Effect of laser amplitude and phase fluctuations on optical bistability

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A generalized Fokker-Planck equation is derived for a system undergoing optical bistability when the dominant source of fluctuations is due to fluctuations in the incident field rather than spontaneous emission. The fluctuations are treated as a time-dependent Gaussian process whose properties are determined by the individual laser's characteristics. Both amplitude and phase fluctuations are included on the incident laser field. The effect of fluctuations of the incident laser on optical bistability are different for amplitude and phase fluctuations and are different for absorptive and dispersive optical bistability. The combination of incident laser phase fluctuations and dispersive optical bistability leads to large amplitude fluctuations near the turning points of the optical bistability curve and causes the system to make a transition from one branch of the optical bistability curve to the other before reaching the mean-field turning points. We take the high-Q cavity limit of our generalized Fokker-Planck equation, solve the linearized equations, and provide criteria for the magnitude of fluctuations in optical bistability in terms of laser and cavity parameters. The magnitude of incident laser fluctuations are typically orders of magnitude larger than fluctuations due to spontaneous emission and are therefore important for practical applications of optical bistability devices.

I. INTRODUCTION

The standard model[1,2] for optical bistability (OB) consists of a system of N two-level systems in a cavity driven by an external field of specified amplitude and phase. The only source of fluctuations in the model is the spontaneous emission of the two-level systems in the cavity. The dimensionless parameter that measures the ratio of spontaneous emission to mean-field terms is q $(\equiv \mathcal{N}/n_s)$, where n_s is the saturation number of photons in the cavity, $\mathcal{N}$ $(\equiv N/N_T)$, and N_T is the laser inversion number of two-level systems at threshold. Typical values of q for present OB systems are usually less than 10^{-6}. In a recent paper[3] we showed that the driving laser frequency fluctuations cause fluctuations in the phase of the field in the cavity that are typically many orders of magnitude larger than the fluctuations caused by spontaneous emission. In the present paper, we show that the effect of driving laser fluctuations on the amplitude of the cavity field are usually much more important than on the phase of the cavity field because the cavity amplitude undergoes critical slowing down and the amplitude fluctuations diverge as the mean-field turning points of the OB curve are approached. The fluctuations of the cavity field due to both amplitude and phase fluctuations in the driving laser will cause the system to make a transition from one branch of the OB curve to the other branch before the mean-field turning points are reached. Furthermore, even if the driving laser has only phase fluctuations and if we consider dispersive OB, then the effect of driving laser phase fluctuations is very large on the cavity field amplitude fluctuations and has only a small effect on the cavity field phase fluctuations. We also include driving laser amplitude fluctuations in this paper. The fluctuations of the field in the cavity depend on the relative relaxation rates of amplitude and phase fluctuations of the driving laser and the degree of dispersion relative to absorption in the OB cavity.

In Ref. 3 we considered only the high-Q cavity limit but in this paper we derive the master equation for the general case which includes the region where neither the high-Q or low-Q cavity limits are valid both for absorptive and dispersive OB. The generalization is important because the region in parameter space where neither the high- or low-Q limit is valid is much larger than the region where the limits are valid. In Ref. 3 we gave a heuristic derivation for the case of driving laser phase fluctuations in the high-Q cavity absorptive OB limit. In the present paper we give the essential steps in the derivation of the master equation for OB with driving laser fluctuations for the general case using an approach due to Stratonovich.[4] In Sec. II we present the Hamiltonian for N two-level systems in a cavity. Then we add the external driving laser field by means of a unitary transformation which is valid for the case where both the amplitude and phase of the driving laser field are time dependent. We next obtain the Liouville equation for the density matrix in the presence of radiation and matter reservoirs and obtain the Maxwell-Bloch equations by means of the self-consistent-field approximation. We make the time dependence of the driving laser field a Gaussian stochastic process in Sec. III and specify the conditions on the stochastic time dependence in order that the Stratonovich[4] derivation of the generalized Fokker-Planck equation is valid. The resultant Fokker-Planck equation has a nonlinear, nondiagonal diffusion tensor whose magnitude depends on the relative amount of dispersive and absorptive OB. In Sec. IV we consider the high-Q cavity limit with both driving laser amplitude and phase fluctuations and for arbitrary dispersion. We solve the linearized equations for both phase and amplitude fluctuations for the cavity field and for arbitrary dispersion. In Sec. V we discuss extensions of the results of this paper which are possible using the same or similar techniques to those developed in this paper.

II. HAMILTONIAN AND SELF-CONSISTENT-FIELD APPROXIMATION

The Hamiltonian for N two-level systems interacting with radiation in a cavity is

$$H = H_m + H_F + H_{\text{int}} , \tag{2.1}$$

$$H_m \equiv \tfrac{1}{2}\hbar\omega_A \sum_{\alpha=1}^{N} s_\alpha \equiv \tfrac{1}{2}\hbar\omega_A S , \tag{2.2a}$$

$$H_F \equiv \hbar\omega_c a^\dagger a , \tag{2.2b}$$

$$H_{\text{int}} \equiv \hbar(\mu P^+ a + \mu^* P^- a^\dagger) , \tag{2.2c}$$

where $P^\pm = \sum_\alpha e^{\pm i \vec{k}\cdot\vec{X}_\alpha} \sigma_\alpha^\pm$ and the commutation relations for the collective operators are $[P^+,P^-]=S$ and $[S,P^\pm]=\pm 2P^\pm$. The commutation relations for the individual two-level systems are $s_\alpha=[\sigma_\alpha^+,\sigma_\alpha^-]$ and $[\sigma_\alpha^-,\sigma_\alpha^+]_+=1$ and operators of different two-level systems commute with each other. The frequencies are ω_A the two-level frequency difference, ω_c the cavity frequency, and $h\mu \equiv \vec{\epsilon}\cdot\vec{\mathscr{p}}^*(2\pi h\omega_c/V)^{1/2}$ where $\vec{\epsilon}$ is the unit vector in the direction of the cavity field, $\vec{\mathscr{p}} \equiv \langle + \,|\, e\vec{r} \,|\, - \rangle$ is the transition dipole moment between the states of the two-level atom, and V is the volume of the cavity. For a more complete discussion of the Hamiltonian see Ref. 5. We can add an external field α to a by means of the unitary transformation $D(\alpha)$,

$$D^{-1}(\alpha) a D(\alpha) = a + \alpha \equiv A , \quad \text{with } D(\alpha) \equiv \exp(\alpha a^\dagger - \alpha^* a) . \tag{2.3}$$

