RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 29, NUMBER 1 JANUARY 1984

Stochasticity and reconnection in Hamiltonian systems

J. E. Howard* and S. M. Hohs
University of California, Berkeley, California 94720
(Received 1 August 1983)

A general class of Hamiltonian systems is studied in which neighboring phase-space islands are shifted in phase. This leads to reconnection of Kolmogorov-Arnold-Moser level curves and necessitates a reexamination of the island overlap criterion for the breakdown of adiabatic barriers between island chains. An analytic reconnection threshold is derived from an averaged Hamiltonian and found to agree closely with numerical surfaces of section for a model mapping. Numerous applications to physical problems are indicated.

In general, reconnection might be defined as a topological rearrangement of level curves in which critical points do not change their type (in contrast to bifurcations, in which critical points may be created or destroyed, or change type.) Reconnection plays an important role in a variety of physical problems, including rf acceleration in particle accelerators,[1] motion of magnetic field lines,[2] particle motion in two-dimensional potentials,[3] wave-particle interactions,[4,5] laser-plasma coupling,[6] and possibly the free-electron laser.[7] In many of these cases involving nonintegrable systems one is concerned with the extent to which the motion is stochastic or regular. For example, in ion or electron cyclotron resonance heating, regular phase-space curves, called Kolmogorov-Arnold-Moser (KAM) curves, can present barriers to stochastic heating. Stochasticity of magnetic field lines can lead to rapid particle and energy loss in fusion devices. In this paper we show that there is an intimate relationship between stochasticity and reconnection, with the result that reconnection can effectively destroy an adiabatic barrier. The reconnection threshold is derived for a general class of Hamiltonian systems and found to agree closely with numerically computed KAM barriers for a model mapping.

Many dynamical systems of current interest are particular cases of the radial twist mapping

$$x' = x - K \sin\theta, \qquad \theta' = \theta + f(x') \tag{1}$$

where K is a constant and $f(x)$ is analytic in some domain. For example, $f(x) = x$ gives the Taylor-Chirikov map,[8] while $f(x) = 1/x$ yields the Fermi map.[9] The general case $f(x) = x^n$ has also been studied.[10] Now consider functions $f(x)$ whose inverse is multivalued, so that $f(x) - 2\pi n = 0$ has multiple roots, corresponding to families of island chains. If $f(x;\alpha)$ depends continuously on the parameter α such that $f' = \partial f/\partial x$ at fixed x changes sign as α is varied, then pairs of island chains merge whenever $f' = 0$. From the tangent map

$$L = \begin{bmatrix} 1 & -K\cos\theta \\ f' & 1 - Kf'\cos\theta \end{bmatrix} \tag{2}$$

it may be shown that the rotational sense of islands of any order is $-\operatorname{sgn}(f')$. The merging of such counter-rotating islands is necessarily accompanied by the reconnection of their separatrices, which can happen in one of two ways. Since the period-one islands lie only at $\theta = 0$ or π, the island centers are either aligned or shifted by π; similar restrictions apply approximately to higher-order islands.

When aligned counter-rotating islands merge, a two-dimensional vortex is formed. When staggered islands merge, their separatrices form a chain of loops, as depicted in Fig. 1. The latter mode of reconnection is the only possible one for period-one islands, since $\mathrm{Tr}L - 2 = -Kf'\cos\theta$ changes sign with f'. This scenario was first observed by Symon and Sessler[1] in calculating beam stacking in particle accelerators; their mapping can in fact be put in the form (1). Mappings of this form also occur in our studies of multifrequency electron-cyclotron-resonance heating[11] where they arise in calculating resonance overlap in a four-dimensional phase space.

Examination of a number of occurences of reconnection in two-dimensional potentials $V_i = V(x,y;\alpha_i)$ suggests that this always involves the merging of two separatrices. Thus,

