Detrapping stochastic particle instability for electron motion in combined longitudinal wiggler and radiation wave fields

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The relativistic motion of an electron is calculated in the combined fields of the longitudinal magnetic wiggler field $[B_0+B_w\sin(k_0z)]\hat{e}_z$ and constant-amplitude, circularly polarized primary and secondary electromagnetic waves propagating in the z direction. It is shown that the presence of the secondary electromagnetic wave can detrap electrons near the separatrix of the primary wave or near the bottom of the primary-wave potential well. The results obtained are also applicable to the electron cyclotron maser (gyrotron) in the limit $B_w=0$ and $k_0=0$.

I. INTRODUCTION

Stochastic instabilities can develop in systems where the particle motion is described by certain classes of nonlinear oscillator equations of motion. Analytic and numerical techniques have been developed that describe essential features of stochastic instabilities[1–9] that occur under many different physical circumstances. Particularly noteworthy is the development of secular variations of the particle action or energy for classes of particles which in the absence of the appropriate perturbation force undergo nonlinear periodic motion. This nonlinear periodic motion can be greatly modified by the stochastic instability and develop chaotic features.

In this paper, we consider the possible development of stochastic instability in circumstances relevant to sustained free-electron-laser (FEL) radiation generation in a longitudinal magnetic wiggler configuration.[10] In particular, we consider a tenuous relativistic electron beam with negligibly small equilibrium self-fields propagating in the z direction through a steady, radiation field with two monochromatic wave components. The detrapping of electrons from the primary-wave potential well, due to stochastic instability, is investigated. To briefly summarize, the relativistic electrons travel along the z direction in the combined fields of a longitudinal magnetic wiggler[10] [Eq. (5)], a constant-amplitude primary transverse electromagnetic wave $(\delta E,\omega,k)$ propagating in the z direction [Eqs. (1) and (2)], as well as a secondary (parasitic) transverse electromagnetic wave $(\delta E_1,\omega_1,k_1)$ propagating in the z direction [Eqs. (3) and (4)]. The dynamical equation of motion for an electron in the above field configuration reduces to the driven pendulum equation (23). By analogy with the stochastic instability previously studied for a free-electron laser with helical transverse wiggler field,[7–9] we make use of the techniques developed by Zaslavskii and Filonenko[2] to determine the region where the electrons are detrapped from the primary-wave potential well.

In Secs. II and III, the dynamical equation of motion is obtained for an electron in the electromagnetic field configuration described by Eqs. (1)–(5). In Sec. IV, the conditions are derived for electron detrapping near the separatrix of the primary wave and near the bottom of the primary-wave potential well. The results obtained in Sec. IV are also applicable to the electron cyclotron maser (gyrotron). Finally, in Sec. V, the results are summarized.

II. ELECTROMAGNETIC FIELD CONFIGURATION AND BASIC ASSUMPTIONS

In the present analysis we examine the relativistic motion of an electron in the combined fields of a *longitudinal* wiggler magnetic field, a primary circularly polarized transverse electromagnetic wave propagating in the z direction, and a secondary circularly polarized transverse electromagnetic wave with frequency and wave number close to that of the primary wave. The electron beam density is assumed to be sufficiently low that equilibrium self-fields are negligibly small, and all spatial variations of field quantities are taken to be in the z direction. In addition, a laser oscillator configuration is assumed in which the steady-state amplitudes of the primary wave (δE) and secondary wave (δE_1) have negligibly small spatial variation. The electromagnetic field of the primary wave is given by

$$\delta\vec{E}(\vec{x},t)=-\delta E[\sin(kz-\omega t)\vec{e}_x+\cos(kz-\omega t)\vec{e}_y]\ , \tag{1}$$

$$\delta\vec{B}(\vec{x},t)=\left[\frac{ck\delta E}{\omega}\right][\cos(kz-\omega t)\vec{e}_x-\sin(kz-\omega t)\vec{e}_y]\ , \tag{2}$$

and the electromagnetic field of the secondary wave is given by

$$\delta\vec{E}_1(\vec{x},t)=-\delta E_1[\sin(k_1z-\omega_1t)\vec{e}_x+\cos(k_1z-\omega_1t)\vec{e}_y]\ , \tag{3}$$

$$\delta\vec{B}_1(\vec{x},t)=\left[\frac{ck_1\delta E_1}{\omega_1}\right][\cos(k_1z-\omega_1t)\vec{e}_x-\sin(k_1z-\omega_1t)\vec{e}_y]\ . \tag{4}$$

The longitudinal magnetic field is assumed to be of the form[10]

$$\vec{B}^0(\vec{x})=[B_0+B_w\sin(k_0z)]\vec{e}_z , \quad (5)$$

where $\lambda_0=2\pi/k_0=\text{const}$ is the wiggler wavelength, and $B_w=\text{const}$ is the wiggler amplitude. Equation (5) is a valid approximation near the axis of the multiple-mirror configuration for electrons with sufficiently small orbital radius r that $k_0^2r^2 \ll 1$. In what follows, it is also assumed that the relative ordering of field amplitudes is given by

$$|B_0| > |B_w| \gg |\delta E| > |\delta E_1| . \quad (6)$$

Before the electrons enter the interaction region, the initial conditions are taken to be the following: axial momentum p_{z0}, transverse momentum $p_{\perp 0}$, and energy

$$E_0=\gamma_0 mc^2=(c^2p_{z0}^2+c^2p_{\perp 0}^2+m^2c^4)^{1/2} ,$$

where

$$\gamma_0^2=(1-v_{\perp 0}^2/c^2-v_{z0}^2/c^2)^{-1} .$$

It is necessary for the electrons to enter the interaction region with an initial transverse momentum, since it is this excess transverse momentum that serves to drive the free-electron laser instability for the longitudinal wiggler configuration in Eq. (5).[10]