Since $D(\alpha)$ commutes with matter operators the net effect of $D(\alpha)$ is to replace a by A in the Hamiltonian Eqs. (2.2b) and (2.2c) as was first pointed out in Ref. 6. In this paper we want to consider the case where the amplitude and phase of α depend on time, in particular the case where $\alpha(t)$ is stochastic. Equation (2.3) is valid when α is a function of time; however, when we consider operators such as $\dot{a}$ and $\dot{A}$ we have extra terms due to the time dependence of $\alpha(t)$ in the unitary transformation $D[\alpha(t)]$. After a straightforward calculation using the commutation relations $[a,a^\dagger]=1$ and the identity

$$\begin{aligned} D(\alpha) &= \exp(\alpha a^\dagger - \alpha^* a) \\ &= \exp(-\tfrac{1}{2}\,|\alpha|^2)\exp(\alpha a^\dagger)\exp(-\alpha^* a) \end{aligned} \tag{2.4}$$

we obtain

$$\begin{aligned} \dot{A} &= D^{-1}(\alpha)\dot{a}D(\alpha) + \dot{D}^{-1}(\alpha) a D(\alpha) + D^{-1}(\alpha) a \dot{D}(\alpha) \\ &= D^{-1}(\alpha)\dot{a}D(\alpha) - \dot{\alpha}[a^\dagger, A] = D^{-1}(\alpha)\dot{a}D(\alpha) + \dot{\alpha}(t) . \end{aligned} \tag{2.5}$$

Consequently, the net effect of adding a time-dependent external field $\alpha(t)$ to the N two-level systems interacting with radiation in a cavity is to replace a by A in the Hamiltonian H and add the term $\dot{\alpha}(t)$ to the time derivative of A. In the Liouville equation (as we show below) we can add a commutator that gives rise to the correct $\dot{\alpha}$ and $\dot{\alpha}^*$ terms. The Liouville equation for N two-level systems interacting with radiation in a cavity is

$$\frac{\partial F_N}{\partial t} + i\hbar^{-1}[F_N, H] = 0 . \tag{2.6}$$

In the presence of the matter reservoir Λ_A and the field reservoir Λ_F we have

$$\frac{\partial F_N}{\partial t} + i\hbar^{-1}[F_N, H] = \Lambda_A F_N + \Lambda_F F_N . \tag{2.7}$$

When we transform Eq. (2.6) with the unitary transformation $D(\alpha)$ the Hamiltonian H becomes[6]

$$H = H_0 + \Delta H, \quad H_0 \equiv \hbar\Omega(A^\dagger A + \tfrac{1}{2}S) \tag{2.8}$$

and

$$\Delta H \equiv \hbar\delta_F A^\dagger A + \tfrac{1}{2}\hbar\delta_A S + i\hbar(\mu P^+ A + \mu^* P A^\dagger) . \tag{2.9}$$

The frequencies are Ω the frequency of the incident laser, $\delta_A \equiv \omega_A - \Omega$ the atomic detuning, and $\delta_F \equiv \omega_c - \Omega$ the cavity detuning. Next we transform Eq. (2.7) to the interaction representation[6] using H_0 and obtain

$$\frac{\partial F_N}{\partial t} = -i\mathscr{L}F_N + \Lambda_A F_N + \Lambda_F F_N + [\dot{\alpha}A^\dagger - \dot{\alpha}^* A, F_N] , \tag{2.10}$$

where

$$\Lambda_F F_N \equiv \kappa[(A-\alpha), F_N(A^\dagger - \alpha^*)] + \text{H.c.} , \tag{2.11}$$

where H.c. is the Hermitian conjugate. The last term in Eq. (2.10), which vanishes if α is time independent, yields the $\dot{\alpha}$ as required by Eq. (2.5). The radiation reservoir acts[6] only on the cavity field operators $a = A - \alpha$ and $a^\dagger = A^\dagger - \alpha^*$. The unitary transformation has no effect on Λ_A. The definition of $\mathscr{L}$ is $[\,\cdots, \Delta H]$.

We obtain the equations of motion for A, $A^\dagger$, $P^\pm$, and S in the self-consistent-field approximation by multiplying Eq. (2.10) by each operator in turn, tracing over the matter and field variables, and neglecting radiation-matter correlations. The resultant equations are

$$\langle \dot{A} \rangle + (\kappa + i\delta_F)\langle A \rangle = \kappa\alpha - i\mu^*\langle P \rangle + \dot{\alpha} , \tag{2.12a}$$

$$\langle \dot{P}^- \rangle + (\gamma_\perp + i\delta_A)\langle P^- \rangle = i\mu\langle A \rangle \langle S \rangle , \tag{2.12b}$$

$$\langle \dot{S} \rangle + \gamma_{||}(\langle S \rangle + N) = 2i(\mu^*\langle A^\dagger \rangle\langle P^- \rangle - \mu\langle A \rangle\langle P^+ \rangle) , \tag{2.12c}$$

where $\langle (\cdots) \rangle \equiv \text{Tr}[(\cdots)F_N]$. The equation for $\langle \dot{A}^\dagger \rangle$ is the complex conjugate of Eq. (2.12a) and the equation for $\langle \dot{P}^+ \rangle$ is the complex conjugate of Eq. (2.12b). $\gamma_\perp$ ($\gamma_{||}$) is the inverse polarization (population) relaxation time. On the right-hand side of Eqs. (2.12) we have neglected the radiation-matter correlations and consequently for $\alpha=0$ we have the Maxwell-Bloch equations. The neglect of radiation-matter correlations causes incoherent spontaneous emission to be absent from Eqs. (2.12). The steady-state solutions of Eqs. (2.12) are