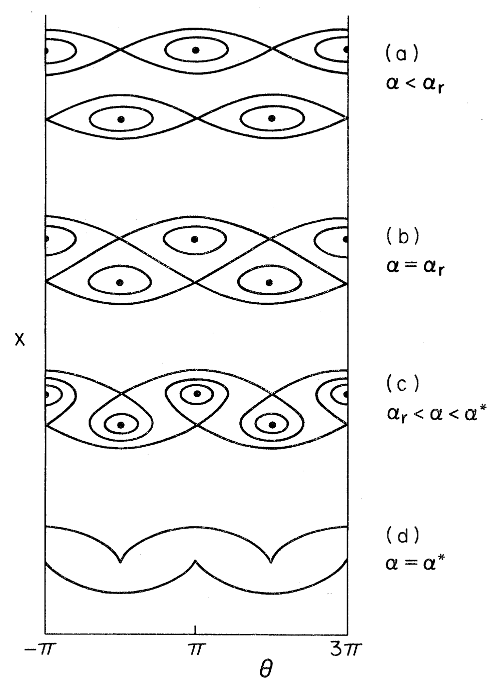

FIG. 1. Reconnection scenario for logistic twist map (two periods are shown for clarity).

a necessary condition for reconnection is that $V_1 = V_2$. For example, it may be shown that the Hénon-Heiles potential[12] is a reconnection point of the more general Hall-McNamara potential.[3] In the case of the radial twist map (1) a reconnection threshold may be derived using the averaged Hamiltonian[13]

$$\overline{H}(x,\theta) = \int_{x_1}^{x} [f(\xi) - 2\pi n]\, d\xi - K\cos\theta \ , \tag{3}$$

which yields a continuous approximation to the mapping in the vicinity of a fixed point. Reconnection occurs when the upper and lower separatrices have the same value of $\overline{H}$. With no loss of generality, suppose that the upper island chain in Fig. 1 has x points at $\pm\pi$, so that the lower separatrix passes through x points at $\theta = 0$ and 2π. From Eq. (3) the lower and upper separatrices are given by $\overline{H}_{\rm ls} = +K$ and

$$\overline{H}_{\rm us} = \int_{x_1}^{x_2} [f(\xi) - 2\pi n]\, d\xi - K \ . \tag{4}$$

Equating $\overline{H}_{\rm us} = \overline{H}_{\rm ls}$ then gives the reconnection threshold

$$K(\alpha) = 1/2 \int_{x_1(\alpha)}^{x_2(\alpha)} [f(\xi;\alpha) - 2\pi n]\, d\xi \ . \tag{5}$$

As we shall see, this simple formula often gives a useful estimate for the breakdown of an adiabatic barrier.

We have chosen for detailed study the "logistic twist map," for which

$$f(x) = x - \alpha x^2 \ , \tag{6}$$

where $\alpha > 0$. The behavior of this mapping is representative of the general class (1) and may be regarded as a paradigm reconnecting system. The period-one fixed points are located at

$$x_n^{\pm} = \frac{1}{2\alpha}(1 \pm \sqrt{1 - 8\pi n\alpha}) \ . \tag{7}$$

For positive n both roots are positive real for $0 \leq 8\pi n\alpha \leq 1$, coalescing when $\alpha^* = (8\pi n)^{-1}$ at $x_n^* = 4\pi n$. The x_n^- reduce to the standard mapping island centers as $\alpha \to 0$; the