III. EQUATIONS OF MOTION

In this section the relativistic Lorentz force equation for an electron moving in the electromagnetic field configuration given by Eqs. (1)—(5) is used to determine the coupled equations of motion for the electron energy and the slowly varying phase of the ponderomotive bunching force. The components of the relativistic Lorentz force equation are given by

$$\frac{dp_x}{dt}=-e\frac{v_y}{c}[B_0+B_w\sin(k_0z)]+e\,\delta E\left[1-\frac{kv_z}{\omega}\right]\sin(kz-\omega t)+e\,\delta E_1\left[1-\frac{k_1v_z}{\omega_1}\right]\sin(k_1z-\omega_1t) , \quad (7)$$

$$\frac{dp_y}{dt}=\frac{ev_x}{c}[B_0+B_w\sin(k_0z)]+e\,\delta E\left[1-\frac{kv_z}{\omega}\right]\cos(kz-\omega t)+e\,\delta E_1\left[1-\frac{k_1v_z}{\omega_1}\right]\cos(k_1z-\omega_1t) , \quad (8)$$

$$\frac{dp_z}{dt}=e\left[\frac{kv_x}{\omega}\delta E\sin(kz-\omega t)+\frac{kv_y}{\omega}\delta E\cos(kz-\omega t)+\frac{k_1v_x}{\omega_1}\delta E_1\sin(k_1z-\omega_1t)+\frac{k_1v_y}{\omega_1}\delta E_1\cos(k_1z-\omega_1t)\right] , \quad (9)$$

and

$$\frac{dE}{dt}=e[v_x\delta E\sin(kz-\omega t)+v_y\delta E\cos(kz-\omega t)+v_x\delta E_1\sin(k_1z-\omega_1t)+v_y\delta E_1\cos(k_1z-\omega_1t)] , \quad (10)$$

where

$$E=\gamma mc^2=mc^2(1-v_\perp^2/c^2-v_z^2/c^2)^{-1/2}$$

is the electron energy.

To express the equations of motion in a useful form, we define $p_+\equiv p_x+ip_y$ and combine Eqs. (7) and (8) to give

$$\frac{d}{dt}\{p_+\exp[-i\sigma(t)]\}=ie\,\delta E\left[1-\frac{kv_z}{\omega}\right]\exp\{-i[kz-\omega t+\sigma(t)]\}+ie\,\delta E_1\left[1-\frac{k_1v_x}{\omega_1}\right]\exp\{-i[k_1z-\omega_1t+\sigma(t)]\} , \quad (11)$$

where

$$\sigma(t)\equiv\int_0^t dt[eB_0+eB_w\sin(k_0z)]c/E .$$

Assuming that

$$|p_{\perp 0}| \gg \left|e\delta E\int_0^t dt(1-kv_z/\omega)\exp\{-i[kz-\omega t+\sigma(t)]\}+e\delta E_1\int_0^t dt(1-k_1v_z/\omega_1)\exp\{-i[k_1z-\omega_1t+\sigma(t)]\}\right| , \quad (12)$$

it is straightforward to show that the approximate solution to Eq. (11) is

$$p_+=p_{\perp 0}\exp[i\phi+i\sigma(t)] , \quad (13)$$

where ϕ is the initial $(t=0)$ phase of the transverse momentum. From Eq. (13), it follows that the magnitude of the transverse momentum remains approximately constant, although the individual x and y components of the momentum can be strongly modulated by the factor $\exp[i\sigma(t)]$, thereby resulting in the generation of high-frequency radiation.

In order to further simplify Eq. (13), we define

$$\omega_b = eB_0/mc \text{ and } \zeta = \int_0^t dt/\gamma \ . \tag{14}$$

Moreover, in the wiggler contribution to the expression for $\sigma(t)$, we approximate $v_z \simeq v_{z0}$ and $\gamma \simeq \gamma_0$. This gives

$$\sigma(t) = \omega_b \zeta + \frac{eB_w}{p_{z0}k_0 c}[1-\cos(k_0 z)] \ . \tag{15}$$

Rewriting Eqs. (9) and (10) in terms of p_+ and $p_+^* = p_x - ip_y$ gives

$$\frac{dp_z}{dt} = \frac{ie}{2m\gamma}\left[\delta E\, p_+^* \frac{k}{\omega}\exp[-i(kz-\omega t)] - \delta E\, p_+ \frac{k}{\omega}\exp[i(kz-\omega t)] + \delta E_1 p_+^* \frac{k_1}{\omega_1}\exp[-i(k_1 z - \omega_1 t)] \right.$$
$$\left. -\delta E\, p_+ \frac{k_1}{\omega_1}\exp[i(k_1 z - \omega_1 t)]\right] \tag{16}$$

and

$$\frac{dE}{dt} = \frac{ie}{2m\gamma}\{\delta E\, p_+^* \exp[-i(kz-\omega t)] - \delta E\, p_+ \exp[i(kz-\omega t)] + \delta E_1 p_+^* \exp[-i(k_1 z-\omega_1 t)]$$
$$-\delta E_1 p_+ \exp[i(k_1 z - \omega_1 t)]\} \ . \tag{17}$$

