Wannier's predictions which has been established here is connected primarily with the use of drift tubes for the measurement of rate coefficients for ion-molecule reactions. In such measurements, the effects of longitudinal and transverse diffusion of the ions should be considered, but the diffusion coefficients above the thermal region are known for only a few ionic species. When the necessary coefficients are not available, the Wannier equations may be used to calculate values of D_L and D_T which should be at least approximately correct, except for cases where resonant charge transfer occurs.

ACKNOWLEDGMENT

We wish to thank Professor E. A. Mason of Brown University for suggesting the form of Eqs. (4)–(8), which allowed this test of Wannier's theory to be made.

*Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under Grant No. AFOSR-70-1933, and Project SQUID which is supported by the Office of Naval Research, Department of the Navy, under Contract No. N00014-67-A-0226-0005, NR-098-038.

[1]E. W. McDaniel, *Collision Phenomena in Ionized Gases* (Wiley, New York, 1964), Sec. 10-2.

[2]G. H. Wannier, Bell System Tech. J. 32, 170 (1953).

[3]E. W. McDaniel, V. Čermák, A. Dalgarno, E. E. Ferguson, and L. Friedman, *Ion-Molecule Reactions* (Wiley, New York, 1970), Sec. 2-4.

[4]J. T. Moseley, I. R. Gatland, D. W. Martin, and E. W. McDaniel, Phys. Rev. 178, 234 (1969).

[5]R. M. Snuggs, D. J. Volz, J. H. Schummers, D. W. Martin, and E. W. McDaniel, Phys. Rev. (to be published).

[6]J. T. Moseley, R. M. Snuggs, D. W. Martin, and E. W. McDaniel, Phys. Rev. 178, 240 (1969).

[7]J. T. Moseley, R. M. Snuggs, D. W. Martin, and E. W. McDaniel, Phys. Rev. Letters 21, 873 (1968).

[8]I. A. Fleming, R. J. Tunnicliffe, and J. A. Rees, J. Phys. D 2, 551 (1969).

[9]T. M. Miller, J. T. Moseley, D. W. Martin, and E. W. McDaniel, Phys. Rev. 173, 115 (1968).

[10]J. Dutton, F. Llewellyn-Jones, W. D. Rees, and E. M. Williams, Phil. Trans. Roy. Soc. London A 259, 339 (1966).

[11]I. A. Fleming, R. J. Tunnicliffe, and J. A. Rees, J. Phys. B 2, 780 (1969).

[12]H. R. Skullerud, *Proceedings of the Seventh International Conference on Ionization Phenomena in Gases, Belgrade*, 1965, edited by B. Perovic and D. Tosie (Gradevinska Knjiga Publishing House, Belgrade, Yugoslavia, 1966), Vol. I, p. 50.

[13]H. R. Skullerud (private communication).

[14]H. R. Skullerud, J. Phys. B 2, 86 (1969).

[15]R. N. Varney, Phys. Rev. Letters 25, 843 (1970).

[16]I. R. Gatland and E. W. McDaniel, Phys. Rev. Letters 25, 1603 (1970).

[17]The effects of the variation of v_d as other than the first power of E/N at high E/N are clearly revealed in the upper curve in Fig. 1 and in the lower curve in Fig. 4. Both the calculated values of ND and the experimental data show a decrease in the slope of the curve at the higher values of E/N shown.

PHYSICAL REVIEW A VOLUME 3, NUMBER 3 MARCH 1971

Quantum-Electrodynamic Treatment of Spontaneous Emission in the Presence of an Applied Field*

C. R. Stroud, Jr.

The Institute of Optics, University of Rochester, Rochester, New York 14627

(Received 21 September 1970)

Spontaneous emission by a "two-level atom" interacting with a highly excited field mode is described using quantum electrodynamics without time-dependent perturbation theory. It is found that both the line shape and the Lamb shift are affected by the "applied field." Detailed solutions are presented as a function of the population of the "applied-field mode" and as a function of the detuning.

I. INTRODUCTION

There have been a number of treatments of the phenomena which occur when spontaneous and stimulated emission proceed simultaneously in the same population.[1–6] These treatments have used various forms of semiclassical radiation theory. Two papers in particular give an extensive treatment using a form of semiclassical theory which includes the effects of radiation reaction on the radiating atom.[5,6]

These improved semiclassical theories have predicted certain nonlinear effects which occur when spontaneous and stimulated emission occur simultaneously in the same population.[5,6] These effects involve changes in the line shape and in the radiative frequency shift, as well as various transient phenomena which occur when the applied field is suddenly turned on. Thus far these phenomena have not been observed experimentally, but an experiment is now in progress which will attempt to observe them.

Predictions from these improved semiclassical theories do not, in general, agree in every detail with quantum electrodynamics, though the discrepancies seem to be entirely in the fine details which have not yet been experimentally checked. These discrepancies have, in fact, been suggested as possible experimental checks of quantum electrodynamics. One difficulty with any such check is that, although the semiclassical treatments have been worked out in great detail, the quantum-electrodynamic treatment has not been carried out in such detail. This paper represents a first step in that quantum treatment.

The method which we will use to treat these problems is to separate out one field mode, which is assumed to be initially highly populated, from the remainder of the field modes, which are assumed to be initially unpopulated. We will take the single field mode interacting with a "two-level atom" to be a quantum system and find its eigenstates and eigenenergies. We will then allow this atom-plus-field-mode system (AFS) to interact with the remainder of the field and determine the nature of the spontaneous transitions between states of the AFS. These spontaneous transitions will be studied by making the Wigner-Weisskopf essential-states approximation and then solving the time-dependent Schrödinger equation with Laplace transforms.