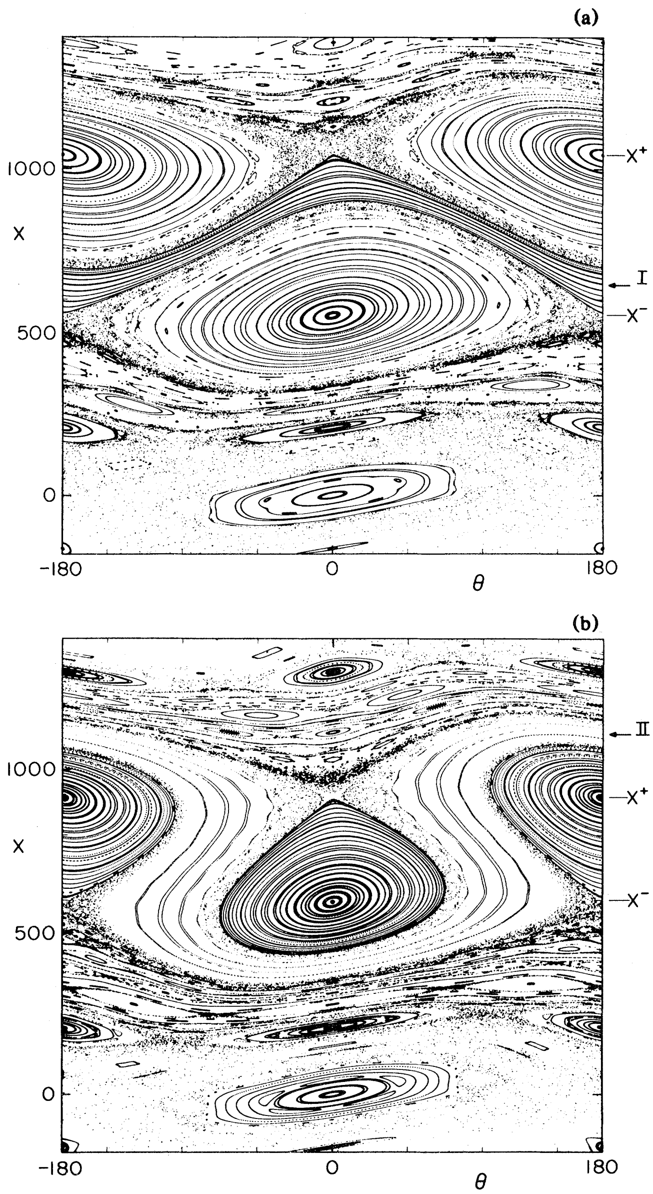

FIG. 2. Surfaces of section for logistic twist map for $K = 1.5$ (a) before reconnection ($\alpha = 0.036$) and (b) after reconnection ($\alpha = 0.038$). Both x and θ are in degrees.

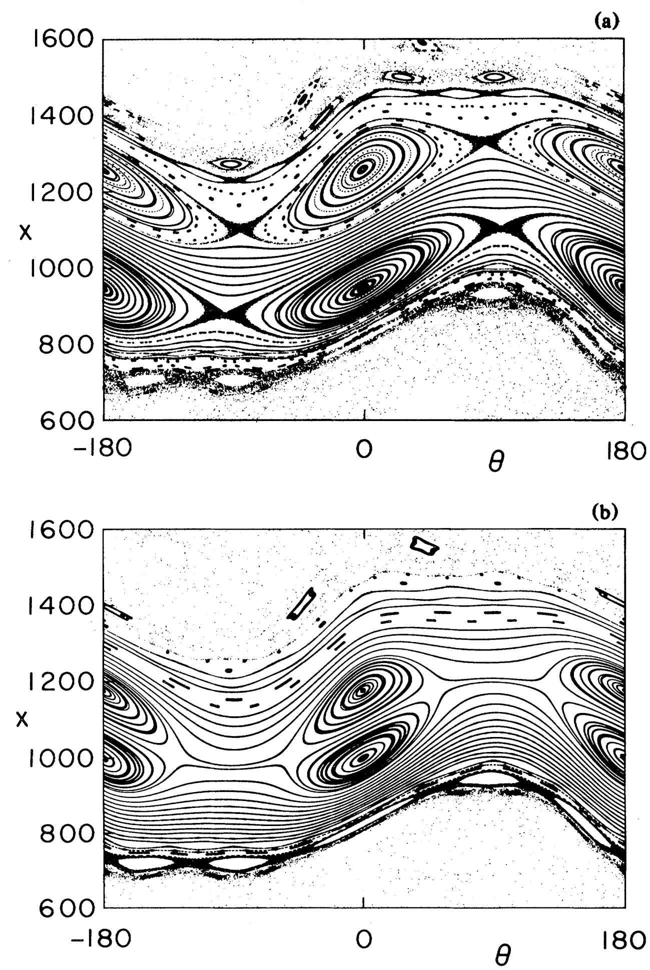

FIG. 3. Surfaces of section for $K = 4$, showing vortex formation. (a) $\alpha = 0.0260$; (b) $\alpha = 0.026\,35$.

x_n^+ islands are born at $x=\infty$, descending and merging sequentially with the rising x_n^- as α is increased (with the exception of the $n=0^-$ island, which is unperturbed). Identifying $x_1=x_n^-$ and $x_2=x_n^+$ and evaluating the integral in (7) gives the reconnection threshold

$$K_r(\alpha)=\frac{(1-8\pi n\alpha)^{3/2}}{12\alpha^2} \quad . \tag{8}$$

Equation (8) has been verified by visual inspection of numerical surfaces of section for the case $n=1$ and $K \leqslant 2.5$; above this value the stochastic layers surrounding the islands obscure the reconnection process. Figure 2 shows typical phase plots in the vicinity of the $n=1$ islands before and after reconnection. Notice the band of KAM curves separating the upper and lower islands in Fig. 2(a), which form a "type-I" barrier to orbits initialized in the stochastic region near $x=0$. After reconnection a second "type-II" barrier exists, with the upper island now topologically below the previously lower island.