Substituting Eq. (15) into Eq. (13), expanding the exponential factors in a series of ordinary Bessel functions $J_l(x)$, and substituting the resulting expression into Eqs. (16) and (17) give (for harmonic component l)

$$\frac{dp_z}{dt} = \frac{ep_{\perp 0}}{m\gamma_0} J_{-l}\left[\frac{eB_w}{ck_0 p_{z0}}\right]\left[\delta E \frac{k}{\omega}\sin\psi + \delta E_1 \frac{k_1}{\omega_1}\sin\psi_1\right] , \tag{18}$$

$$\frac{dE}{dt} = \frac{ep_{\perp 0}}{m\gamma_0} J_{-l}\left[\frac{eB_w}{ck_0 p_{z0}}\right](\delta E \sin\psi + \delta E_1 \sin\psi_1) \ . \tag{19}$$

In Eqs. (18) and (19) we have approximated $\gamma \simeq \gamma_0$ on the right-hand side and retained only those terms with the slowly varying phases (ψ,ψ_1) of the ponderomotive bunching force. The phases (ψ,ψ_1) are defined by $(l=0,1,2,\ldots)$

$$\psi \equiv kz - \omega t + \omega_b \zeta + lk_0 z + \phi + l\pi/2 + eB_w/ck_0 p_{z0} \ , \tag{20}$$

$$\psi_1 \equiv \frac{(k_1+lk_0)}{(k+lk_0)}\left[\psi - \phi - \frac{l\pi}{2} - \frac{eB_w}{ck_0 p_{z0}}\right] + \left[\frac{(k_1+lk_0)(\omega-\omega_{c0})}{(k+lk_0)} - (\omega_1 - \omega_{c0})\right]t + \frac{eB_w}{ck_0 p_{z0}} + \phi + \frac{l\pi}{2} \ . \tag{21}$$

Here $\omega_{c0} \equiv eB_0/mc\gamma_0$ is the relativistic cyclotron frequency in the average solenoidal magnetic field B_0. Differentiating Eq. (20) with respect to time t gives

$$\frac{d\psi}{dt} = (k+lk_0)v_z - \omega + \frac{\omega_b}{\gamma} = \frac{(k+lk_0)p_z/m + \omega_b}{\gamma} - \omega \ . \tag{22}$$

Equations (19) and (22) give the desired dynamical equations of motion for the electron energy E and the phase ψ of the primary-wave bunching force with radiation emission occurring at the lth harmonic of the wiggler magnetic field wave number k_0.

In order to obtain a solution to Eqs. (19) and (22), we differentiate Eq. (22) with respect to time t and substitute Eqs. (18) and (19) into the resulting expression. In normalized variables, this yields the equation of motion

$$\frac{d^2\psi}{d\tau^2} + \sin\psi = -\delta_1 \sin[\hat{k}_1(\psi - V_p\tau + \alpha_1)] \ , \tag{23}$$

where $\tau \equiv \hat{\omega} t$, $\delta_1 \equiv \hat{\omega}_1^2/\hat{\omega}^2$, $V_p \equiv \Delta\omega_1/\hat{k}_1\hat{\omega}$, and

$$\hat{\omega}^2 \equiv \frac{ep_{\perp 0}\delta E}{c^2 m^2 \gamma_0^2} J_{-l}\left[\frac{eB_w}{ck_0 p_{z0}}\right] \times [(k+lk_0)v_{z0} + \omega_{c0} - c^2 k(k+lk_0)/\omega] \ ,$$

$$\hat{\omega}_1^2 \equiv \frac{ep_{\perp 0}\delta E_1}{c^2 m^2 \gamma_0^2} J_{-l}\left[\frac{eB_w}{ck_0 p_{z0}}\right] \times [(k_1+lk_0)v_{z0} + \omega_{c0} - c^2 k_1(k_1+lk_0)/\omega_1] \ ,$$

$$\hat{k}_1 \equiv (k_1+lk_0)/(k+lk_0) \ , \tag{24}$$

$$\alpha_1 \equiv \left[\frac{eB_w}{ck_0 p_{z0}} + \phi + \frac{l\pi}{2}\right]\frac{(1-\hat{k}_1)}{\hat{k}_1} \ ,$$

$$\Delta\omega_1 \equiv (\omega_1 - \omega_{c0}) - (k_1+lk_0)(\omega-\omega_{c0})/(k+lk_0) \ .$$

Equation (23) is of the form of a driven pendulum equation which, in the absence of the secondary wave ($\delta_1 = 0$), is a conservative equation. In the presence of the secondary wave ($\delta_1 \neq 0$), the right-hand side of Eq. (23), when averaged over the lowest-order motion, can lead to secular changes in the electron energy and result in stochastic electron motion and a concomitant detrapping of electrons from the primary-wave ponderomotive potential well.