$$\langle S\rangle = -N(1+\hat{\delta}_A^2)(1+\hat{\delta}_A^2+|\langle x\rangle|^2)^{-1}, \quad (2.13a)$$

$$\langle \dot{P}^-\rangle = -iN\langle x\rangle(\gamma_{\|}/\gamma_\perp)^{1/2}2^{-1}(1-i\hat{\delta}_A) \times(1+\hat{\delta}_A^2+|\langle x\rangle|^2)^{-1}, \quad (2.13b)$$

$$\langle y\rangle = \langle x\rangle\left[\left[1+\frac{\mathcal{N}}{1+\hat{\delta}_A^2+|\langle x\rangle|^2}\right] - i\left[\frac{\mathcal{N}\hat{\delta}_A}{1+\hat{\delta}_A^2+|\langle x\rangle|^2}-\bar{\delta}_F\right]\right], \quad (2.13c)$$

where $\hat{\delta}_A \equiv (\delta_A/\gamma_\perp)$, $\bar{\delta}_F \equiv (\delta_F/\kappa)$, $\mathcal{N} \equiv (|\mu|^2 N/\gamma_\perp\kappa)$, $\langle x\rangle \equiv 2|\mu|\langle A\rangle/(\gamma_{\|}\gamma_\perp)^{1/2}$, and $\langle y\rangle \equiv 2|\mu|\alpha/(\gamma_{\|}\gamma_\perp)^{1/2}$. The steady-state solutions Eqs. (2.13) are the standard[1,2] solutions for OB in the absence of fluctuations. In Sec. III we treat Eqs. (2.12) as Langevin equations with $\dot{\alpha}$ as a rapidly varying stochastic driving term.

III. MASTER EQUATION

We obtain the generalized Fokker-Planck equation for our model of OB with a stochastic driving laser by first obtaining a Langevin description. We now reinterpret Eqs. (2.12) by dropping the average values and we take the variables x, x^*, $P^\pm$, and S as classical stochastic variables whose statistical properties are determined by the statistical properties of $\dot{\alpha}(t)$. In this paper we are considering the cases where the fluctuations due to incoherent spontaneous emission are small compared to fluctuations on the driving laser. The stochastic properties of $\dot{\alpha}(t)$ may be due to phase fluctuations, amplitude fluctuations, or both. In any case we find an amplitude phase variable description of $\dot{\alpha}$ is the most convenient so we introduce amplitude and phase variables for both matter and radiation. When we substitute the definitions of the amplitude and phase variables,

$$P^\pm \equiv \pm i\mathcal{P}e^{\pm i\psi}, \quad x \equiv re^{-i\phi}, \quad y \equiv \mathcal{E}e^{-i\theta},$$

into Eqs. (2.12) we obtain

$$\dot{\xi}_l = f_l(\xi) + \sum_m g_{lm}\eta_m(t), \quad (3.1)$$

where $d\xi_1 = dr$, $d\xi_2 = r\,d\phi$, $d\xi_3 = d\mathcal{P}$, $d\xi_4 = \mathcal{P}\,d\psi$, and $d\xi_5 = dS$. The definitions of the $f_l(\xi)$ are

$$f_1 = -\kappa r + \kappa\mathcal{E}\cos(\theta-\phi) - 2|\mu|^2(\gamma_{\|}\gamma_\perp)^{-1/2}\mathcal{P}\cos(\psi-\phi), \quad (3.2a)$$

$$f_2 = r\delta_F + \kappa\mathcal{E}\sin(\theta-\phi) - 2|\mu|^2(\gamma_{\|}\gamma_\perp)^{-1/2}\mathcal{P}\sin(\psi-\phi), \quad (3.2b)$$

$$f_3 = -\gamma_\perp\mathcal{P} - (\gamma_{\|}\gamma_\perp)^{1/2}2^{-1}Sr\cos(\phi-\psi), \quad (3.2c)$$

$$f_4 = \delta_A\mathcal{P} - (\gamma_{\|}\gamma_\perp)^{1/2}2^{-1}Sr\sin(\phi-\psi), \quad (3.2d)$$

$$f_5 = -\gamma_{\|}(S+N) + 2(\gamma_{\|}\gamma_\perp)^{1/2}r\mathcal{P}\cos(\theta-\psi). \quad (3.2e)$$

The definitions of the g_{lm} are

$$g_{11} = \Gamma_u^{1/2}\mathcal{E}\cos(\theta-\phi), \quad g_{21} = \Gamma_u^{1/2}\mathcal{E}\sin(\theta-\phi),$$
$$g_{12} = -\Gamma_\theta^{1/2}\mathcal{E}\sin(\theta-\phi), \quad g_{22} = \Gamma_\theta^{1/2}\mathcal{E}\cos(\theta-\phi), \quad (3.3)$$

where $\mathcal{E} \equiv e^u$, $\eta_1(t) = \Gamma_u^{-1/2}(\dot{\mathcal{E}}/\mathcal{E}) \equiv \Gamma_u^{-1/2}\dot{u}$, and $\eta_2(t) = \Gamma_\theta^{-1/2}\dot{\theta}$. The constants Γ_u and Γ_θ are defined below and are introduced in Eq. (3.3) to make the coefficient of $\delta(t)$ in Eq. (3.5) dimensionless. For Eq. (3.1) to be a Langevin equation we need two sets of conditions on the time dependence of $\dot{\theta}$ and $(\dot{\mathcal{E}}/\mathcal{E})$. First, the correlation times

$$t_\theta \equiv \langle\dot{\theta}^2\rangle^{-1}\int_0^\infty \langle\dot{\theta}(\tau)\dot{\theta}\rangle d\tau$$

and (3.4)

$$t_u \equiv \langle\dot{u}^2\rangle^{-1}\int_0^\infty \langle\dot{u}(\tau)\dot{u}\rangle d\tau = t_{\mathcal{E}}$$

must be shorter than the times κ^{-1}, $\gamma_{\|}^{-1}$, $\gamma_\perp^{-1}$, $(\mu\alpha)^{-1}$, and $[|\mu|^2N(\gamma_{\|}\gamma_\perp)^{-1/2}]^{-1}$. The averages in Eq. (3.4) are over the time-dependent Gaussian stochastic processes which determine $\dot{\theta}$ and $\dot{\mathcal{E}}$. This is just a form of the Brownian motion condition that the stochastic force varies more rapidly in time than the remaining forces on the Brownian particle. The second set of conditions[7] require the correlation times t_θ and t_u to be small compared with the modulation, i.e., $\delta_\theta t_\theta \ll 1$ and $\delta_u t_u \ll 1$ where $\delta_\theta \equiv \langle\dot{\theta}^2\rangle^{1/2}$ and $\delta_u \equiv \langle\dot{u}^2\rangle^{1/2}$. The second set of conditions lead to a Markovian description of the effects of the driving laser fluctuations and justify a generalized Fokker-Planck equation.