II. CALCULATIONS

The Hamiltonian for a "two-level atom" interacting with the field can be written in the form

$$H = H_0 - (e/mc)\,\vec{A}'(0)\cdot\vec{p} + H_f', \tag{1}$$

where

$$H_0 \equiv H_a - (e/mc)\,\vec{A}_0(0)\cdot\vec{p} + H_{f0}. \tag{2}$$

We have denoted the Hamiltonian for the atom alone by H_a and have divided the field into two parts,

$$\vec{A}(r) = \vec{A}_0(r) + \vec{A}'(r). \tag{3}$$

The first part, $\vec{A}_0(r)$, is a single highly populated mode. Initially, we will take this mode to be resonant with the transition between the two levels of our atom. The other part of the field, $\vec{A}'(r)$, is simply all of the field except the one populated mode $\vec{A}_0(r)$.

Writing the Hamiltonian this way requires two approximations, the dipole approximation in evaluating the field at the center of mass of the atom, and the weak-field approximation in dropping the diamagnetic term which is quadratic in the field.

We will begin by finding the exact eigenstates of the AFS Hamiltonian H_0. The basis states for the two-level atom satisfy

$$H_a|u\rangle = \hbar\Omega|u\rangle, \qquad H_a|l\rangle = 0|l\rangle, \tag{4}$$

and the resonant field states satisfy

$$H_{f0}|n\rangle = n\hbar\Omega|n\rangle. \tag{5}$$

We will choose for our field-mode expansion[7]

$$\vec{A}(r) = \left(\frac{2\pi\hbar c}{V}\right)^{1/2} \sum_{k,\lambda} \frac{\hat{\epsilon}_\lambda}{k^{1/2}}\left(e^{ikr}a_{k,\lambda} + e^{-ikr}a_{k,\lambda}^\dagger\right), \tag{6}$$

where V is the volume of the cavity in which the field modes are defined.[8] Neglecting non-energy-conserving transitions of the type in which a photon is given off when the atom goes from the lower state to the upper state, or absorbed when the atom goes from the upper to the lower state, we can write down the eigenvalues by diagonalizing a two-by-two matrix.[9] Choosing as a basis for our representation products of the states defined in Eqs. (4) and (5), the eigenstates are

$$|n,\pm\rangle = 2^{-1/2}\left(|n-1\rangle|u\rangle \pm |n\rangle|l\rangle\right) \tag{7}$$

and the corresponding eigenvalues are

$$E_n^{\pm} = \hbar[n\Omega \pm \epsilon(n)]. \tag{8}$$

The interaction energy $\epsilon(n)$ is

$$\epsilon(n) \equiv (1/\hbar)\left|\langle n-1|\langle u| - (e/mc)\,\vec{A}_0(0)\cdot\vec{p}\,|n\rangle|l\rangle\right| = (1/\hbar)\left| -i\,\hat{\epsilon}\cdot\vec{\mu}\,(2\pi\hbar\Omega n/V)^{1/2}\right|, \tag{9}$$

with

$$\vec{\mu} \equiv \langle u|e\vec{r}|l\rangle,$$

the dipole matrix element.

Then for reasonable field strengths,

$$\epsilon(n) \ll \Omega, \tag{10}$$

the spectrum of H_0 consists of a series of closely spaced doublets [see Fig. 1(a)]. The doublet splitting is nearly equal in adjacent doublets if $n \gg 1$. Thus in what we do subsequently, we will use the approximation

$$\epsilon(n) = \epsilon(n-1) = \epsilon, \tag{11}$$

and will restrict our considerations to fields for which this is a good approximation.[10]

Evaluating probabilities for transitions from one doublet to the other, we are led to predict the spectrum of Fig. 1(b). This level splitting is just the ac Stark effect.[11] This transition-probability argument can only give the spectral lines and their relative intensities and not the radiative effects we are looking for. To obtain the line shape and frequency shifts we must consider the complete Hamiltonian, Eq. (1), and solve the equation of motion for the

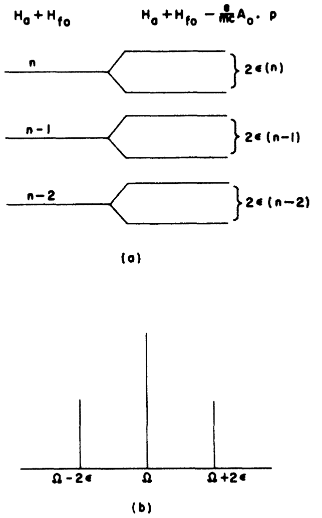

FIG. 1. (a) Energy levels of H_0 without the interaction Hamiltonian. The doublet splitting is equal for large n. (b) Emission spectrum of the atom-plus-field-mode system.

state vector of the entire system. We will assume that the AFS is prepared at time $t=0$ in a linear combination of the states $|n, \pm\rangle$, and that all the remaining field states are unoccupied. The state vector then develops in time according to Schrödinger's equation into some linear combination of the states $|n-1, \pm\rangle$ with the accompanying spontaneous emission of a photon into a state $|k, \lambda\rangle$. Accordingly, the state vector will be written as

$$|\Psi(t)\rangle = \{b^+(t)|n+\rangle e^{-i\epsilon t} + b^-(t)|n-\rangle e^{i\epsilon t} + \sum_{k,\lambda} [c^+(k,\lambda,t)|n-1, +, k, \lambda\rangle e^{-i\epsilon t} + c^-(k,\lambda,t) \times |n-1, -, k, \lambda\rangle e^{i\epsilon t}] e^{i(kc-\Omega)t}\} e^{-in\Omega t}, \quad (12)$$

with the amplitudes, the b's and the c's, determined by Schrödinger's equation.

