# Characterization of experimental (noisy) strange attractors

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In experiments involving deterministic chaotic signals, contamination by random noise is unavoidable. We discuss a practical method that disentangles the deterministic chaos from the random part. The method yields a characterization of the strange attractors together with an estimate of the size of random noise.

In recent papers[1–3] it has been suggested that strange attractors can be characterized by the correlation exponent $\nu$. This exponent is defined on the basis of a long-time series $\{\vec{X}_i\}_{i=1}^N$ of points on the attractor by considering the correlation integral

$$C(l)=\frac{1}{N^2}\sum_{i\neq j}\Theta(l-|\vec{X}_i-\vec{X}_j|)\ , \tag{1}$$

where $\Theta$ is the Heaviside function. This correlation integral counts the number of pairs whose distance $|\vec{X}_i-\vec{X}_j|$ is smaller than $l$. It has been shown that $C(l)$ scales as

$$C(l)\sim l^\nu\ , \tag{2}$$

and that the exponent $\nu$ can serve as a satisfactory measure of the strange attractor. We have argued that $\nu\leq\sigma\leq D$, where $D$ and $\sigma$ are the fractal and information dimension, respectively,[4,5] with equalities obtaining only when the fractal is uniform (i.e., no "seniority" effects in the langauge of Refs. 1–3). In a later work[6] we have shown that, in fact, $\nu$ is one of an infinite set of dimensions that characterize probabilistic fractals, but that it is singled out by the ease of its actual calculation on the basis of time series.

In fact, characterizing the attractor with the exponent $\nu$ rather than with the fractal dimension $D$ has a definite advantage for experimental applications. The numerical estimate of $D$ calls for partitioning phase space into boxes of size $l$, and then counting the number of boxes which contain a piece of the attractor. Such "box counting" algorithms are extremely slowly converging even for low dimensional attractors ($D<2$), and are quite impractical for higher dimensional attractors ($D>2$).[7] They call for a measurement of a prohibitively large number of points of a time signal. In contrast, the algorithms to calculate $\nu$ are efficiently converging even with a relatively small number of experimental points in a time series, and even at high dimensions.[3]

A practical question that has been raised, however, is how to determine $\nu$ in experimental situations, where unavoidable noise smears the fractal structure of the strange attractor. The purpose of this Brief Report is to give a practical answer to this question, with the hope that the proposed algorithm will be used in the analysis of the various experiments which reveal strange attractors.

The basic idea is that when we have a deterministic motion on a strange attractor, the existence of noise will not ruin the fractal structure, but will cause fuzziness on length scales that are smaller or equal to the noise strength.[8–10] To

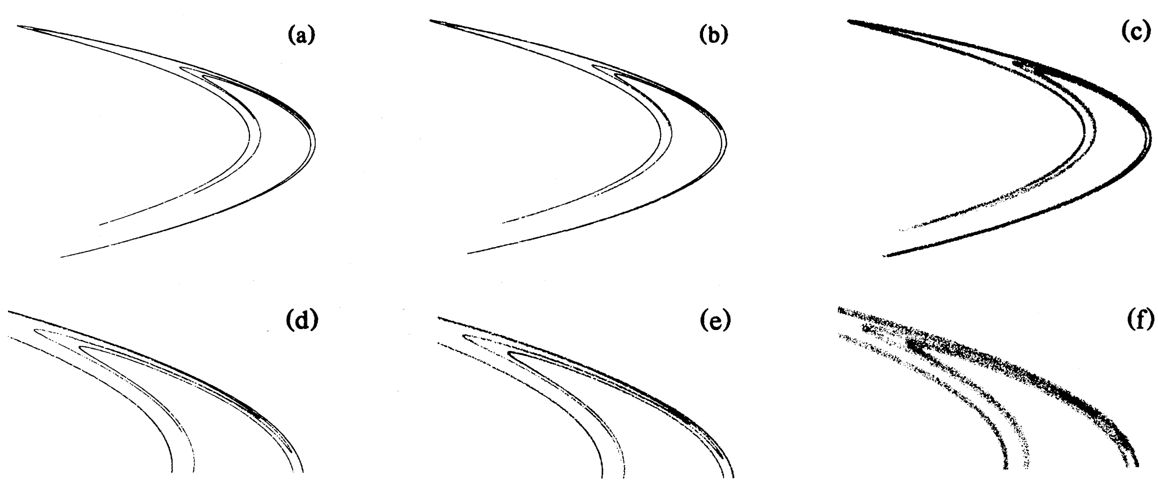


FIG. 1. Attractor of the Hénon map without noise and with a random jitter added to each iteration. (a) No noise; (b) random jitter chosen from the interval [ −0.001, 0.001]; (c) random jitter chosen from the interval [ −0.05, 0.05]; (d) blowup of (a); (e) blowup of (b); (f) blowup of (c).