When the two sets of conditions on the time parameters are met we can write

$$\langle\eta_l\eta_m(t)\rangle = \delta_{lm}\delta(t). \quad (3.5)$$

We can now obtain the generalized Fokker-Planck equation for the probability distribution $\rho(r,\phi,\mathcal{P},\psi,S,t)$ because Eqs. (3.1) and (3.5) are the Langevin equations for which the Stratonovich[4] derivation applies. Since the physical noise source is external to the system and is not strictly white noise, the Stratonovich prescription is the correct limit.[8] In actual situations, frequently some of the conditions on t_θ and t_u will not be satisfied (in particular $\delta_\theta t_\theta \ll 1$ may be violated) and a Markovian description with a Fokker-Planck equation will not be possible. In this paper we consider only the cases where all the conditions are satisfied and we have then

$$\frac{\partial\rho}{\partial t} = -\sum_{\alpha=1}^{5}\frac{\partial}{\partial x_\alpha}k_\alpha\rho + \frac{1}{2}\sum_{\alpha,\beta}^{2}\frac{\partial^2}{\partial x_\alpha\partial x_\beta}k_{\alpha\beta}\rho, \quad (3.6)$$

where

$$k_1 = f_1 + \frac{1}{2}\sum_m g_{mj}\frac{\partial g_{1j}}{\partial x_m} = f_1 + \langle\mathcal{E}^2\rangle(2r)^{-1}[\Gamma_u\sin^2(\theta-\phi) + \Gamma_\theta\cos^2(\theta-\phi)], \quad (3.7a)$$

$$k_2 = f_2 + \frac{1}{2}\sum_m g_{mj}\frac{\partial g_{2j}}{\partial x_m} = f_2 + \langle\mathcal{E}^2\rangle(2r)^{-1}[(\Gamma_\theta - \Gamma_u)\sin(\theta-\phi)\cos(\theta-\phi)], \quad (3.7b)$$

$$k_3 = f_3, \quad k_4 = f_4, \quad k_5 = f_5, \quad (3.7c)$$

and

$$k_{11}=\sum_j g_{1j}g_{1j} = \langle \mathscr{E}^2\rangle \Gamma_u \cos^2(\theta-\phi)+\langle \mathscr{E}^2\rangle \Gamma_\theta \sin^2(\theta-\phi) \,, \qquad (3.8a)$$

$$k_{22}=\sum_j g_{2j}g_{2j} = \langle \mathscr{E}^2\rangle \Gamma_u \sin^2(\theta-\phi)+\langle \mathscr{E}^2\rangle \Gamma_\theta \cos^2(\theta-\phi) \,, \qquad (3.8b)$$

$$k_{12}=k_{21}=g_{11}g_{21}+g_{12}g_{22} = \langle \mathscr{E}^2\rangle (\Gamma_u-\Gamma_\theta)\sin(\theta-\phi)\cos(\theta-\phi) \,. \qquad (3.8c)$$

The inverse relaxation times Γ_θ and Γ_u are $\Gamma_\theta=\delta_\theta^2 t_\theta \equiv T_\theta^{-1}$, $\Gamma_u=\delta_u^2 t_u \equiv T_u^{-1}$, and $\Gamma_{\mathscr{E}}=\mathscr{E}^2\Gamma_u$. Another form of our Markov conditions $\delta_\theta t_\theta \ll 1$ and $\delta_u t_u \ll 1$ can be written using T_θ and T_u, i.e, $t_\theta T_\theta^{-1} \ll 1$ and $t_u T_u^{-1} \ll 1$ which are the conditions for the Born-Markov approximation. Equation (3.6) is too lengthy to write out explicitly and since the drift terms are just the Maxwell-Bloch equations, we display only the diffusion terms explicitly:

$$\frac{\partial \rho}{\partial t}+\sum_\alpha \frac{\partial}{\partial x_\alpha}k_\alpha \rho=\frac{\langle \mathscr{E}^2\rangle}{2}\left[\frac{\partial^2}{\partial r^2}[\Gamma_u\cos^2(\theta-\phi)+\Gamma_\theta \sin^2(\theta-\phi)]+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}[\Gamma_u \sin^2(\theta-\phi)+\Gamma_\theta\cos^2(\theta-\phi)] + \left[\frac{1}{r}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}+\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial \phi}\right][(\Gamma_u-\Gamma_\theta)\sin(\theta-\phi)\cos(\theta-\phi)]\right]\rho \,, \qquad (3.9)$$

where the normalization of ρ is $\int \rho \, dr\, d\phi\, d\mathscr{P}\, d\psi\, dS=1$. The two independent parameters that determine the diffusion terms of Eq. (3.9) are (Γ_u/Γ_θ) and $[\sin(\theta-\phi)/\cos(\theta-\phi)]$. When phase fluctuations or jitter dominate amplitude fluctuations we have $\Gamma_\theta \gg \Gamma_u$. In absorptive OB $\langle \cos(\theta-\phi)\rangle \gg \langle \sin(\theta-\phi)\rangle$, while in dispersive OB $\langle \sin(\theta-\phi)\rangle \gg \langle\cos(\theta-\phi)\rangle$.

