec{p}'\cdot\vec{r}} d\vec{r} .$$

The function $\psi_{\vec{p}}(\vec{r})$ is the wave function for a scattering problem between unit-mass particles interacting with the potential. As the reduced mass is $\frac{1}{2}$, the energy of the system is p^2. The analog in three dimensions of the function $f(\vec{p}', \vec{p})$ defined above reduces to the scattering amplitude multiplied by (-4π) when its arguments are of equal magnitude. This is not the case in two dimensions. If the scattering amplitude in the two-dimensional problem is denoted $\tilde{f}(\vec{p}', \vec{p})$, then it can be shown that

$$f(\vec{p}', \vec{p}) = -(8\pi p)^{1/2} e^{-\pi i/4} \tilde{f}(\vec{p}', \vec{p}) \quad \text{for } p = p' .$$

As in three dimensions, the first-order chemical potential and self-energies are obtained from the effective interaction Γ according to[5]

$$\mu = n_0\Gamma(0, 0, 0) = n_0 f(0, 0) + n_0 \text{P. P.} \int |f(0, \vec{q})|^2 \left(\frac{1}{2\mu - q^2} + \frac{1}{q^2} \right) \frac{d\vec{q}}{(2\pi)^2} , \tag{4}$$

$$\Sigma_{02}(\vec{p}) = n_0\Gamma(0, \vec{p}, 0) = n_0 f^*(\vec{p}, 0) + n_0 \text{P. P.} \int f^*(\vec{p}, \vec{q}) f(0, \vec{q}) \left(\frac{1}{2\mu - q^2} + \frac{1}{q^2} \right) \frac{d\vec{q}}{(2\pi)^2} , \tag{5}$$

$$\Sigma_{20}(\vec{p}) = n_0\Gamma(\vec{p}, 0, 0) = n_0 f(\vec{p}, 0) + n_0 \text{P. P.} \int f^*(0, \vec{q}) f(\vec{p}, \vec{q}) \left(\frac{1}{2\mu - q^2} + \frac{1}{q^2} \right) \frac{d\vec{q}}{(2\pi)^2} , \tag{6}$$

$$\Sigma_{11}(\vec{p}, p^0) = n_0\Gamma(\tfrac{1}{2}\vec{p}, \tfrac{1}{2}\vec{p}, p) + n_0\Gamma(-\tfrac{1}{2}\vec{p}, \tfrac{1}{2}\vec{p}, p)$$

$$= 2n_0 f_s(\tfrac{1}{2}\vec{p}, \tfrac{1}{2}\vec{p}) + 2n_0 \int |f_s(\tfrac{1}{2}\vec{p}, \vec{q})|^2 \left(\frac{1}{p^0 + 2\mu - \frac{1}{4}p^2 - q^2 + i\delta} + \frac{1}{q^2 - \frac{1}{4}p^2 + i\delta} \right) \frac{d\vec{q}}{(2\pi)^2} , \tag{7}$$

where

$$f_s(\vec{p}', \vec{p}) = \tfrac{1}{2}[f(\vec{p}', \vec{p}) + f(-\vec{p}', \vec{p})], \tag{8}$$

and P. P. denotes principal part.

For hard disks of diameter a and unit mass, it is shown in Appendix A that the function $f(\vec{K}', \vec{K})$ is given by

$$f(\vec{K}', \vec{K}) = -2\pi K a \sum_{\nu=0}^{\infty} c_\nu e^{i\delta_\nu} J_\nu(K'a) \times [\cos\delta_\nu J_{\nu+1}(Ka) - \sin\delta_\nu Y_{\nu+1}(Ka)] \cos\nu\theta,$$

where

$c_\nu = 1$ for $\nu = 0$

$= 2$ otherwise,

J_ν and Y_ν are the cylindrical Bessel functions of the first and second kind, θ is the angle between $\vec{K}$ and $\vec{K}'$, and the sine and cosine of the phase shifts δ_ν are given by

$$\sin\delta_\nu = \frac{J_\nu(Ka)}{[J_\nu^2(Ka)+Y_\nu^2(Ka)]^{1/2}},$$
$$\cos\delta_\nu = \frac{Y_\nu(Ka)}{[J_\nu^2(Ka)+Y_\nu^2(Ka)]^{1/2}}. \tag{9}$$