Finally, we reiterate that several approximations have been made in deriving Eq. (23). First, Eq. (12) must be satisfied. Taking $v_z \simeq v_{z0}$, making use of Eq. (15), and retaining only the slowly varying phases (ψ,ψ_1), Eq. (12) can be expressed as

$$|p_{\perp 0}| \gg \left| e\,\delta E \left[1-\frac{kv_{z0}}{\omega}\right] J_{-l}\left[\frac{eB_w}{ck_0p_{z0}}\right]\frac{\sin\psi}{d\psi/dt} + e\,\delta E_1\left[1-\frac{k_1v_{z0}}{\omega_1}\right] J_{-l}\left[\frac{eB_w}{ck_0p_{z0}}\right]\frac{\sin\psi_1}{d\psi_1/dt}\right| . \tag{25}$$

Also, in retaining only the axial component of the magnetic field in Eq. (5), it has been assumed that the influence of the lowest-order radial magnetic field[10]

$$B_r^0=\tfrac{1}{2}B_w k_0 r\cos(k_0 z) \tag{26}$$

on the electron motion and the ponderomotive bunching phases (ψ,ψ_1) is negligibly small. It can be shown that the effects of B_r^0 on ψ and ψ_1 are negligibly small provided

$$1\gg \frac{\omega_b}{k_0v_{z0}\gamma_0}\frac{k}{2k_0}\left[\frac{p_{\perp 0}}{p_{z0}}\right]^2 \left|\sum_{n=-\infty}^{\infty}\frac{J_n^2\left[\dfrac{eB_w}{ck_0p_{z0}}\right]}{(n+\omega_b/k_0v_{z0}\gamma_0)^2}\right| , \tag{27}$$

$$1\gg \frac{\omega_b}{k_0v_{z0}\gamma_0}\frac{k_1}{2k_0}\left[\frac{p_{\perp 0}}{p_{z0}}\right]^2 \left|\sum_{n=-\infty}^{\infty}\frac{J_n^2\left[\dfrac{eB_w}{ck_0p_{z0}}\right]}{(n+\omega_b/k_0v_{z0}\gamma_0)^2}\right| . \tag{28}$$

In Eqs. (27) and (28), it is assumed that system parameters are well removed from beam-cyclotron resonance so that the denominators do not vanish (i.e., $\omega_b/\gamma_0 \neq -nk_0v_{z0}$).

IV. STOCHASTIC INSTABILITY

In this section we determine the region of stochastic instability for Eq. (23) in the limit $\delta_1 \ll 1$. Multiplying Eq. (23) by $d\psi/d\tau$ gives

$$\frac{dH_0}{d\tau}=\frac{d}{d\tau}\left[\frac{1}{2}\left[\frac{d\psi}{d\tau}\right]^2-\cos\psi\right] = -\frac{d\psi}{d\tau}\delta_1\sin[\hat{k}_1(\psi-V_p\tau+\alpha_1)] . \tag{29}$$

In lowest order ($\delta_1=0$), Eq. (29) gives the conserved energy

$$H_0=\frac{1}{2}\left[\frac{d\psi}{d\tau}\right]^2-\cos\psi=\text{const} . \tag{30}$$

Equation (30) can also be expressed as

$$\frac{1}{4}\left[\frac{d\psi}{d\tau}\right]^2=\kappa^2-\sin^2\left[\frac{\psi}{2}\right] , \tag{31}$$

where

$$\kappa^2=\tfrac{1}{2}(1+H_0) . \tag{32}$$

The solution to Eq. (31) can be expressed in terms of the elliptic integrals, $F(\eta,\kappa)$ and $E(\eta,\kappa)$, where

$$F(\eta,\kappa)=\int_0^\eta \frac{d\eta'}{(1-\kappa^2\sin^2\eta')^{1/2}} , \tag{33}$$

$$E(\eta,\kappa)=\int_0^\eta d\eta'(1-\kappa^2\sin^2\eta')^{1/2} . \tag{34}$$

In the present analysis, Eq. (31) is solved assuming trapped electron orbits with $\kappa^2<1$. Introducing the coordinate η defined by

$$\kappa\sin\eta=\sin\left[\frac{\psi}{2}\right] , \tag{35}$$

Eq. (31) can be expressed as

$$\left[\frac{d\eta}{d\tau}\right]^2=(1-\kappa^2\sin^2\eta) , \tag{36}$$

which has the solution (neglecting initial conditions)

$$F(\eta,\kappa)=\tau . \tag{37}$$

Here $\eta=\sin^{-1}[(1/\kappa)\sin(\psi/2)]$, and $F(\eta,\kappa)$ is the elliptic integral of the first kind defined in Eq. (33). Several properties of the trapped electron motion can be determined directly from Eqs. (31), (35), and (37). For example, it is readily shown that the normalized velocity is given by

$$\frac{d\psi}{d\tau}=2\kappa\,\mathrm{cn}\tau , \tag{38}$$

where $\mathrm{cn}\tau=(1-\mathrm{sn}^2\tau)^{1/2}$, and $\mathrm{sn}\tau=\sin\eta\equiv(1/\kappa)\sin(\psi/2)$ is the inverse function to the elliptic integral

$$F(\sin^{-1}[\kappa^{-1}\sin(\psi/2)],\kappa) .$$

For subsequent analysis of the stochastic instability, it is useful to express properties of the trapped electron motion in terms of action-angle variables (I,θ). Defining, in the usual manner,

$$I=I(H_0)=\frac{1}{2\pi}\oint\frac{d\psi}{d\tau}d\psi ,$$
$$\theta(\psi,I)=\frac{\partial}{\partial I}S(\psi,I) , \tag{39}$$
$$S(\psi,I)=\frac{1}{2\pi}\int^\psi\frac{d\psi}{d\tau}d\psi ,$$