Three special cases of the diffusion operator in Eq. (3.9) are thermal fluctuations in the driving laser where $\Gamma_u=\Gamma_\theta\equiv\Gamma$ and we have

$$\frac{\langle \mathscr{E}^2\rangle}{2}\Gamma\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial}{\partial\phi^2}\right]\rho \,. \qquad (3.10)$$

Second, in absorptive OB $\langle \sin(\theta-\phi)\rangle$ vanishes and the right-hand side of Eq. (3.9) is approximately

$$\frac{\langle \mathscr{E}^2\rangle}{2}\left[\Gamma_u\frac{\partial^2}{\partial r^2}+\Gamma_\theta\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]\rho \,. \qquad (3.11)$$

Third, one of the most important cases is where $\Gamma_\theta \gg \Gamma_u$, there is some dispersion and the Γ_θ terms are the dominant diffusion terms on the right-hand side of Eq. (3.9) which becomes

$$\frac{\langle \mathscr{E}^2\rangle}{2}\Gamma_\theta\left[\left[\frac{\partial}{\partial r}\sin(\theta-\phi)-\frac{1}{r}\frac{\partial}{\partial\phi}\cos(\theta-\phi)\right]^2\right]\rho \,. \qquad (3.12)$$

The most important consequence of Eq. (3.12) is that in the practically important case where Γ_θ is large and we have some dispersion, the cavity amplitude fluctuations caused by the driving laser phase fluctuations Γ_θ are appreciable. In pure absorptive OB the driving laser phase fluctuations have very little effect on the amplitude fluctuations of the cavity field. Thus, the combination of driving laser phase fluctuations with some dispersive OB leads to enhanced fluctuation effects which will cause the system to jump from one branch of the OB curve to the other branch before the mean-field turning points are reached.

IV. HIGH-Q CAVITY LIMIT

Since the full set of equations [Eq. (3.6)] contains so many terms and so many parameters, we find it useful to consider the effect of driving laser fluctuations in particular cases. Consequently, in this section we obtain the master equation for the high-Q limit and solve the master equation after linearization. In order to adiabatically eliminate the matter variables in Eq. (3.6) to obtain the master equation in the high-Q limit we require the condition $(\kappa/\gamma_\perp) \ll \mathscr{N}^{-1}$ be satisfied. The adiabatic elimination of the matter variables leads to the master equation for

$$R(r,\phi,\tau)\equiv \int \rho(r,\phi,\mathscr{P},\psi,S,\tau)d\mathscr{P}\, d\psi\, dS \;;$$

$$\frac{\partial R}{\partial \tau}=-\frac{\partial}{\partial r}\left[-r+\langle\mathscr{E}\rangle\cos(\theta-\phi)-\frac{\mathscr{N}r}{1+\hat{\delta}_A^2+r^2}+\frac{\langle \mathscr{E}^2\rangle}{2r}[\bar{\Gamma}_u\sin^2(\theta-\phi)+\Gamma_\theta\cos^2(\theta-\phi)]\right]R$$
$$-\frac{1}{r}\frac{\partial}{\partial\phi}\left[r\bar{\delta}_F+\langle\mathscr{E}\rangle\sin(\theta-\phi)-\frac{\mathscr{N}\hat{\delta}_A r}{1+\hat{\delta}_A^2+r^2}\right]R$$
$$+\frac{\langle \mathscr{E}^2\rangle}{2}\left[\frac{\partial^2}{\partial r^2}[\bar{\Gamma}_u\cos^2(\theta-\phi)+\bar{\Gamma}_\theta\sin^2(\theta-\phi)]+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}[\bar{\Gamma}_u\sin^2(\theta-\phi)+\bar{\Gamma}_\theta\cos^2(\theta-\phi)] + \frac{2}{r}\frac{\partial^2}{\partial\phi\partial r}[(\bar{\Gamma}_u-\bar{\Gamma}_\theta)\sin(\theta-\phi)\cos(\theta-\phi)]\right]R \,, \qquad (4.1)$$

where $\tau \equiv \kappa t$ and a frequency with an overbar has been made dimensionless by dividing by κ. The fluctuation-dependent ϕ drift term of Eq. (3.6) has been exactly canceled by a corresponding term in the diffusion coefficient. Equation (4.1) does not satisfy detailed balance and thus we do not know the exact stationary state, let alone the time-dependent solutions. One of the purposes of the present paper is to show that the amplitude and phase fluctuations on the driving laser have different effects on the amplitude and phase of the cavity field, which we can do by simplifying further. We first obtain the master equation for the phase variable $R(\phi,\tau)$ by integrating Eq. (4.2) over the amplitude variable r:

$$\frac{\partial R(\phi,\tau)}{\partial\tau} = -\frac{\partial}{\partial\phi}\left[\bar{\delta}_F + \langle \mathcal{E}\rangle \left\langle \frac{1}{r}\right\rangle_\phi \sin(\theta-\phi) - \left\langle \frac{\mathcal{N}\hat{\delta}_A}{1+\hat{\delta}_A^2+r^2}\right\rangle_\phi + \langle \mathcal{E}^2\rangle \left\langle \frac{1}{r^2}\right\rangle_\phi (\bar{\Gamma}_u - \bar{\Gamma}_\theta)\sin(\theta-\phi)\cos(\theta-\phi)\right] R(\phi,\tau)$$
$$+\frac{\langle \mathcal{E}^2\rangle}{2}\frac{\partial^2}{\partial\phi^2}\left[\left\langle \frac{1}{r^2}\right\rangle_\phi [\bar{\Gamma}_u \sin^2(\theta-\phi) + \bar{\Gamma}_\theta \cos^2(\theta-\phi)]\right] R(\phi,\tau)\,, \tag{4.2}$$

where the notation $\langle(\cdots)\rangle_\phi$ means the integration of $(\cdots)$ over r in the conditional probability holding ϕ constant. The nonlinearity of the drift and diffusion coefficients make Eq. (4.2) very difficult to solve, therefore we linearize Eq. (4.2) about the mean-field solution and obtain

$$\frac{\partial R(\chi,\tau)}{\partial\tau} = \frac{\partial}{\partial\chi}\left[\lambda_\phi \chi + D_\phi \frac{\partial}{\partial\chi}\right] R(\chi,\tau)\,, \tag{4.3}$$