The following limiting behaviors will be useful:

$$f(\vec{K}', 0) = 0 \quad \text{for all } \vec{K}', \tag{10}$$
$$f(0, \vec{K}) \to 2\pi(\ln Ka)^{-1} \quad \text{for } Ka \ll 1, \tag{11}$$
$$f(\vec{K}', \vec{K}) \to 2\pi(\ln Ka)^{-1} \quad \text{for } K'a \text{ and } Ka \ll 1. \tag{12}$$

This limiting behavior is to be contrasted with that of the analogous function in three dimensions which is asymptotically equal to a constant value for $K'a$ and Ka much less than unity.

Substituting Eq. (10) into Eq. (4) for the chemical potential, one obtains the homogeneous (as opposed to inhomogeneous in three dimensions) integral equation for μ

$$\mu = n_0 \text{P. P.} \int \frac{d\vec{q}}{(2\pi)^2} \left| f(0, \vec{q}) \right|^2 \left(\frac{1}{2\mu - q^2} + \frac{1}{q^2} \right).$$

The major contribution to this integral comes from values of qa which are much less than $(2\mu a^2)^{1/2}$ which will be shown to be much less than unity. Thus the limiting behavior of $f(0, \vec{q})$ given by Eq. (11) may be substituted into the above integral. In the limit of $n_0 a^2 \ll 1$, one obtains

$$\mu = -4\pi n(\ln na^2)^{-1}\{1 + O[(\ln na^2)^{-1}]\}, \tag{13}$$

where the result that n_0 is equal to n in lowest order has been anticipated. As stated above, the quantity na^2 is small compared to unity in the low-density limit. The ground-state energy per unit area is obtained from

$$E(n)/A = \int_0^n \mu(n')\, dn',$$

which yields for the ground-state energy per particle in the low-density limit

$$E/N = -2\pi n(\ln na^2)^{-1}\{1 + O[(\ln na^2)^{-1}]\}. \tag{14}$$

It may be seen from Eqs. (13) and (14) that the expansion parameter in this problem is the small quantity $(\ln na^2)^{-1}$.

The self-energies $\Sigma_{20}(\vec{p})$, $\Sigma_{20}(\vec{p})$ are easily obtained in the limit $pa \ll 1$. Substituting Eqs. (10)–(12) into Eqs. (5) and (6) for the self-energies and comparing with Eq. (4) for the chemical potential, one obtains

$$\Sigma_{20}(\vec{p}) \to \Sigma_{02}(\vec{p}) \to \mu = -4\pi n(\ln na^2)^{-1} \quad \text{for } pa \ll 1.$$

The evaluation of $\Sigma_{11}(\vec{p}, p^0)$ given by Eq. (7) is somewhat more complicated. We leave to Appendix B the demonstration that

$$\Sigma_{11}(\vec{p}, p^0) \to 2\mu = -8\pi n(\ln na^2)^{-1}$$

for

$$pa \ll p_c a \equiv [-16\pi na^2(\ln na^2)^{-1}] \ll 1, \tag{15}$$

and for the important values of p^0 which are shown to be

$$p^0 \approx \tfrac{1}{2} p_c p \quad \text{for } p < p_c.$$

It should be noted from the intimate relation between the effective interaction Γ and the self-energies, given in Eqs. (5)–(7), and the limiting expressions for these self-energies given above that the effective interaction in the limit of small momentum transfers depends on the density as $(\ln na^2)^{-1}$ as stated previously.

Using the self-energies given above, the Green's function can be obtained from the well-known expression of Beliaev.[5] It has the form, for $p < p_c$,

$$G(p) = u_p^2/(p^0 - \omega_p + i\delta) - v_p^2/(p^0 + \omega_p - i\delta), \tag{16}$$

where

$$u_p^2 = (\epsilon_p + \mu + \omega_p)/2\omega_p, \quad v_p^2 = (\epsilon_p + \mu - \omega_p)/2\omega_p, \tag{17}$$

and where ω_p, the excitation spectrum, is given by

$$\omega_p = (\epsilon_p^2 + 2\epsilon_p\mu)^{1/2}$$
$$= [\tfrac{1}{4}p^4 - 4\pi n p^2(\ln na^2)^{-1}]^{1/2}.$$

In the long-wavelength limit, the spectrum is phononlike with a speed of propagation

$$c = [-4\pi n(\ln na^2)^{-1}]^{1/2}.$$

The spectrum is seen to change from phononlike to free particlelike in the vicinity of the momentum p_c defined in Eq. (15).