we find

$$I(H_0)=\frac{8}{\pi}[E(\pi/2,\kappa)-(1-\kappa^2)F(\pi/2,\kappa)] , \tag{40}$$

where $\kappa^2=(1/2)(1+H_0)$, and $F(\eta,\kappa)$ and $E(\eta,\kappa)$ are defined in Eqs. (33) and (34). The unperturbed equation of motion (23) (for $\delta_1=0$) can be expressed in the new variables (I,θ) as

$$\frac{dI}{d\tau}=0,\quad \frac{d\theta}{d\tau}=\frac{\omega_T(I)}{\hat{\omega}} , \tag{41}$$

where $\hat{\omega}$ is defined in Eq. (24), and the frequency $\omega_T(I)$ is determined from $\omega_T(I)/\hat{\omega}=\partial H_0(I)/\partial I$, i.e.,

$$\omega_T(I)=\frac{\pi}{2F(\pi/2,\kappa)}\hat{\omega}\ . \tag{42}$$

Near the bottom of the potential well, $H_0\to-1$, $\kappa^2\to0$, $F(\pi/2,\kappa)\to\pi/2$, and therefore $\omega_T(I)\to\hat{\omega}$, as expected from Eq. (23) with $\delta_1=0$. On the other hand, near the top of the potential well, $H_0\to+1$, $\kappa^2\to1$, $F(\pi/2,\kappa)\to\infty$, and $\omega_T(I)\to0$.

For future reference, the normalized velocity can be expressed as

$$\frac{d\psi}{d\tau}=2\kappa\,\mathrm{cn}\tau=8\frac{\omega_T}{\hat{\omega}}\sum_{n=1}^{\infty}\frac{a^{n-1/2}}{1+a^{2n-1}}\cos[(2n-1)\omega_T t]\ . \tag{43}$$

The quantity a in Eq. (43) is defined by

$$\begin{aligned}&a\equiv\exp(-\pi F'/F)\ ,\\&F'\equiv F(\pi/2,(1-\kappa^2)^{1/2}),\quad F\equiv F(\pi/2,\kappa)\ .\end{aligned} \tag{44}$$

Near the top of the potential well, where $H_0\to1$, the electron motion becomes stochastic in the presence of the perturbation δ_1. Defining $H_0=1-\Delta H$, where $\Delta H\ll1$ near the separatrix, we find $\kappa^2\to1$, $\omega_T(I)\to0$, and

$$\begin{aligned}&F\simeq\tfrac{1}{2}\ln(32/\Delta H)\ ,\\&F'\simeq\pi/2\ ,\\&\omega_T\simeq\pi\hat{\omega}[\ln(32/\Delta H)]^{-1}\ ,\\&a\simeq\exp(-\pi\omega_T/\hat{\omega})\ ,\end{aligned} \tag{45}$$

for small $\Delta H\ll1$.

In what follows, the leading-order correction to the electron motion is retained on the right-hand side of Eq. (23) in an iterative sense. For consideration of the stochastic instability that develops near the separatrix, it is particularly convenient to examine the motion in action-angle variables. Correct to order δ_1, we find

$$\frac{dI}{d\tau}=\frac{dI}{dH_0}\frac{dH_0}{d\tau}=\frac{\hat{\omega}}{\omega_T}\frac{dH_0}{d\tau}\ , \tag{46}$$

where $\omega_T=\omega_T(I)$, and

$$\frac{dH_0}{d\tau}=-\frac{d\psi}{d\tau}\delta_1\sin[\hat{k}_1(\psi-V_p\tau+\alpha_1)]\ . \tag{47}$$

Equation (46) then becomes

$$\frac{dI}{d\tau}=-\delta_1\frac{\hat{\omega}}{\omega_T}\frac{d\psi}{d\tau}\sin[\hat{k}_1(\psi-V_p\tau+\alpha_1)]\ . \tag{48}$$

It is well known that near the separatrix Eq. (48) can lead to a stochastic instability that is manifest by a secular change in the action I and a systematic departure of the electron from the potential well. Near the separatrix with $H_0\to1$, it follows from Eqs. (30) and (43) that the electron is moving with approximately constant normalized velocity $d\psi/d\tau\simeq2$ for a short time of order $\hat{\tau}=\hat{\omega}^{-1}$. Moreover, this feature of the electron motion recurs with frequency $\omega_T(I)\ll\hat{\omega}$, and can lead to a significant change in the action I in Eq. (48).