While there is at present no theoretical method for predicting the existence or destruction of KAM barriers between staggered islands, some insight may be gained through study of the heteroclinic orbits joining the x points of the upper and lower island chains. Figure 1(b) is an integrable approximation to the map when the heteroclinic orbits are dense and form a separatrix. When the island chains are separated by KAM curves they cannot be joined by heteroclinic orbits and when they are joined by heteroclinic orbits they cannot be separated by KAM curves. It remains to be proven whether in all cases there are either KAM surfaces or heteroclinic orbits.

The second mode of reconnection, vortex formation, is shown in Fig. 3 for the period-two aligned islands. The

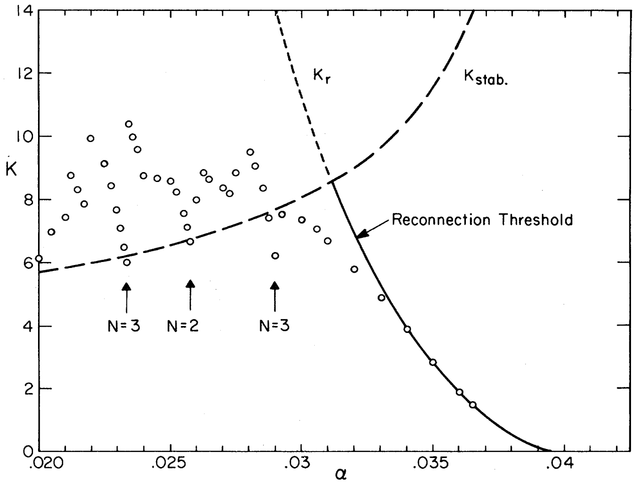

FIG. 4. Computed adiabatic barriers between $n=1$ islands compared with reconnection threshold $K_r(\alpha)$.

upper and lower elliptic fixed points are located at

$$x_m^{\pm}=\frac{1}{2\alpha}(1\pm\sqrt{1-4\pi m\alpha}) \quad , \tag{9}$$

where m is an odd integer. As α is increased, the x points move together nearly vertically, joining at the reconnection point, after which they move apart horizontally. As α is increased further, each vortex shrinks as a unit, vanishing when $\alpha^*=(4\pi m)^{-1}$, an apparently previously unobserved kind of "multifurcation." Although a reconnection threshold may be obtained by deriving an averaged Hamiltonian

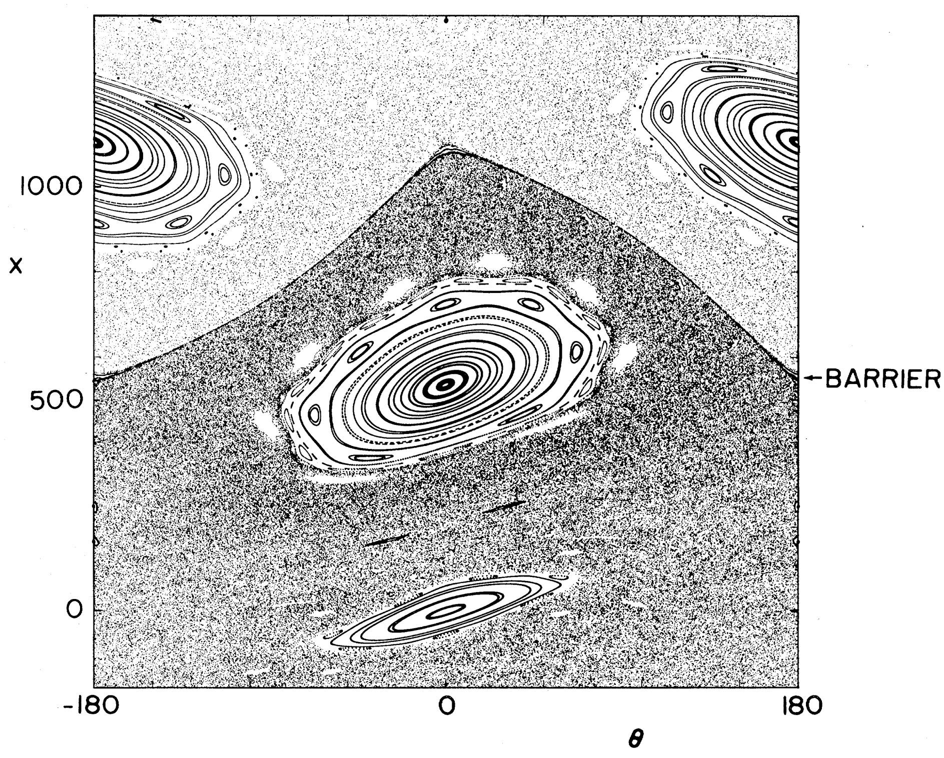

FIG. 5. Surface of section near barrier breakdown for $\alpha=0.035$ and $K=2.82$. Note the very thick stochastic layer.