where $\chi \equiv \phi - \theta$ and $\lambda_\phi \equiv (\langle \mathcal{E}\rangle / r_s)\langle \cos\chi\rangle_s = 1 + \mathcal{N}(1+\hat{\delta}_A^2 + r_s^2)^{-1}$. The diffusion coefficient D_ϕ is

$$D_\phi \equiv \tfrac{1}{2}\langle \mathcal{E}^2\rangle r_s^{-2}(\bar{\Gamma}_u \langle \sin\chi\rangle_s^2 + \bar{\Gamma}_\theta \langle\cos\chi\rangle_s^2)$$
$$= \tfrac{1}{2}[1+\langle(\Delta\mathcal{E})^2\rangle \mathcal{E}_0^{-2}]\{\bar{\Gamma}_u[\mathcal{N}(1+\hat{\delta}_A^2+r_s^2)^{-1} - \bar{\delta}_F]^2 + \bar{\Gamma}_\theta[1+\mathcal{N}(1+\hat{\delta}_A^2+r_s^2)^{-1}]^{-2}\}\,, \tag{4.4}$$

where r_s is the solution for the amplitude in Eq. (2.13); $\mathcal{E} = \mathcal{E}_0 + \Delta\mathcal{E}$ where $\langle \Delta\mathcal{E}\rangle = 0$ and thus $\langle \mathcal{E}^2\rangle = \mathcal{E}_0^2 + \langle(\Delta\mathcal{E})^2\rangle$. We used the mean-field results[2(d)]

$$(\mathcal{E}_0/r_s)\langle\cos\chi\rangle_s = 1 + \mathcal{N}(1+\hat{\delta}_A^2 + r_s^2)^{-1}\,, \tag{4.5a}$$

$$(\mathcal{E}_0/r_s)\langle\sin\chi\rangle_s = \mathcal{N}\hat{\delta}_A(1+\hat{\delta}_A^2+r_s^2)^{-1} - \bar{\delta}_F\,. \tag{4.5b}$$

We neglected the usually small contribution to λ_ϕ of the drift term in Eq. (4.2) proportional to $(\bar{\Gamma}_u - \bar{\Gamma}_\theta)$, because for most parameter ranges, it makes only a small contribution. For a solution and full analysis of Eq. (4.3) see Ref. 3, where Eq. (6) is the same functional form as the present Eq. (4.3). The results of Ref. 3 can be used with the present more general definitions of λ_ϕ and D_ϕ, Eq. (4.4). In Ref. 3 we found that the parameter that measures the importance of phase fluctuations in the cavity is (D_ϕ/λ_ϕ). For $(D_\phi/\lambda_\phi) \ll 1$, fluctuations of the phase are relatively unimportant. When (D_ϕ/λ_ϕ) is of the order of unity or greater, the phase fluctuations become important and phase fluctuations can cause a transition from one branch of the OB curve to the other branch. Note, however, that the phase fluctuations remain bounded and do not undergo critical slowing down even at the turning points of the OB curve. In the frequently occurring case where the $\bar{\Gamma}_\theta$ term is larger than the $\bar{\Gamma}_u$ term in Eq. (4.4) we have

$$D_\phi/\lambda_\phi \to \tfrac{1}{2}[1+\langle(\Delta\mathcal{E})^2\rangle\mathcal{E}_0^{-2}][1+\mathcal{N}(1+\hat{\delta}_A^2+r_s^2)^{-1}]\,.$$

For a driving laser linewidth of the order of a kilohertz or less, for a typical range of parameters, we have $D_\phi/\lambda_\phi < 10^{-2}$–$10^{-1}$ and the phase fluctuations on the driving laser have relatively little effect on the cavity field phase variable. However, if the linewidth of the driving laser exceeds a few kilohertz the phase fluctuations induced in the cavity field by the driving laser will cause the system to jump from one branch of the OB curve to the other well before reaching the mean-field turning points. The above limit on D_ϕ/λ_ϕ is a lower limit because if we have some dispersive OB and $\bar{\Gamma}_u$ is not zero, there will be an amplitude fluctuations component added to the $\bar{\Gamma}_\theta$ component thus increasing D_ϕ/λ_ϕ.

We obtain the master equation for the amplitude r by integrating Eq. (4.1) over ϕ which leads to the following equation:

$$\frac{\partial R(r,\tau)}{\partial\tau} = -\frac{\partial}{\partial r}\left[-r + \langle\mathcal{E}\rangle\langle\cos(\theta-\phi)\rangle_r - \frac{\mathcal{N}r}{1+\hat{\delta}_A^2+r^2} + \frac{1}{2}\frac{\langle\mathcal{E}^2\rangle}{r}(\bar{\Gamma}_u\langle\sin^2(\theta-\phi)\rangle_r + \bar{\Gamma}_\theta\langle\cos^2(\theta-\phi)\rangle_r)\right] R$$
$$+\frac{\langle\mathcal{E}^2\rangle}{2}\frac{\partial}{\partial r^2}[\bar{\Gamma}_u\langle\cos^2(\theta-\phi)\rangle_r + \bar{\Gamma}_\theta\langle\sin^2(\theta-\phi)\rangle_r] R\,, \tag{4.6}$$

where the notation $\langle(\cdots)\rangle_r$ indicates the integration of $(\cdots)$ over ϕ in the condition probability holding r constant. When we linearize Eq. (4.6) about the mean-field stationary state we obtain

$$\frac{\partial R(r,\tau)}{\partial \tau}=\frac{\partial}{\partial r}\left[\lambda_r(r-r_s)+D_r\frac{\partial}{\partial r}\right]R(r,\tau)\ , \tag{4.7}$$

where

$$\lambda_r\equiv 1+\frac{\mathcal{N}(1-r_s^2+\hat{\delta}_A^2)}{(1+\hat{\delta}_A^2+r_s^2)^2}+\frac{1}{2}\left[\overline{\Gamma}_u\left[\frac{\mathcal{N}\hat{\delta}_A}{1+\hat{\delta}_A^2+r_s^2}-\overline{\delta}_F\right]^2+\overline{\Gamma}_\theta\left[1+\frac{\mathcal{N}}{1+\hat{\delta}_A+r_s^2}\right]^2\right]\ , \tag{4.8}$$