We now examine the implications of Eq. (48) near the separatrix keeping in mind that the sine term in large parentheses on the right-hand side of Eq. (48) generally represents a high-frequency modulation. Making use of the lowest-order expression for the normalized velocity $d\psi/d\tau$, it follows that

$$\begin{aligned}\frac{dI}{d\tau}=-4\delta_1\sum_{n=1}^{\infty}\frac{a^{n-1/2}}{1+a^{2n-1}}\Bigg[&\sin\left[\hat{k}_1\psi+\hat{k}_1\alpha_1+(2n-1)\frac{\omega_T\tau}{\hat{\omega}}-\hat{k}_1V_p\tau\right]\\&+\sin\left[\hat{k}_1\psi+\hat{k}_1\alpha_1-(2n-1)\frac{\omega_T\tau}{\hat{\omega}}-\hat{k}_1V_p\tau\right]\Bigg]\ .\end{aligned} \tag{49}$$

Near the separatrix, the first sine term in large parentheses on the right-hand side of Eq. (49) acts as a nearly constant driving term for some high harmonic number $s\gg1$ satisfying the resonance condition

$$2s\omega_T(I_s)/\hat{\omega}\simeq\hat{k}_1V_p\ , \tag{50}$$

or equivalently,

$$\begin{aligned}\omega_T(I_s)&\simeq\frac{\hat{\omega}\hat{k}_1V_p}{2s}\\&=\frac{(\omega_1-\omega_{c0})(k+lk_0)-(\omega-\omega_{c0})(k_1+lk_0)}{2s(k+lk_0)}\ .\end{aligned} \tag{51}$$

Here, I_s is the action corresponding to the resonance condition for harmonic number s. From Eq. (51), it follows that the separation between the adjacent resonances s and $s+1$ is

$$\begin{aligned}\delta_s&\equiv\omega_T(I_s)-\omega_T(I_{s+1})\\&\simeq\hat{\omega}\hat{k}_1V_p/2s^2\simeq2\omega_T^2(I_s)/\hat{\omega}\hat{k}_1V_p\\&=\frac{2\omega_T^2(I_s)(k+lk_0)}{(\omega_1-\omega_{c0})(k+lk_0)-(\omega-\omega_{c0})(k_1+lk_0)}\ .\end{aligned} \tag{52}$$

On the other hand, for a small change in the action ΔI_s, the characteristic frequency width of the sth resonance can be expressed as

$$\Delta\omega_T(I_s)=\left|\frac{d\omega_T(I_s)}{dI_s}\right|\Delta I_s\ ,$$

where $\Delta\omega_T(I_s)\ll\omega_T(I_s)$ has been assumed. The condition for the appearance of stochastic instability[2] is $\Delta\omega_T(I_s)\gg\delta_s$, or

$$\left| \frac{d\omega_T(I_s)}{dI_s} \Delta I_s \right| \gg \frac{2\omega_T^2(I_s)}{\hat{\omega} \hat{k}_1 V_p} . \quad (53)$$

To determine the size of ΔI_s, we express $\omega_T(I)$ as $\omega_T(I_s) + \Delta\omega_T(I_s)$ and integrate Eq. (49) over a time interval of duration $\hat{\tau} = \hat{\omega}^{-1}$ in the vicinity of the sth resonance defined in Eq. (51). In an order-of-magnitude sense, this gives

$$\Delta I_s \approx 2\delta_1 \hat{\omega} \frac{a^{s-1/2}}{1+a^{2s-1}} \left| s \frac{d\omega_T(I_s)}{dI_s} \Delta I_s \right|^{-1} . \quad (54)$$

Solving Eq. (54) for ΔI_s then gives

$$\Delta I_s \approx \left| \frac{4\delta_1 a^{s-1/2}/(1+a^{2s-1})}{|d\omega_T(I_s)/dI_s|} \frac{\omega_T(I_s)}{\hat{k}_1 V_p} \right|^{1/2} , \quad (55)$$

where Eq. (50) has been used to eliminate s. Substituting Eq. (55) into Eq. (53) then gives the condition for stochastic instability to occur,

$$\delta_1 \left| \frac{d\omega_T(I_s)}{dI_s} \right| \left[\frac{a^{s-1/2}}{1+a^{2s-1}} \right] \gg \frac{\omega_T^3(I_s)}{\hat{\omega}} \left| \frac{(k+lk_0)}{(\omega_1-\omega_{c0})(k+lk_0)-(\omega-\omega_{c0})(k_1+lk_0)} \right| = \frac{\omega_T^3(I_s)}{\hat{\omega}^2 \hat{k}_1 V_p} . \quad (56)$$

The various factors in Eq. (56) are now estimated near the separatrix where $H_0 \to 1$ and $\omega_T(I_s) \ll \hat{\omega}$. From Eqs. (45) and (51) it follows that

$$a^s \simeq \exp\left[-\frac{\pi}{2} \hat{k}_1 V_p \right] . \quad (57)$$

Also, from Eq. (45), $\ln[32/(1-H_0)] = \pi\hat{\omega}/\omega_T$ gives

$$\frac{\partial H_0/\partial I}{1-H_0} = -\pi \frac{\hat{\omega}}{\omega_T^2(I)} \frac{d\omega_T(I)}{dI} . \quad (58)$$

Using the fact that $\partial H_0/\partial I = \omega_T(I)/\hat{\omega}$ yields

$$\frac{\hat{\omega}^2}{\omega_T^3(I)} \frac{d\omega_T(I)}{dI} = -\frac{1}{32\pi} \exp[\pi\hat{\omega}/\omega_T(I)] . \quad (59)$$

Substituting Eqs. (57) and (59) into Eq. (56) then gives

$$\frac{\delta_1}{32\pi} \hat{k}_1 V_p \frac{\exp(\pi\hat{\omega}/\omega_T - \pi\hat{k}_1 V_p/2)}{1+\exp(-\pi\hat{k}_1 V_p)} \gg 1 . \quad (60)$$