for the period-two islands, it is easier in this case to work directly with the period-two mapping equations. Requiring that the x points merge then gives

$$K_r(\alpha) = \frac{1}{\alpha}\sqrt{1-4\pi m\alpha}\ . \tag{10}$$

This result has been verified visually to more than five significant figures. We have also observed reconnection of higher-order islands up to period six, similar to those seen by Fukuyama.[4] In general, even-order islands form vortices and odd-order islands make loops.

Extensive numerical calculations of the type-I barrier have been carried out for the $n = 1$ islands by fixing α and following single orbits for increasing K until a breakthrough was observed. The results are shown in Fig. 4 for $0.020 \leq \alpha \leq 0.040$, along with the reconnection threshold (8). The dashed line is the bifurcation threshold $K_{\mathrm{bif}} = 4(1-8\pi n\alpha)^{-1/2}$ (found by setting $\mathrm{Tr}L = -2$), above which the notion of reconnection becomes meaningless. The most striking feature of this figure is the convergence of the barrier data to the reconnection curve for $\alpha \geq 0.031$ (the island chains merge when $\alpha = 1/8\pi = 0.039\,79$). While one would expect approximate agreement as the stochastic layers diminish with decreasing K, the convergence is rapid, even in the presence of a thick stochastic layer. For example, at $\alpha = 0.035$, where Fig. 5 reveals a very thick stochastic layer near the barrier ($K = 2.82$), the agreement with K_r is better than one part in 10^3; at $\alpha = 0.0365$ the relative difference is only 3×10^{-6}! The sharp minima in the barrier data at $\alpha \approx 0.0235$ and 0.0290 are due to reconnection of period-three islands, while the dip near $\alpha = 0.0255$ is a consequence of the period-two vortices depicted in Fig. 3. It may be shown that the barrier data also lie near the reconnection thresholds in these regions. The close agreement between K_b and K_r is surprising both because K_r was calculated from an approximate Hamiltonian and because of the apparent lack of influence of the stochastic layers. These and other questions leave considerable scope for future work.

We would like to thank A. J. Lichtenberg and M. A. Lieberman for helpful discussions. This work was supported by the Office of Naval Research Contract No. N00014-79-C-0674.

---

*Present address: TRW Energy Development Group, 1 Space Park, Redondo Beach, CA 90278.

[1]K. R. Symon and A. M. Sessler, in *Proceedings of the International Conference on High-Energy Accelerators and Instrumentation* (CERN, Geneva, 1956), p. 44.

[2]J. B. Taylor, Phys. Rev. Lett. 33, 1139 (1974).

[3]L. S. Hall and B. McNamara (unpublished).

[4]A. Fukuyama, in *Intrinsic Stochasticity in Plasmas,* edited by G. Laval and D. Gresilon (Les Editions de Physiques, Courtaboeuf, Orsay, France, 1979), p. 207.

[5]C. F. F. Karney, Phys. Fluids 21, 1584 (1978).

[6]A. B. Langdon and B. F. Lasinsky, Phys. Rev. Lett. 34, 934 (1975).

[7]N. M. Kroll, P. L. Morton, and M. N. Rosenbluth, in *Free Electron Generators of Coherent Radiation,* edited by S. F. Jacobs *et al.* (Addison-Wesley, Reading, MA 1980).

[8]B. V. Chirikov, Phys. Rep. 52, 265 (1979).

[9]M. A. Lieberman and A. J. Lichtenberg, Phys. Rev. A 5, 1852 (1972).

[10]A. J. Lichtenberg, M. A. Lieberman, and R. H. Cohen, Physica D 1, 291 (1980).

[11]J. E. Howard, A. J. Lichtenberg, M. A. Lieberman, and R. H. Cohen (unpublished).

[12]M. Hénon and C. Heiles, Astron. J. 69, 73 (1964).

[13]A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer Verlag, New York, 1983).