Substituting Eq. (12) into Schrödinger's time-dependent equation, we find that the amplitudes satisfy the following set of coupled equations:

$$i\hbar \dot{b}^{\pm}(t) e^{\mp i\epsilon t} = \sum_{k,\lambda} [\langle n\pm|H_{\text{int}}|n-1, +, k, \lambda\rangle c^+(k,\lambda,t) \times e^{-i\epsilon t} + \langle n\pm|H_{\text{int}}|n-1, -, k, \lambda\rangle \times c^-(k,\lambda,t) e^{i\epsilon t}] e^{-i(kc-\Omega)t}, \quad (13)$$

$$i\hbar \dot{c}^{\pm}(k,\lambda,t) e^{\mp i\epsilon t} e^{-i(kc-\Omega)t} = \langle n-1, \pm, k, \lambda|H_{\text{int}}|n+\rangle \times b^+(t) e^{-i\epsilon t} + \langle n-1, \pm, k, \lambda|H_{\text{int}}|n, -\rangle \times b^-(t) e^{i\epsilon t}, \quad (14)$$

where

$$H_{\text{int}} \equiv -(e/mc)\, \vec{A}'(0) \cdot \vec{p}. \quad (15)$$

We will solve this set of equations by use of Laplace transforms.[12] Defining the transformed amplitudes by

$$B(s) \equiv \int_0^\infty e^{-st}\, b(t)\, dt, \quad (16)$$

we can reduce Eqs. (13) and (14) to algebraic equations for the transformed variables. These algebraic equations are easily solved, giving the transform of the amplitude $b^+(t)$ as

$$B^+(s) = \frac{(s - 2i\epsilon + q)\, b^+(0) - q\, b^-(0)}{(s+q-i\epsilon-\gamma)(s+q-i\epsilon+\gamma)}, \quad (17)$$

where

$$q \equiv \sum_{k,\lambda} \left[\frac{u^2(k,\lambda)}{s+ikc-i\Omega} + \frac{u^2(k,\lambda)}{s+ikc-i\Omega-2i\epsilon}\right] \quad (18)$$

and

$$\gamma \equiv (q^2 - \epsilon^2)^{1/2}. \quad (19)$$

The matrix elements $u(k, \lambda)$ are

$$\begin{aligned} u(k,\lambda) &\equiv (1/i\hbar)\langle n\ +|H_{\text{int}}|n-1, -, k, \lambda\rangle \\ &= -(1/i\hbar)\langle n+|H_{\text{int}}|n-1, +, k, \lambda\rangle \\ &= -(1/i\hbar)\langle n\ -|H_{\text{int}}|n-1, +, k, \lambda\rangle \\ &= (1/i\hbar)\langle n\ -|H_{\text{int}}|n-1, -, k, \lambda\rangle \\ &= \tfrac{1}{2}\vec{\mu}\cdot\hat{\epsilon}_\lambda\, \Omega(2\pi/V\hbar kc)^{1/2}. \end{aligned} \quad (20)$$

In order to understand these rather messy expressions and make the necessary approximations we will first look at the limit $\epsilon \to 0$ in detail. In that limit the applied field is decoupled from the atom and we are simply describing ordinary spontaneous emission. In this limit

$$B^+(s) \to \frac{b^+(0)s + q\big(b^+(0) - b^-(0)\big)}{s(s+2q)}, \quad (21)$$

$$q \to 2\sum_{k,\lambda} \frac{u^2(k,\lambda)}{s+ikc-i\Omega}. \quad (22)$$

If the system is assumed to be prepared at time $t=0$ in the state

$$|\Psi(t=0)\rangle = |n-1, u\rangle, \quad (23)$$

then

$$b^+(0) = b^-(0) = 1/\sqrt{2}, \quad (24)$$

and then

$$B^+(s) = [\sqrt{2}(s+2q)]^{-1}. \quad (25)$$

Even in the limit $\epsilon = 0$, the expression for q is rather complicated. We will leave its explicit evaluation to the Appendix. In the Appendix we show that, except for a negligibly small correction, q is independent of s and equal to

$$q(\epsilon = 0) = \tfrac{1}{4}A + \tfrac{1}{2}i\Delta, \tag{26}$$

where A is the Einstein A coefficient and Δ is the Lamb shift which we had to renormalize in the usual way.

Substitution of Eq. (25) into (24) and inversion of the Laplace transform gives the usual exponential decay and the Lamb shift

$$b^+(t) = (1/\sqrt{2})\, e^{-At/2}\, e^{-i\Delta t}. \tag{27}$$

Returning to Eqs. (13)–(15), we find

$$\dot{b}^+(t)\, e^{-i\epsilon t} = \dot{b}^-(t)\, e^{i\epsilon t}, \tag{28}$$

which in the case $\epsilon = 0$ reduces to $\dot{b}^+(t) = \dot{b}^-(t)$. Integrating this using the initial conditions of Eq. (23), one obtains

$$b^-(t) = b^+(t) = (1/\sqrt{2})\, e^{-At/2}\, e^{-i\Delta t}. \tag{29}$$

Substitution of these results back into Eq. (14) determines the amplitudes of the lower states:

$$c^{\pm}(k, \lambda, t) = \frac{\pm\sqrt{2}\, u(k, \lambda)}{i(kc - \Omega - \Delta) - \frac{1}{2}A}\left(e^{-At/2}\, e^{i(kc-\Omega-\Delta)t} - 1\right). \tag{30}$$

The probability that a photon is in the mode k, λ, regardless of the state of the atom, is then given by

$$P(k, \lambda) = |c^+(k, \lambda, t)|^2 + |c^-(k, \lambda, t)|^2$$
$$= \frac{2u^2(k, \lambda)}{(kc - \Omega - \Delta)^2 + \frac{1}{4}A^2}\, \left|e^{-At/2}\, e^{i(kc-\Omega-\Delta)t} - 1\right|^2$$
$$\to \frac{2u^2(k, \lambda)}{(kc - \Omega - \Delta)^2 + \frac{1}{4}A^2} \text{ as } t \to \infty. \tag{31}$$

This is just the usual Lorentzian centered about the Lamb-shifted line with the natural linewidth.