The depletion of the zero-momentum state is obtained from $G(p)$ according to the expression

$$n - n_0 = i(2\pi)^{-3} \int d\vec{p} \int_C dp^0 G(p),$$

where the contour C contains the real axis and is closed in the upper half-plane. Upon substituting Eqs. (16) and (17) into the above expression, one obtains for the condensate density

$$n_0 = n[1 + (\ln na^2)^{-1}].$$

Finally, the ground-state energy, chemical potential, and Green's function are related by the equation

$$E/A - \tfrac{1}{2}n\mu = i(2\pi)^{-3} \int d\vec{p} \int_C \tfrac{1}{2}(p^0 + \epsilon_p) G(p)\, dp^0.$$

In three dimensions, the analog of this equation can be used to obtain the next approximation for the energy and chemical potential from the Green's function.[6] Unfortunately this is not true in two dimensions which can be demonstrated as follows: Substituting Eq. (16) for the Green's function into the above, carrying out the integration, and using Eq. (13) yields

$$E/A - \tfrac{1}{2}n\mu = -\pi n^2(\ln na^2)^{-2}\,. \tag{18}$$

Let

$$E/A = -2\pi n^2(\ln na^2)^{-1}[1+\gamma(\ln na^2)^{-1}],$$

so that

$$\mu = \frac{d(E/A)}{dn} = -4\pi n(\ln na^2)^{-1} + 2\pi n(\ln na^2)^{-2}(1-2\gamma) + O[(\ln na^2)^{-3}]\,.$$

Substituting these expressions into Eq. (18) yields an identity which is independent of γ.

SUMMARY AND DISCUSSION

Properties of the ground state of a two-dimensional system of hard-disk bosons of diameter a have been calculated in the low-density limit. The ground-state energy per particle, chemical potential, and condensate density were found to be

$$E/N = -(2\pi\hbar^2 n/m\,\ln na^2)[1+O(1/\ln na^2)\,],$$
$$\mu = -(4\pi\hbar^2 n/m\,\ln na^2)[1+O(1/\ln na^2)\,],$$
$$n_0 = n\{1+1/\ln na^2 + O[(1/\ln na^2)^2]\}$$

upon restoring $\hbar$ and m. The excitation spectrum for long wavelengths was found to be phononlike. For a system of finite size, the above quantities will vary smoothly with temperature in the limit of vanishing temperatures. In particular, the phononlike behavior of the excitation spectrum indicates that for $T \ll T_0/\ln N$, the specific heat will be quadratic in temperature. Corrections to the above quantities due to finite-size effects are smaller than the above by the ratio $\ln na^2/\ln N$.

The expansion parameter of the system is $-1/\ln na^2$. Taking the hard-core diameter a to be 2.2 Å and the density to lie in the range appropriate to real helium monolayers,[4] 0.06–0.08 Å^{-2}, one finds that the expansion parameter varies from slightly larger to slightly less than unity. This is in contrast with the expansion parameter in the three-dimensional system which, when evaluated at the density of liquid helium, is approximately 21. Thus the motivation for obtaining the next few terms in the above series for the two-dimensional system is far greater than in the three-dimensional case.