$$D_r\equiv\frac{r_s^2}{2}[1+\langle(\Delta\mathcal{E})^2\rangle\mathcal{E}_0^{-2}]\left[\overline{\Gamma}_\theta\left[\frac{\mathcal{N}\hat{\delta}_A}{1+\hat{\delta}_A^2+r_s^2}-\overline{\delta}_F\right]^2+\overline{\Gamma}_u\left[1+\frac{\mathcal{N}}{1+\hat{\delta}_A^2+r_s^2}\right]^2\right]\ , \tag{4.9}$$

and r_s is the amplitude solution of Eq. (2.13), i.e.,

$$\mathcal{E}_0^2=r_s^2\left[\left[1+\frac{\mathcal{N}}{1+\hat{\delta}_A^2+r_s^2}\right]^2+\left[\frac{\mathcal{N}\hat{\delta}_A}{1+\hat{\delta}_A^2+r_s^2}-\overline{\delta}_F\right]^2\right]\ . \tag{4.10}$$

The Green's function for Eq. (4.7) is

$$G[(r-r_s),\tau;(r-r_s)_0,0]=Z^{-1}\exp\left[\frac{-\lambda_r}{2D_r}\left[\frac{[(r-r_s)-(r-r_s)_0\exp(-\lambda_r\tau)]^2}{[1-\exp(-2\lambda_r\tau)]}\right]\right]\ , \tag{4.11}$$

where

$$Z^{-1}\equiv\{2\pi D_r[1-\exp(-2\lambda_r\tau)]\lambda_r^{-1}\}^{1/2}\ ,$$

and where $(r-r_s)_0$ is the initial value of the deviation of r from the mean-field stationary state. The initial value of the amplitude deviation is forgotten on the time scale λ_r^{-1}, and the stationary state is reached on the same time scale. Equation (4.7) represents a competition between the amplitude relaxing to the stationary state measured by λ_r and amplitude diffusion measured by D_r. If $\lambda_r \gg D_r$ the steady state is reached before there is an appreciable spread in $r-r_s$ due to fluctuations and the dispersion $\sigma_r^2\equiv D_r/\lambda_r$ in the steady state is small. Thus r remains close to r_s with small fluctuations in the steady state. On the other hand, if $D_r \gg \lambda_r$, then there is an appreciable spread in $r-r_s$ before the steady state is achieved and the steady-state dispersion is large. We can see the effect of $D_r \gg \lambda_r$ most clearly by considering the interval $D_r^{-1}<\tau<\lambda_r^{-1}$ in Eq. (4.11) which becomes

$$G[(r-r_s),\tau;(r-r_s)_0,0]$$
$$=(4\pi D_r)^{-1/2}\exp[-(r-r_0)^2/4D_r\tau]\ .$$

Thus we see that for $D_r>\lambda_r$ the amplitude diffuses instead of relaxing to r_s until τ becomes of the order λ_r^{-1}. The preceding discussion applies to the phase variable[3] with D_ϕ instead of D_r and λ_ϕ instead of λ_r. The important new result for the amplitude is that $\lambda_r\rightarrow 0$ at the turning points, whereas λ_ϕ is always greater than zero even on unstable states. The amplitude variable consequently undergoes critical slowing and fluctuations in the amplitude grow without limit as the turning points of the OB curve are approached.

The role of driving laser fluctuations is important because the D_r due to driving laser fluctuations is usually orders of magnitude larger than D_r for incoherent spontaneous emission. Consequently, the inequality $D_r>\lambda_r$ is satisfied farther from the mean-field turning points and the discontinuous jumps from one branch of the OB curve to the other branch will occur over a spread of values of the driving laser field rather than just the two values of $\mathcal{E}$ at the mean-field turning points. The combination of dispersive OB where $\langle\sin(\theta-\phi)\rangle_s \gg \langle\cos(\theta-\phi)\rangle_s$, which is the easier case to achieve experimentally, and $\overline{\Gamma}_\theta>\overline{\Gamma}_u$, which is the more common case in lasers, make D_r large and thus the amplitude fluctuations more pronounced. The spectrum of the amplitude fluctuations is the Fourier transform of the two time amplitude correlation function $\langle\Delta r(\tau)\Delta r\rangle$ which we obtain in the following manner:

$$\langle\Delta r(\tau)\Delta r\rangle\equiv\int(r-r_s)G[(r-r_s),\tau;(r-r_s)_0,0]$$
$$\times(r-r_s)_0P_s[(r-r_s)_0]$$
$$\times d(r-r)_s d(r-r_s)_0$$
$$=e^{-\lambda_r\tau}(D_r/\lambda_r)\ , \tag{4.12}$$

where $\Delta r\equiv r-r_s$ and where we used Eq. (4.11) for G and P_s, the steady-state amplitude distribution function. The spectrum of the amplitude fluctuations in the linear approximation is thus a Lorentzian of width λ_r. When we substitute Eq. (4.9) in Eq. (4.12) we obtain the relative fluctuations

$$\frac{\langle \Delta r(\tau)\Delta r\rangle}{r_s^2} = \frac{[1+\langle(\Delta\mathscr{E})^2\rangle\mathscr{E}_0^{-2}]}{2\lambda_r}\left[\overline{\Gamma}_\theta\left[\frac{\mathscr{N}\hat{\delta}_A}{1+\hat{\delta}_A^2+r_s^2}-\overline{\delta}_F\right]^2+\overline{\Gamma}_u\left[1+\frac{\mathscr{N}}{1+\hat{\delta}_A^2+r_s^2}\right]^2\right]\times e^{-\lambda_r\tau}\,. \tag{4.13}$$