Now, having seen the manner of solution for ordinary spontaneous emission, we can return to the problem of spontaneous emission in the presence of an applied field. By precisely the same argument which we give in the Appendix for neglecting the s dependence of q in the absence of ϵ, we can still neglect this small s dependence. Further, since $\epsilon \ll \Omega$, its appearance in q represents only a small correction to q, which is itself small compared with Ω. Thus to a good approximation we have

$$q = \tfrac{1}{4}A - \tfrac{1}{2}i\Delta \tag{26$'$}$$

even for nonzero ϵ.

We can then easily invert Eq. (17) to give

$$b^+(t) = (2\sqrt{2}\,\lambda)^{-1}\left[(-q - i\epsilon + \gamma)\, e^{(-q+i\epsilon+\lambda)t} + (q - i\epsilon + \gamma)\, e^{(-q+i\epsilon-\gamma)t}\right], \tag{32}$$

where

$$\gamma \equiv (q^2 - \epsilon^2)^{1/2}. \tag{19}$$

Using Eq. (27), we can also solve for

$$b^-(t) = (2\sqrt{2}\,\gamma)^{-1}\left[(-q + i\epsilon + \gamma)\, e^{(-q-i\epsilon+\gamma)t} + (q + i\epsilon + \gamma)\, e^{(-q-i\epsilon-\gamma)t}\right], \tag{33}$$

and from Eq. (14) we can then determine

$$c^{\pm}(k, \lambda, t) = \frac{u(k, \lambda)}{\sqrt{2}\,\gamma}\left\{\frac{(\gamma - q)(e^{(-q+\gamma+ikc-i\Omega\pm i\epsilon)t} - 1)}{i(kc - \Omega - \pm\epsilon) - q + \gamma} + \frac{(\gamma + q)(e^{(-q-\lambda+ikc-i\Omega\pm i\epsilon)t} - 1)}{i(kc - \Omega \pm \epsilon) - q - \gamma}\right\}. \tag{34}$$

From Eqs. (25) and (31) we see that the amplitude of the upper state does not in general decay as $e^{-At/2}$ but with a decay constant equal to $\frac{1}{4}A$ plus the real part of γ. In the limit $\epsilon = 0$ the real part of γ is $\frac{1}{4}A$ and we have the normal decay rate $\frac{1}{2}A$, but when $\epsilon \gg \frac{1}{4}A$, the real part of γ vanishes and the decay rate is $\frac{1}{4}A$, one-half the usual decay rate.

Further, if we go to the long-time limit in Eq. (34) and calculate the probability of a phonon in the k, λ mode regardless of the state of the atom, we find

$$P(k, \lambda) \equiv \lim_{t\to\infty}\left[|c^+(k, \lambda, t)|^2 + |c^-(k, \lambda, t)|^2\right]$$
$$= \frac{u^2(k, \lambda)}{2|\gamma|^2}\left\{\left|\frac{\gamma - q}{i(kc - \Omega + \epsilon) - q + \gamma} + \frac{\gamma + q}{i(kc - \Omega + \epsilon) - q - \gamma}\right|^2 + \left|\frac{\gamma - q}{i(kc - \Omega - \epsilon) - q + \gamma} + \frac{\gamma + q}{i(kc - \Omega - \epsilon) - q - \gamma}\right|^2\right\}, \tag{35}$$

which in the limit

$$\epsilon \gg \tfrac{1}{4}A, \quad \gamma \to i\epsilon \tag{36}$$

becomes

$$P(k, \lambda) \to \frac{u^2(k, \lambda)}{2}\left\{\frac{1}{(kc - \Omega - \frac{1}{2}\Delta + 2\epsilon)^2 + \frac{1}{16}A^2} + \frac{2}{(kc - \Omega - \frac{1}{2}\Delta)^2 + \frac{1}{16}A^2}\right.$$

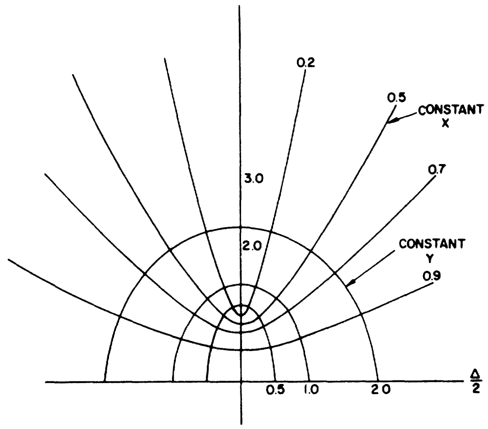

FIG. 2. Curves of constant decay rate and frequency shift as a function of the field strength and the Lamb shift. All quantities are measured in units of one-fourth the Einstein A coefficient.

$$+\left.\frac{1}{(kc-\Omega-\frac{1}{2}\Delta-2\epsilon)^2+\frac{1}{16}A^2}\right\} . \tag{37}$$

Thus the spectrum of the spontaneously emitted photons is the expected ac Stark-effect triplet, but it is centered about the atomic transition frequency plus one-half the Lamb shift, and each component has a natural linewidth which is one-half the normal width.