Expressed in terms of the energy bandwidth $\Delta H = 1 - H_0$, the condition in Eq. (60) for stochastic instability becomes

$$\frac{\delta_1}{\pi} \hat{k}_1 V_p \frac{\exp[-\pi\hat{k}_1 V_p/2]}{1+\exp[-\pi\hat{k}_1 V_p]} \gg \Delta H . \quad (61)$$

Because $\delta_1 \ll 1$, it follows from Eq. (61) that the detrapping of the electrons will be most pronounced when $\hat{k}_1 V_p \simeq 1$, or from Eq. (51) when

$$\hat{\omega} \simeq \frac{(\omega_1-\omega_{c0})(k+lk_0)-(\omega-\omega_{c0})(k_1+lk_0)}{(k+lk_0)} . \quad (62)$$

Making use of the expression for $\hat{\omega}$ given in Eq. (24), the condition in Eq. (62) can be expressed as

$$\left[\frac{e\delta E p_{\perp 0}}{c^2 m^2 \gamma_0^2} J_{-l} \left(\frac{eB_w}{ck_0 p_{z0}} \right) [(k+lk_0)v_{z0} + \omega_{c0} - c^2 k(k+lk_0)/\omega] \right]^{1/2} \simeq \frac{(\omega_1-\omega_{c0})(k+lk_0)-(\omega-\omega_{c0})(k_1+lk_0)}{(k+lk_0)} . \quad (63)$$

In the limit of zero wiggler magnetic field with $B_w = 0$, $k_0 = 0$, and $l = 0$, the above analysis holds for the electron cyclotron maser interaction. The parameter regime for detrapping of the electrons for the cyclotron maser is then given by [Eq. (63)]

$$\left[\frac{e\delta E p_{\perp 0}}{c^2 m^2 \gamma_0^2} (kv_{z0} + \omega_{c0} - c^2 k^2/\omega) \right]^{1/2} \simeq (\omega_1 - \omega_{c0}) - (\omega - \omega_{c0}) k_1/k . \quad (64)$$

For the case $\omega_1 \gg k_1 c$, $\omega \gg kc$, and $k_1/k = 1$ (gyrotron), the condition given in Eq. (64) becomes

$$\hat{\omega} = \left[\frac{e\delta E p_{\perp 0} \omega_{c0}}{c^2 m^2 \gamma_0^2} \right]^{1/2} \simeq \omega_1 - \omega . \quad (65)$$

Equation (65) indicates that if the difference in frequency between the primary and secondary waves in a gyrotron is close to the electron bounce frequency $\hat{\omega}$ in the primary wave, then the electrons will detrap from the primary-wave potential well, leading to a decrease in output power at the primary-wave frequency.

Finally, we examine the condition for stochastic instability for an electron deeply trapped in the primary-wave potential well, i.e., $H_0 \to -1$ and $\kappa^2 \ll 1$. For this case, the quantities given in Eqs. (42) and (44) become

$$\begin{aligned} F &\simeq \pi/2 + \pi\kappa^2/8 , \\ F' &\simeq \ln(4/\kappa) , \\ \omega_T(I) &\simeq \hat{\omega}(1-\kappa^2/4) , \\ a &\simeq \kappa^2/16 . \end{aligned} \quad (66)$$

Because $a \ll 1$, the equation for the unperturbed normalized velocity [Eq. (43)] becomes ($n = 1, \omega_T \simeq \hat{\omega}$),

$$\frac{d\psi}{d\tau} \simeq 2\kappa \cos(\omega_T \tau/\hat{\omega}) . \quad (67)$$

Equation (67) gives for ψ

$$\psi \simeq 2\kappa \sin(\omega_T \tau/\hat{\omega}) \ . \tag{68}$$

Substituting Eq. (68) into Eq. (49), and expanding in a series of ordinary Bessel functions $J_q(x)$ yields

$$\frac{dI}{d\tau} = -4\delta_1 \sum_{n=1}^{\infty} \sum_{q=-\infty}^{\infty} \frac{a^{n-1/2}}{1+a^{2n-1}} J_q(2\kappa\hat{k}_1) \left\{ \sin\left[\left[\frac{q\omega_T}{\hat{\omega}} + (2n-1)\frac{\omega_T}{\hat{\omega}} - \hat{k}_1 V_p \right]\tau + \hat{k}_1\alpha_1 \right] + \sin\left[\left[\frac{q\omega_T}{\hat{\omega}} - (2n-1)\frac{\omega_T}{\hat{\omega}} - \hat{k}_1 V_p \right]\tau + \hat{k}_1\alpha_1 \right] \right\} . \tag{69}$$

Near the bottom of the well, both sine terms in large square brackets in Eq. (69) can act as nearly constant driving terms for some harmonic numbers $n=s$ and $q=r$ satisfying the resonance conditions

$$\omega_T(I_{r,s}) = \hat{\omega}\hat{k}_1 V_p/(r+2s-1) \ , \tag{70}$$

or

$$\omega_T(I_{r,s}) = \hat{\omega}\hat{k}_1 V_p/(r-2s+1) \ .$$

Here, $I_{r,s}$ is the action corresponding to the resonance condition for harmonic numbers (r,s). From Eq. (70), it follows that the separation between adjacent resonances s and $s+1$, and r and $r+1$ is

$$\delta_{r,s} = \frac{3\hat{\omega}\hat{k}_1 V_p}{(r+2s-1)(r+2s+2)} = \frac{3\omega_T^2}{3\omega_T + \hat{\omega}\hat{k}_1 V_p}$$

or

$$\delta_{r,s} = \frac{-\hat{\omega}\hat{k}_1 V_p}{(r-2s)(r-2s+1)} = \frac{\omega_T^2}{\omega_T - \hat{\omega}\hat{k}_1 V_p} \ . \tag{71}$$