At the turning points of the OB curve the relative fluctuations diverge. Fluctuations will become important when the relative fluctuations become of order unity or greater. Since $\lambda_r \to 0$ as the turning points are approached the fluctuations will always eventually become important. However, the larger the factor in large square brackets in Eq. (4.13) is, the further from the turning points will be the place where fluctuations start to cause transitions from one branch to the other. The relative fluctuations in Eq. (4.13) depend on seven dimensionless parameters so it is difficult to analyze. There are a few inequalities the parameters must satisfy, such as $(\kappa/\gamma_\perp) \ll \mathscr{N}^{-1}$ the high-Q limit, $\mathscr{N}/(1+\hat{\delta}_A^2) > 8$ the requirement for the existence of OB, and δ_F has to be less than the mode spacing in the cavity. Otherwise, there are few restrictions on the variables. We can obtain upper bounds on the effect of fluctuations by considering the bounds on the various diffusion coefficients by taking $\langle \sin^2\chi\rangle$ or $\langle\cos^2\chi\rangle$ equal to unity. We find the following bounds: $D_r^d \lesssim \langle \mathscr{E}^2\rangle(\Gamma_\theta/\kappa)$, $D_r^a \lesssim \langle \mathscr{E}^2\rangle(\Gamma_u/\kappa)$, $D_\phi^d \lesssim (\langle \mathscr{E}^2\rangle/r_s^2)(\Gamma_u/\kappa)$, and $D_\phi^a \lesssim (\langle \mathscr{E}^2\rangle/r_s^2)(\Gamma_\theta/\kappa)$ where the superscript d (a) means dispersive (absorptive) OB. For high-Q cavities the radiation relaxation frequency κ is of the order of 10^6. The frequency Γ_θ ($\Gamma_{\mathscr{E}} = \langle \mathscr{E}^2\rangle\Gamma_u$) is the width of the driving laser due to phase (amplitude) fluctuations, respectively. Thus, unless the linewidth of the driving laser is of the order of a kilohertz or less, then for large intensities $\langle \mathscr{E}^2\rangle$ we have dimensionless diffusion constants which are of the order of unity or greater which will lead to large fluctuations from one branch of the OB curve to the other. The magnitude of Γ_θ is $(\delta_\theta^2 t_\theta/\kappa)$. The existence of a Fokker-Planck equation for driving laser noise requires $\delta_\theta t_\theta \ll 1$. Consequently if we assume $\delta_\theta t_\theta \sim 10^{-1}$ then if δ_θ, which is $\langle(\Delta\dot{\theta})^2\rangle^{1/2}$, is comparable to or greater than about 10^4 Hz, the effect of driving laser phase fluctuations on OB fluctuations will be large. Similarly, if $\delta_{\mathscr{E}}$, which is $\langle(\Delta\dot{\mathscr{E}})^2\rangle$, is comparable to or greater than about 10^4 Hz, the effect of driving laser amplitude fluctuations will be large.

V. DISCUSSION AND CONCLUSIONS

There are various straightforward extensions of the results of this paper which are possible using techniques the same as or similar to those we used in Secs. III and IV. We obtain the low-Q cavity limit for the case of time-independent driving laser, i.e., α independent of t, when $(\gamma_\perp/\kappa) \ll \mathscr{N}^{-1}$ by eliminating the radiation variables adiabatically, resulting in three equations for the matter variables $P^\pm$ and S. If α depends on time stochastically, the adiabatic solution of Eq. (2.12a) is

$$\langle A\rangle = (\kappa + i\delta_F)^{-1}(\kappa\alpha - i\mu^*\langle P^-\rangle + \dot{\alpha}) \tag{5.1}$$

when the additional conditions $t_\theta, t_u \gg \kappa^{-1}$ are satisfied. After the elimination of $\langle A\rangle$ the three matter equations for $\langle P^\pm\rangle$ and $\langle S\rangle$ are Langevin equations for the matter variables with stochastic "forces" proportional to the stochastic functions $\dot{\alpha}(t)$ and $\dot{\alpha}^*(t)$. The subsequent derivation proceeds in the same manner as the high-Q limit in Sec. IV. Alternately, if t_θ and t_u satisfy the inequality $\kappa^{-1} \gg t_\theta, t_u$ then the stochasticity itself is eliminated adiabatically and the only sources of fluctuations remaining in the problem are those due to incoherent spontaneous emissions which are treated in Ref. 2(b). Ideally, one would like to solve the full nonlinear Maxwell-Bloch equations with fluctuations, i.e., Eq. (3.5), but this would be extremely difficult. Even the linearized form of Eq. (3.5) is very difficult because it requires the simultaneous diagonalization of the drift and diffusion operators. It is possible to compute the spectra of the linearized form of Eq. (3.5) using the method of Lugiato,[9] which consists of diagonalizing the drift terms and calculating the time-independent quadratic correlation functions for the steady state. The spectra[9] are determined by the linearized Maxwell-Bloch dynamics and steady-state quadratic correlation functions of the dynamical variables. All of the results of this paper and all the results mentioned so far in this section are based on the assumption that the stochastic process in the driving laser satisfies the condition of fast modulation,[7] i.e., $\delta_\theta t_\theta \ll 1$ and $\delta_{\mathscr{E}} t_{\mathscr{E}} \ll 1$. The assumption of Gaussian processes and fast modulation leads to Fokker-Planck master equations. Frequently the stochasticity of the driving laser will undergo slow modulation,[7] especially for the phase variable. It is not possible to get a Fokker-Planck equation when the stochasticity is slowly modulated because the process becomes non-Markovian and any master equation will have to have memory, i.e., a time-dependent kernel. In the lowest order of the slow modulation limit, one solves the Maxwell-Bloch equations with α and α^* as time-independent parameters. Next, one averages the solution of the Maxwell-Bloch equations over $P(\alpha,\alpha^*)$, the distribution function of the stochastic variables. Higher-order corrections depend on the time dependence of the Green's function of the laser fluctuations. The spectra of the linearized Maxwell-Bloch equations can be explicitly calculated in the slow modulation limit.

All of the results derived and discussed in this paper have been based on noise in the driving laser. However, there is another way that noise can come into the problem, namely, the laser cavity itself can have jitter. The cavity jitter problem is fundamentally more difficult because the noise is internal rather than external. The overall approach in this paper is applicable for the most part and both the high- and low-Q cavity limits can be carried out explicitly.

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