We can understand this a little better by calculating the probability that the atom is in its upper state and has not emitted the spontaneous-emission photon:

$$P(u,n-1)=\tfrac{1}{2}\left|b^+(t)\,e^{-i\epsilon t}-b^-(t)\,e^{i\epsilon t}\right|^2$$

$$=e^{-At/2}\cos^2(\tfrac{1}{2}\epsilon t). \tag{38}$$

Thus, on the average, the atom spends only half as much time in the excited state when there is a strong applied field. Since the spontaneous emission can take place only from the atomic excited state, its effective rate is halved by the applied field.

For general applied fields which do not fall into the weak-field or the strong-field asymptotic limit we can determine the appropriate decay constant and frequency shift by writing

$$\gamma\equiv\tfrac{1}{2}[(\tfrac{1}{2}A-i\Delta)^2-4\epsilon^2]^{1/2}\equiv x+iy \tag{39}$$

and then determining that x and y satisfy

$$\epsilon^2/(\tfrac{1}{16}A^2-x^2)-\tfrac{1}{4}\Delta^2/x^2=1 \tag{40}$$

and

$$\epsilon^2/(\tfrac{1}{16}A^2+y^2)+\tfrac{1}{4}\Delta^2/y^2=1, \tag{41}$$

equations for a hyperbola and an ellipse[12] (see Fig. 2). Figure 3 illustrates the line shape for various applied fields on resonance when the Lamb shift is negligible. Figures 4–7 illustrate the line shape when there is a Lamb shift just equal to the natural linewidth with various applied fields. In Fig. 4 the ordinary Lorentizian natural line shape is centered at the Lamb-shifted frequency. As we apply a gradually increasing field, we see in Fig. 5 two sharp spikes appear on the basically Lorentizian profile. These are due to the contributions of the first and third terms in Eq. (35). As the field strength increases we see, in Fig. 6, the contributions of all four terms in Eq. (35). And finally, in Fig. 7, as we approach the asymptotic limit, the components due to the second and third terms merge, giving the Stark-split triplet centered at half the Lamb shift predicted in Eq. (37).

This analysis can be carried through in almost exactly the same form for an off-resonance applied field. The only change is that γ is given, not by

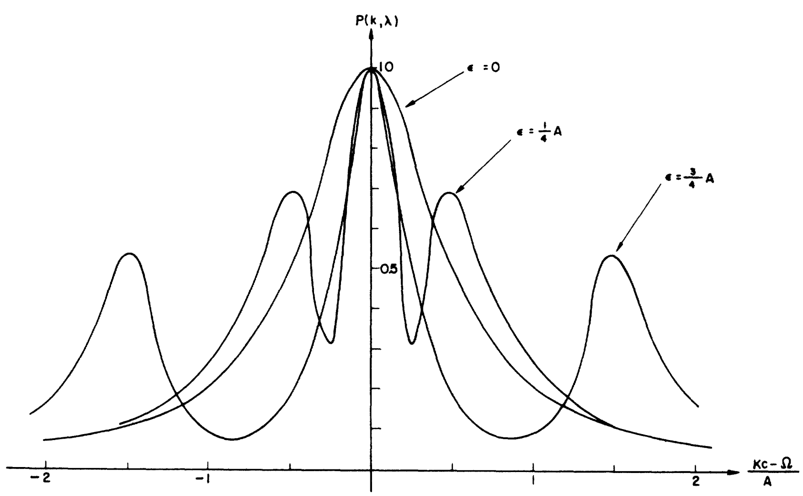

FIG. 3. Fluorescence spectrum for various applied fields when the Lamb shift is neglected.

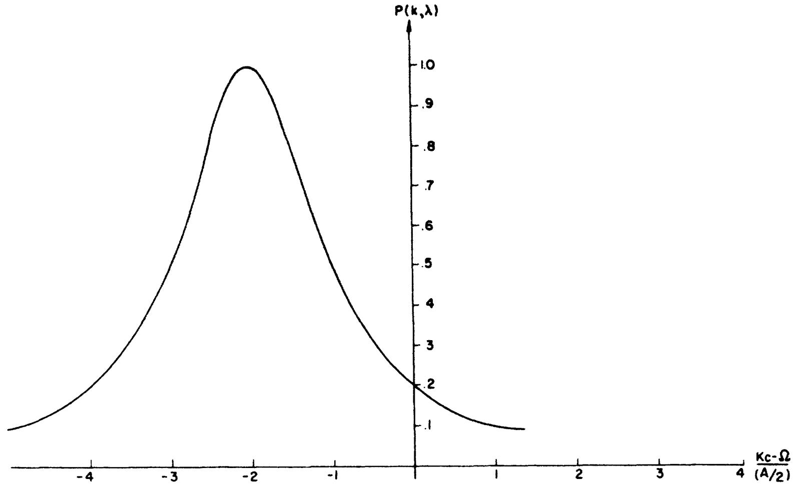

FIG. 4. Line shape for a Lamb shift equal to the natural linewidth and no applied field.

Eq. (39), but instead by

$$\gamma \equiv \tfrac{1}{2}[(\tfrac{1}{2}A - i\Delta + i\alpha)^2 - 4\epsilon^2]^{1/2}, \quad (42)$$

where α is the detuning parameter. Comparing this with Eq. (19), we see that we have simply replaced the Lamb shift Δ by $\Delta - \alpha$, the detuning measured from the Lamb-shifted line. Figure 2 will describe this case if we interpret the horizontal axis to be this new parameter $\Delta - \alpha$. The line shapes will in general be qualitatively different from those in Figs. 4–7 because the amplitudes of the various components will be different.