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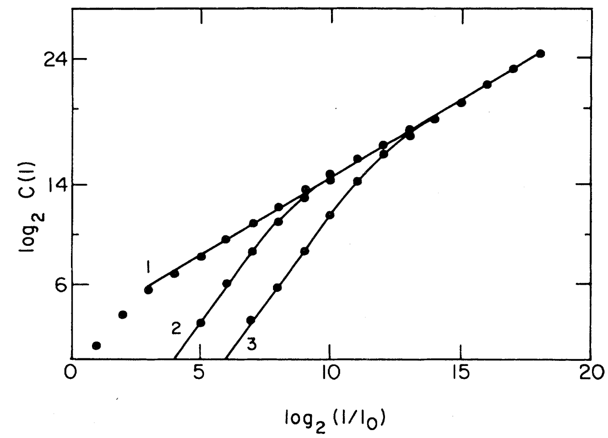


FIG. 2. $\log_2 C(l)$ as a function of $\log_2 l$ for the Hénon map embedded in three dimensions. Curve 1 is for the map without noise and yields $\nu = 1.25$. Curve 2 is for the map with a random jitter chosen from $[-0.5\times10^{-3}, 0.5\times10^{-3}]$. Curve 3 is for the map with a random jitter chosen from $[-0.5\times10^{-2}, 0.5\times10^{-2}]$. Curves 2 and 3 break at length scales that are determined by the noise level, below which the slope is approximately 3.

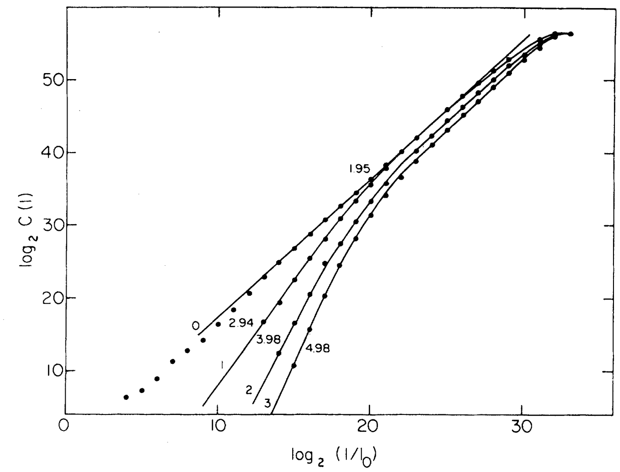


FIG. 3. 600-dimensional system for which $\nu = 1.95$. Shown is $\log_2 C(l)$ as a function of $\log_2 l$ for the Mackey-Glass equation with $\tau = 17$ (Refs. 3 and 5). Curve 0 pertains to the system without noise. For the same level of noise we embed the attractor in three-, four-, and five- dimensional spaces and obtain curves 1, 2, and 3, respectively.

clarify this point we show in Fig. 1(a), the Hénon attractor[11] without any noise and with a random jitter that has been added to each iteration in Figs. 1(b) and 1(c). The size of the jitter has been chosen randomly from the intervals $[-0.001, 0.001]$ and $[-0.05, 0.05]$ in the cases of Figs. 1(b) and 1(c), respectively. On the scale of Figs. 1(a)–1(c) one hardly notices the difference between Figs. 1(a) and 1(b), although in Fig. 1(c) the effect of the noise is already apparent. However, upon magnification [Figs. 1(d), 1(e), and 1(f)] the fuzziness introduced by the noise becomes very clear.

If we now embed the attractor in $d$-dimensional space, we expect that the noisy trajectory will be space filling on length scales smaller than the noise strength. When it is space filling $C(l)$ scales like

$$C(l) \sim l^d \ . \tag{3}$$

We thus expect that a plot of $\log_2 C(l)$ as a function of $\log_2$ will have a slope of $\nu$ down to length scales characterized by the noise strength and then a slope of $d$. A confirmation of this idea in the context of Hénon's map is shown in Fig. 2. This agrees with Zardecki's results[10] for the behavior of the fractal dimension as a function of $l$.

Experimental systems with strange attractors are typically high dimensional systems which, however, possess low dimensional attractors. Moreover, one typically follows only one (or a few) of the multitude of degrees of freedom. It has been argued before[3,12,13] that a knowledge of the time series of one variable is sufficient. If we know $X(t)$ we can reconstruct a $d$-dimensional space from the $d$-dimensional variable $\vec{X}_i$:

$$\vec{X}_i = \{X(t_i),\ X(t_i+\tau),\ \ldots,\ X(t_i+(d-1)\tau)\} \ , \tag{4}$$

and then use $\{\vec{X}_i\}_{i=1}^{N}$ in Eq. (1).

The algorithm to extract $\nu$ in a noisy strange attractor with a given noise level is now described as follows: One reconstructs space with an increasing value of $d$ and plots $\log_2 C(l)$ versus $\log_2 l$. For a series of values of $d$ such that $d > \nu$ one should see then a fan-shaped plot. Above the length scales characterizing the noise strength all curves should be linear with a slope equal to $\nu$. All curves should break at the same value of $l$, below which a slope equal to $d$ should be seen. An example is shown in Fig. 3. Here we have a 600-dimensional system which is generated from the Mackey-Glass delay differential equation.[3,5] For the parameter chosen the strange attractor is characterized by $\nu = 1.95$. The noise strength is $10^{-3}$. We see that all the graphs have a break at the same $l$ (which is precisely of the order of the noise strength) below which they have a slope of $d$.

The advantage of the proposed algorithm is twofold. Firstly, it offers an efficient way to characterize experimental noisy attractors. Secondly, the position of the break in the plots of $\log_2 C(l)$ vs $\log_2 l$ supplies information on the noise level in the system. Thus one knows which length scales belong to the deterministic chaotic motion and which length scales belong to the blurred realm of random processes. This information might be of considerable use for other experimental applications.

We hope that the simplicity of the algorithm and its usefulness would prompt a characterization of experimental strange attractors along these lines.


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