APPENDIX A

In this Appendix, the function

$$f(\vec{K}',\vec{K}) = \int u(\rho)\,\psi_{\vec{K}}(\vec{\rho})e^{-i\vec{K}'\cdot\vec{\rho}}d\vec{\rho} \tag{A1}$$

is evaluated. Standard partial wave analysis applied to the two-dimensional scattering problem yields for the wave function $\psi_{\vec{K}}(\vec{\rho})$ the expansion

$$\psi_{\vec{K}}(\vec{\rho}) = \sum_{\nu=0}^{\infty} c_\nu e^{i\delta_\nu}F_\nu(K\rho)\cos\phi, \tag{A2}$$

where

$c_\nu = 1$ for $\nu = 0$

$= 2$ otherwise,

ϕ is the angle between the vectors $\vec{k}$ and $\vec{\rho}$ and $F_\nu(k\rho)$ satisfies the equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dF_\nu(K\rho)}{d\rho}\right) = \left(-K^2 + u(\rho) + \frac{\nu^2}{\rho^2}\right)F_\nu(K\rho). \tag{A3}$$

Again it should be recalled that the reduced mass is $\frac{1}{2}$. Denoting by γ the angle between $\vec{K}'$ and $\vec{\rho}$, we employ the expansion

$$\exp(-i\vec{K}'\cdot\vec{\rho}) = \sum_{\mu=0}^{\infty} c_\mu J_\mu(K'\rho)\cos\mu\gamma$$
$$= \sum_{\mu=0}^{\infty} c_\mu J_\mu(K'\rho) \times(\cos\mu\theta\cos\mu\phi + \sin\mu\theta\sin\mu\phi), \tag{A4}$$

where $\theta = \phi + \gamma$ is the angle between $\vec{K}$ and $\vec{K}'$. Substituting Eqs. (A2) and (A4) into Eq. (A1) and carrying out the angular integration yields

$$f(\vec{K}',\vec{K}) = 2\pi\sum_{\nu=0}^{\infty} c_\nu e^{i\delta_\nu}\cos\nu\theta \times\int_0^\infty U(\rho)\,F_\nu(K\rho)J_\nu(K'\rho)\rho\,d\rho. \tag{A5}$$

The Bessel function $J_\nu(K'\rho)$ satisfies the differential equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(K'\rho)}{d\rho}\right) = \left(-(K')^2 + \frac{\nu^2}{\rho^2}\right)J_\nu(K'\rho). \tag{A6}$$

Upon multiplying Eq. (A3) by $J_\nu(K'\rho)$ and Eq. (A6) by $F_\nu(K\rho)$, subtracting the equations, and integrating from zero to infinity, one obtains

$$\left(\rho J_\nu(K'\rho)\frac{d}{d\rho}F_\nu(K\rho) - \rho F_\nu(K\rho)\frac{d}{d\rho}J_\nu(K'\rho)\right)_{\rho=0}^{\rho\to\infty}$$
$$= (K'^2 - K^2)\int_0^\infty J_\nu(K'\rho)F_\nu(K\rho)\rho\,d\rho + \int_0^\infty U(\rho)J_\nu(K'\rho)\,F_\nu(K\rho)\rho\,d\rho. \tag{A7}$$

Use is now made of the specific properties of the hard-core interaction. As the interaction is infinitely strong for $\rho < a$, the radial wave function $F_\nu(K\rho)$ vanishes in this region so that

$$\int_0^\infty J_\nu(K'\rho)F_\nu(K\rho)\rho\,d\rho = \int_a^\infty J_\nu(K'\rho)F_\nu(K\rho)\rho\,d\rho. \tag{A8}$$

In addition, the potential vanishes for $\rho > a$ so that in this region $F_\nu(K\rho)$ satisfies the equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}F_\nu(K\rho)\right)=\left(-K^2+\frac{\nu^2}{\rho^2}\right)F_\nu(K\rho)\ .$$

Upon multiplying this equation by $J_\nu(K'\rho)$ and Eq. (A6) by $F_\nu(K\rho)$, subtracting and integrating from a to infinity, one obtains

$$\left[\rho J_\nu(K'\rho)\frac{d}{d\rho}F_\nu(K\rho)-\rho F_\nu(K\rho)\frac{d}{d\rho}J_\nu(K'\rho)\right]_{\rho=a}^{\rho\to\infty}$$

$$=(K'^2-K^2)\int_a^\infty J_\nu(K'\rho)F_\nu(K\rho)\rho\,d\rho$$

$$=(K'^2-K^2)\int_0^\infty J_\nu(K'\rho)F_\nu(K\rho)\rho\,d\rho,$$

where Eq. (A8) has been used to obtain the last line. Subtracting this equation from Eq. (A7) and evaluating the resultant expression at the limit yields