III. DISCUSSION

We have studied the behavior of a two-level atom interacting with a resonant or nearly resonant highly populated field mode in detail without resorting to time-dependent perturbation theory, or some semiclassical approximation. We have found that, in addition to stimulated absorption and emission into this field mode, the system can spontaneously emit a photon into some other field mode. (In a laser this would include the "side light" scattered out of the lasing mode.) The linewidth and the radiative frequency shift of this spontaneously emitted "side light" were found to be functions of the amplitude of the "applied-field mode."

These phenomena do not appear to have been observed experimentally up to this time. The curves of Fig. 2 illustrate some of the difficulties of this kind of experiment. Clearly, to get an appreciably narrowed or shifted line one must have a very intense, very nearly resonant applied field.[13] Further, inhomogeneous broadening is a difficulty encountered in any line-shape or line-shift experiment. This would lead one to expect that an atomic beam would be necessary. In spite of these difficulties, this would appear to be an attractive way to measure the Lamb shift, since it can be measured as the difference between the position of the center

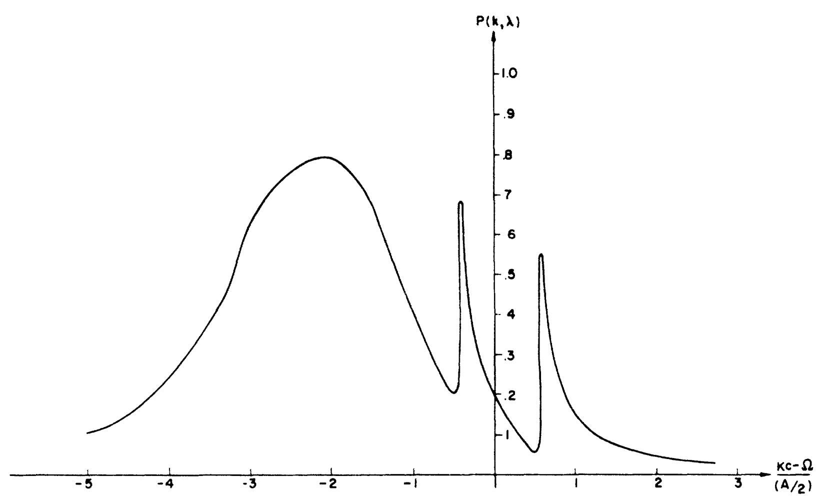

FIG. 5. Line shape for a Lamb shift equal to the natural linewidth and a weak applied field.

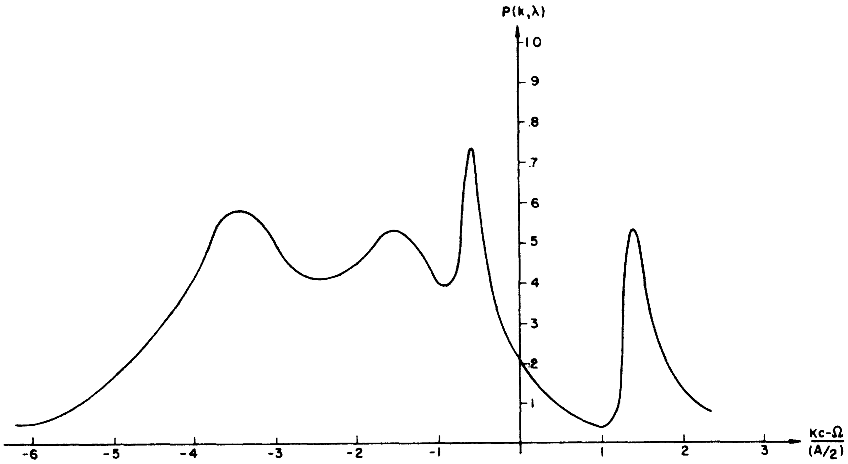

FIG. 6. Line shape for a Lamb shift equal to the natural linewidth and an applied field such that stimulated and spontaneous emission are equally probable.

of the line with and without the field.

Our analysis is incomplete in one important aspect. The spontaneous transition which we describe is between excited states of the atom-plus-field-mode system. We have truncated the problem in considering only a single spontaneous transition. In the real physical case there will be a cascade emitting many spontaneous-emission photons. The present treatment is sufficient to discuss this process if, in fact, the cascade involves the successive emission of photons for which the probability of emission of a given photon is not appreciable until the probability of the preceding photon's having been emitted is nearly 1, i.e., if the cascade is simply a series of transitions each of which is described by present theory. The semiclassical theories[5,6] treat spontaneous emission as a continuous rather than successive process. This leads to qualitatively different results for the long-term solutions. The semiclassical theory predicts that in general the atom may settle down into a certain definite linear combination of the excited and ground states reemitting as much energy as it absorbs, whereas the present theory predicts that there will always be oscillations of the type predicted in Eq. (38). Work is proceeding on the cascade problem, but we are unable to say at the present time how important these corrections are to the above theory.

ACKNOWLEDGMENTS

We would like to thank Professor J. Eberly for some suggestions made during this research, and P. Stroud for numerical work leading to Figs. 3–7.