For a small change in the action $\Delta I_{r,s}$, the characteristic frequency width of the (r,s) resonance can be expressed as

$$\Delta\omega_T(I_{r,s}) = \left[\frac{d\omega_T(I_{r,s})}{dI_{r,s}} \right] \Delta I_{r,s} \ ,$$

where again $\Delta\omega_T(I_{r,s}) \ll \omega_T(I_{r,s})$ has been assumed. The condition for the appearance of stochastic instability is $\Delta\omega_T(I_{r,s}) \gg \delta_{r,s}$, or equivalently

$$\left| \frac{d\omega_T(I_{r,s})}{dI_{r,s}} \Delta I_{r,s} \right| \gg \left| \frac{3\omega_T^2}{3\omega_T + \hat{\omega}\hat{k}_1 V_p} \right|$$

or

$$\left| \frac{d\omega_T(I_{r,s})}{dI_{r,s}} \Delta I_{r,s} \right| \gg \left| \frac{\omega_T^2}{\omega_T - \hat{\omega}\hat{k}_1 V_p} \right| \ . \tag{72}$$

The size of $\Delta I_{r,s}$ is estimated in the same manner as for the case near the separatrix. Integrating Eq. (69) in the vicinity of the (r,s) resonance gives

$$\Delta I_{r,s} \approx 4\delta_1 \hat{\omega} J_r(2\hat{k}_1\kappa) \frac{a^{s-1/2}}{1+a^{2s-1}} \times \begin{cases} 1/(r+2s-1)\Delta\omega_T \ , \\ 1/(r-2s+1)\Delta\omega_T \ . \end{cases} \tag{73}$$

Solving for $\Delta I_{r,s}$ then results in

$$(\Delta I_{r,s})^2 \approx \left| 4\delta_1 \hat{\omega} J_r(2\hat{k}_1\kappa) \frac{a^{s-1/2}}{1+a^{2s-1}} \right| \left| \frac{\omega_T/\hat{\omega}\hat{k}_1 V_p}{d\omega_T/dI_{r,s}} \right| \ , \tag{74}$$

where use has been made of Eq. (70). Substituting Eq. (74) into Eq. (72) we find that the condition for stochastic instability to occur near the bottom of the potential well is given by

$$|\delta_1 J_r(2\hat{k}_1\kappa)(\kappa/4)^{(2s-1)}| \gg \left| \frac{72\hat{k}_1 V_p}{(3+\hat{k}_1 V_p)^2} \right|$$

or

$$|\delta_1 J_r(2\hat{k}_1\kappa)(\kappa/4)^{(2s-1)}| \gg \left| \frac{8\hat{k}_1 V_p}{(1-\hat{k}_1 V_p)^2} \right| \tag{75}$$

where use has been made of $a \simeq \kappa^2/16$ and

$$\frac{d\omega_T}{dI} = -\frac{\hat{\omega}}{4}\frac{d\kappa^2}{dI} = -\frac{\hat{\omega}}{8}\frac{dH_0}{dI} = -\frac{\omega_T}{8} \simeq -\frac{\hat{\omega}}{8} \ .$$

Because $\delta_1 \ll 1$, $\kappa^2 \ll 1$, and $|J_r| \le 1$, the inequality in Eq. (75), subject to the constraint given in Eq. (70), can only be satisfied for $\hat{k}_1 V_p \ll 1$, or equivalently,

$$\frac{(\omega_1-\omega_{c0})(k+lk_0)-(\omega-\omega_{c0})(k_1+lk_0)}{(k+lk_0)} \ll \hat{\omega} \ , \tag{76}$$

which follows from Eq. (24).

In the limit where $B_w=0$, $k_0=0$, and $l=0$, together with $\omega_1 \gg k_1 c$, $\omega \gg kc$, and $k_1/k=1$ (gyrotron), Eq. (76) gives

$$\omega_1 - \omega \ll \hat{\omega} \ . \tag{77}$$

Equation (77) indicates that if the frequency difference between the primary and secondary waves in a gyrotron is much less than the bounce frequency of an electron at the bottom of the potential well, then the deeply trapped electrons can be detrapped by a low-amplitude secondary wave.

V. CONCLUSIONS

To summarize, we have investigated the motion of an electron in the combined fields of a longitudinal magnetic wiggler, and constant-amplitude, circularly polarized primary electromagnetic wave ($\delta E,\omega,k$). It has been shown that the presence of a secondary moderate-amplitude transverse electromagnetic wave ($\delta E_1,\omega_1,k_1$) can lead to a stochastic particle instability in which electrons trapped near the separatrix of the primary wave or near the bottom of the primary-wave potential well can undergo a systematic departure from the potential well. This "detrapping" can result in a significant reduction in power output at the primary-wave frequency. The conditions for onset of stochastic instability have been calculated near the separatrix [Eq. (61)], and near the bottom of the potential well [Eq. (75)]. Equations (61) and (75) are also valid in the limit $B_w=0$ and $k_0=0$, and give the condition for onset of the stochastic instability for the electron cyclotron maser (gyrotron).

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