$$\int U(\rho)J_\nu(K'\rho)F_\nu(K\rho)\rho\,d\rho=aJ_\nu(K'a)\frac{d}{d\rho}F_\nu(K\rho)\Big|_{\rho=a}$$

$$-aF_\nu(Ka)\frac{d}{d\rho}J_\nu(K'\rho)\Big|_{\rho=a}$$

$$=aJ_\nu(K'a)\frac{d}{d\rho}F_\nu(K\rho)\Big|_{\rho=a}\ , \qquad \text{(A9)}$$

as the radial wave function $F_\nu(k\rho)$ vanishes at ρ equal to a. To carry out the derivatives of $F_\nu(k\rho)$, we employ the explicit form

$$F_\nu(K\rho)=\cos\delta_\nu J_\nu(K\rho)-\sin\delta_\nu Y_\nu(K\rho).$$

Lastly, substituting the result for the integral in Eq. (A9) into Eq. (A5) yields

$$f(\vec{K}',\vec{K})=-2\pi(Ka)\sum_{\nu=0}^{\infty}c_\nu e^{i\delta_\nu}J_\nu(K'a)$$

$$\times[\cos\delta_\nu J_{\nu+1}(Ka)-\sin\delta_\nu Y_{\nu+1}(Ka)]\cos\nu\theta\ , \qquad \text{(A10)}$$

as stated in the text.

It is interesting to note in passing that the differential cross section $\sigma(\vartheta)$ is related to $f(\vec{K}',\vec{K})$ by

$$\sigma(\theta)=(8\pi K)^{1/2}\left|f(\vec{K}',\vec{K})\right|^2,\quad K=K'$$

and the total cross section is given by the optical theorem

$$\sigma_T(K)=-K^{-1}\mathrm{Im}f(\vec{K},\vec{K}).$$

If Eq. (A10) in the proper limit is substituted into the above, the result can, after some manipulation, be brought to the form

$$\sigma_T(K)=(4/K)\textstyle\sum_{\nu=0}^{\infty}c_\nu\sin^2\delta_\nu\ .$$

The sines of the phase shifts are given in the text [Eq. (9)] and have, in the limit $Ka\gg1$, the asymptotic behavior

$$\sin\delta_\nu\to\pi(Ka)^{2\nu}/2^{2\nu}\nu!(\nu-1)!\ ,\qquad \nu>0$$

$$\sin\delta_0\to-\pi/2\ln Ka,\qquad \nu=0\ .$$

Thus in the low-energy limit, the total cross section has the form

$$\sigma_T(Ka)\to\pi^2/k(\ln Ka)^2,\quad Ka\ll1$$

and diverges in the limit of zero energy.

APPENDIX B

In this Appendix, the real part of the self-energy $\Sigma_{11}(\vec{p},p^0)$, which is given by

$$\mathrm{Re}\Sigma_{11}(\vec{p},p^0)=2n_0f_s\left(\tfrac{1}{2}\vec{p},\tfrac{1}{2}\vec{p}\right)+2n_0\,\mathrm{P.P.}\int\left|f_s\left(\tfrac{1}{2}\vec{p},\vec{q}\right)\right|^2\left(\frac{1}{p^0+2\mu-\frac{1}{4}p^2-q^2}+\frac{1}{q^2-\frac{1}{4}p^2}\right)\frac{d\vec{q}}{(2\pi)^2}\ , \qquad \text{(B1)}$$

will be calculated. The imaginary part is ignored as it is easily shown to be smaller than the real part by a factor of $1/\ln na^2$.