APPENDIX

In Eqs. (25) and (22) we found

$$B^{+}(s)=[\sqrt{2}\,(s+2q)]^{-1}\ , \tag{25}$$

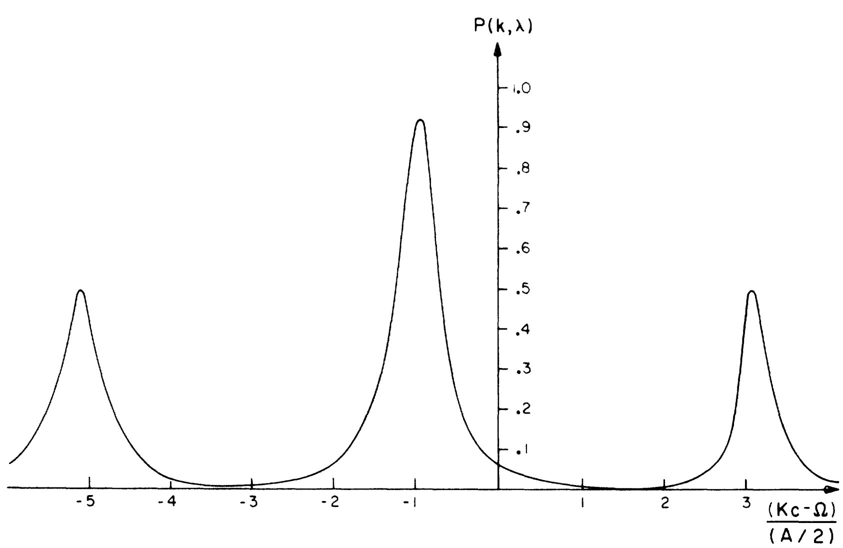

FIG. 7. Line shape for a Lamb shift equal to the natural linewidth and an applied field such that stimulated emission is twice as probable as spontaneous emission.

where

$$q = 2\sum_{k,\lambda} \frac{u^2(k,\lambda)}{s + ikc - i\Omega} \quad . \tag{22'}$$

We desire to obtain the transform of Eq. (25),

$$b^+(t) = \frac{1}{2\pi i} \lim_{\delta\to 0} \int_{-i\infty+\delta}^{i\infty+\delta} e^{st} B^+(s) ds$$

$$= \frac{1}{2\pi i}\frac{1}{\sqrt{2}} \lim_{\delta\to 0} \int_{-i\infty+\delta}^{i\infty+\delta} ds \, \frac{e^{st}}{s + 2q(s)} \quad . \tag{A1}$$

Making a change of variables

$$s = iy + \delta \quad , \tag{A2}$$

the integral becomes

$$b^+(t) = \frac{1}{2\pi i}\frac{1}{\sqrt{2}} \lim_{\delta\to 0} \int_{-\infty}^{\infty} dy \, \frac{e^{(iy+\delta)t}}{y - i\delta - 2iq(iy+\delta)} \quad . \tag{A3}$$

Now the only place where the limit causes any problems is in $q(iy+\delta)$. We must investigate the limit

$$\lim_{\delta\to 0} q(iy+\delta) = 2\lim_{\delta\to 0} \sum_{k,\lambda} \frac{u^2(k,\lambda)}{iy + \delta + ikc - i\Omega} \quad .$$

In order to calculate spontaneous emission we must evaluate the sum in the limit of a very large cavity. In that limit the discrete sum over k becomes an integral over all k space. Then the sum becomes

$$q(iy+\delta) \to -2i\,\frac{V}{(2\pi)^3} \sum_{k,\lambda} \int \frac{d^3k\, u^2(k,\lambda)}{kc + y - \Omega - i\delta} \quad . \tag{A4}$$

By a well-known identity we can write the limit as

$$\lim_{\delta\to 0} q(iy+\delta)$$

$$= -\frac{2iV}{(2\pi)^3} \sum_{\lambda} \left(\mathcal{P}\int \frac{d^3k\, u^2(k,\lambda)}{kc + y - \Omega} + i\pi \int d^3k\, u^2(k,\lambda)\, \delta(kc + y - \Omega) \right), \tag{A5}$$

where $\mathcal{P}$ denotes the Cauchy principal-part integral and the δ is the Dirac δ function. Substituting from Eq. (20), we can carry out the second of these integrations

$$\frac{2\pi V}{(2\pi)^3} \sum_{\lambda} \int d^3k\, u^2(k,\lambda)\delta(kc + y - \Omega) = \frac{\mu^2\Omega^2}{3\hbar c^3}(\Omega + y) \quad . \tag{A6}$$

If we ignore the y, this expression is exactly one-fourth the Einstein A coefficient. That the y is in fact negligible follows because the decay constant and the frequency shift are small compared with Ω. Thus the integral (A3) can have no appreciable contribution from any frequency components e^{iyt} for which y does not satisfy $y \ll \Omega$.

The principal-part integral is divergent in our dipole approximation.[14] This is not surprising since it is just the self-energy of the electron. We must carry out the mass renormalization and cut off the integral at high frequencies to obtain the usual expression for the Lamb shift. After renormalization (R. N.) our expression for the frequency shift is

$$\mathrm{Im}(q)_{\mathrm{R.N.}} = -(\Omega^2\mu^2/3\pi\hbar c^3)(\Omega - y)\ln|Kc/(\Omega - y)| \quad , \tag{A7}$$

where K is the cutoff wave number. Since the frequency shift like the decay constant is small compared with Ω, we have approximately

$$\mathrm{Im}(q)_{\mathrm{R.N.}} = -(\Omega^3\mu^2/3\pi\hbar c^3)\ln|Kc/\Omega| \quad , \tag{A8}$$

which is just one-half the usual expression for the Lamb shift in a "two-level atom."[15]

Finally, then, we see that, for the purposes of substituting into Eq. (25), q is given approximately by

$$q = \tfrac{1}{4}A - i\,\tfrac{1}{2}\Delta \quad , \tag{A9}$$

where

$$A \equiv 4\mu^2\Omega^3/3\hbar c^3 \tag{A10}$$

is the Einstein A coefficient, and

$$\Delta \equiv (2\Omega^3\mu^2/3\pi\hbar c^3)\ln|Kc/\Omega| \tag{A11}$$

is the Lamb shift.