It will be assumed that $pa\ll1$ so that

$$f_s(\tfrac{1}{2}\vec{p},\tfrac{1}{2}\vec{p})\approx2\pi(\ln pa)^{-1}\ ,$$

which follows from Eqs. (8) and (12), and the assumption that pa is small compared to unity. In addition it will be assumed, and later confirmed, that the major contribution to the integral above comes from the region in which $qa\ll1$ so that

$$f_s(\tfrac{1}{2}\vec{p},\vec{q})\approx2\pi(\ln qa)^{-1}\ .$$

Defining the parameters

$$s=p^0a^2+2\mu a^2-t,\quad t=(pa/2)^2\ll1\ , \qquad \text{(B2)}$$

and making a simple change of variable, Eq. (B1) can be written as

$$\mathrm{Re}\Sigma_{11}(\vec{p},p^0)/4\pi n_0=-(\ln pa)^{-1}+2\mathrm{P.P.}$$

$$\times\int_0^\infty\frac{dx}{(\ln x)^2}\left(\frac{1}{x-t}-\frac{1}{x-s}\right)\ . \qquad \text{(B3)}$$

The major contribution to this integral comes from the region of x which is greater than t and less than s or, in terms of q,

$$s>(qa)^2>t\ .$$

The parameter t is assumed at the outset to be less than unity and it will be shown that for the important values of p^0, s is also small compared to unity. Assuming this, the integral has the value

$$1/\ln t-1/\ln s+O[1/(\ln t)^2]+O[1/(\ln s)^2]\ .$$

To order $(\ln t)^{-1}$, the first term in the above expression and the first term in Eq. (B3) cancel leaving

$$\mathrm{Re}\Sigma_{11}(\vec{p}, p^0) = \frac{-8\pi n_0}{\ln[p^0 a^2 + 2\mu a^2 - (pa/2)^2]} .$$

Restricting ourselves further to the regime

$$pa \ll p_c a = (-16\pi n a^2/\ln n a^2)^{1/2} = 2(\mu a^2)^{1/2} ,$$

we find that the third term in the argument of the logarithm can be ignored with respect to the second.

The important values of p^0 are those near the poles of the Green's function where p^0 has the magnitude

$$p^0 = \omega_p \approx \tfrac{1}{2} p_c p \quad \text{for } p \ll p_c .$$

Therefore, the ratio of the first to the second term in the argument of the logarithm is

$$p^0/2\mu \approx p_c p/4\mu = p/p_c \ll 1 .$$

From Eq. (B2) for s, it can now be seen that s is approximately equal to $2\mu a^2$ which, from Eq. (13) of the text, is small compared to unity. Therefore the most important part of the integral in Eq. (B1) does come from qa much less than unity as assumed.

We have found that

$$\mathrm{Re}\Sigma_{11}(\vec{p}, p^0) = -8\pi n_0/\ln 2\mu a^2 = -8\pi n/\ln n a^2 + O[1/(\ln n a^2)^2] = 2\mu + O[1/(\ln n a^2)^2] ,$$

in the region $p \ll p_c$, $p^0 \approx \omega_p$.

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High-Energy Neutron Scattering Measurements on Liquid Helium and Bose Condensation in He II †

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Differential neutron scattering cross sections for liquid ^{4}He have been obtained for momentum transfers along the free-particle excitation curve up to 20.3 Å^{-1}. Momentum transfers in this range are much higher than in previously reported work. The measured recoil energies as well as the angular dependence of the scattering confirm that the neutron scattering, at these high values of κ, is sensitive to the motions of single helium atoms. A temperature dependence in the widths of the cross-section peaks, as well as shape changes in these peaks as the temperature is decreased below T_λ, is consistent with the idea that a narrow condensate component contributes to the scattering for $T < T_\lambda$. Results from a detailed comparison of the measured cross sections with a theory for high-energy neutron scattering are reported. The kinetic energy per liquid-helium atom is deduced from this theory-experiment comparison and is found to be in agreement with computations of the kinetic energy based on thermodynamic phase equilibria considerations. A fractional occupation of the zero-momentum condensate state of $(8.8 \pm 1.3)\%$ at 1.27 °K is also deduced from this theory-experiment comparison.

I. INTRODUCTION

At sufficiently high energy and momentum transfer, neutron scattering from liquid helium can be expected to provide information about the momentum distribution $n(p)$ of individual helium atoms. While any experimental information about $n(p)$ for helium atoms is of interest, it is particularly important to obtain knowledge regarding the shape of $n(p)$ near $p = 0$. The shape of this part of the momentum spectrum bears on the question of the existence of a condensation in momentum space. Much of the theoretical basis for understanding superfluid He II involves the notion of macroscopic occupation of a