---

*Work supported in part by a National Science Foundation grant.

[1]B. R. Mollow, Phys. Rev. 188, 1969 (1969); Phys. Rev. A 2, 76 (1970).

[2]M. C. Newstein, Phys. Rev. 167, 89 (1968).

[3]S. G. Rautian and I. I. Sobelman, Zh. Eksperim. i Teor. Fiz. 41, 456 (1961)[Soviet Phys. JETP 14, 328 (1962)].

[4]R. K. Wangsness, Phys. Rev. 149, 60 (1966).

[5]M. Crisp and E. T. Jaynes, Phys. Rev. 179, 1253 (1969).

[6]C. R. Stroud, Jr. and E. T. Jaynes, Phys. Rev. A 1, 106 (1970).

[7]See J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1967), Sec. 2.3.

[8]The particular shape of the volume and the particular boundary conditions are unimportant since we will later take the volume to be very large so that the sum over

modes is replaced by an integral.

[9]E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963).

[10]When we describe spontaneous emission, we will allow the cavity to become very large. We will assume that in that limit n also becomes very large so that in that limit ϵ approaches a constant finite value representing a finite strength.

[11]An excellent review article on the Stark effect is A. M. Bonch-Bruevich and V. A. Khodovoi, Usp. Fiz. Nauk 93, 71 (1967)[Soviet Phys. Usp. 10, 637 (1968)].

[12]This technique was apparently first used for this type of problem by G. Kallen, in *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1958), Vol. V. It has been used several times since for simple spontaneous emission.

[13]Numerical calculations indicate that the field strength $\epsilon = A$ would require a laser beam tuned to resonance with an intensity of about 1 W/cm^2 for the sodium D lines.

[14]It is interesting to note that this integral is not divergent if we retain the retardation rather than making the dipole approximation, though it does not appear to give the correct answer for the Lamb shift. See C. R. Stroud, Jr., Ph.D. thesis, Washington University, St. Louis, Mo. (unpublished), for a detailed treatment for the $1s$ and $2p$ states in hydrogen. There it is shown that there is a small correction to exponential decay which goes at t^{-2} for sufficiently long times.

[15]We have used exactly the same method as H. A. Bethe, Phys. Rev. 72, 339 (1947), for our renormalization.

PHYSICAL REVIEW A VOLUME 3, NUMBER 3 MARCH 1971

Excitations Radiated from a Thermal Source in Helium II below 0.3K*

R. W. Guernsey,[†] Jr. and K. Luszczynski[‡]
Department of Physics, Washington University, Saint Louis, Missouri 63130
(Received 19 October 1970)

Results are presented on the propagation of excitations generated by a small (~ 1 mm square) pulsed heater immersed in a large (~ 300 cc) sample of He II maintained at temperatures below 300 mK. Small carbon-film detectors placed at 1, 2, 3, and 4 cm from the heater are used to measure the flux of radiation over a wide range of heater-power densities (W_H). The observed signals are free from wall reflections. At the lowest temperatures and small W_H, the fastest excitations propagate without dispersion or attenuation at a velocity of 234 ± 4 m/sec. Spatial attenuation which is observed at higher temperatures corresponds to effective mean free paths for large-angle scattering of 1, 2, and 3cm at 306 ± 6, 272 ± 6, and 254 ± 8 mK, respectively. When W_H exceeds 0.4 W/cm^2, signal shapes reflecting appreciable interactions between radiated excitations are observed. For $W_H > 2.9$ W/cm^2, the observed signal acquires another component which propagates without appreciable dispersion at a velocity of 200 ± 10 m/sec. In these experiments, no evidence has been found of any excitations associated with any upward bend in the He II phonon-dispersion curve resulting in signal velocities in excess of the first-sound velocity.

I. INTRODUCTION

A heater immersed in He II is expected to generate phonons and other excitations. At sample temperatures above about 600 mK, collisions with intrinsic He II excitations thermalize these emissions to produce the density and temperature fluctuations of first and second sound. At lower temperatures, the intrinsic thermal excitations become rare and direct radiation of phonons from the heater to the detector becomes possible. This paper reports on an experimental study of such signals received at distances of 1–4 cm for input energy fluxes between 0.02 and 8 W/cm^2 and at sample temperatures of 120–900 mK.

There is evidence of direct phonon radiation in past experiments on the transmission of heat pulses down a He II filled tube where one end is the heater and the other a thermometric detector.[1] In such a geometry, however, much of the radiation reaches the detector via reflection from the tube wall. The present experiment is designed to eliminate this complication. A small "point" heater and small detectors are positioned far from the walls in a relatively large sample of He II. At low temperatures corresponding to long mean free path, the signal shape should be indicative of the velocity distribution of the emitted excitations.

This work was originally motivated by a desire to determine the amount of dispersion (if any) in the 50–250-GHz range (energy equal to 2.5–12 K and wave number equal to 0.13–0.66 Å^{-1}) of the phonon spectrum. It is known that the He II liquid structure factor $S(K)$, in the low-temperature limit must have a slope of $1/(2mc)$ as the wave number K approaches 0, where m is the atomic mass of He^4 and c the first-sound velocity.[2] Jackson[3] and Miller *et al.*[2] independently pointed out some time ago