Instability of relativistic-electron helical trajectories in combined uniform axial and helical wiggler magnetic fields

W. W. Zachary
Naval Research Laboratory, Washington, D.C. 20375
(Received 4 August 1982; revised manuscript received 23 January 1984)

The stability is investigated of helical trajectories of relativistic electrons in combined axial guide and helical wiggler magnetic fields appropriate for studies of free-electron lasers without using the frequently invoked assumption of evaluating the wiggler field on the laser axis. This is the first analysis of such trajectories which is free of this approximation. It is found that the helical trajectories are unstable with the possible exception of one value of the orbital radius. The instabilities are of two types corresponding to different properties of the eigenvalues of the matrix associated with the linearization of the equations of motion. These properties, in turn, depend upon the relative signs and magnitudes of the physical parameters that occur in the problem. At the exceptional radial value the trajectories can be either unstable or linearly stable depending upon the values of these parameters. For the linearly stable situations it is not known whether the trajectories are nonlinearly stable or unstable. We also prove that the trajectories of one of the instability types are of a particular kind called conditionally stable. This type of unstable trajectory has the property that it is approached by some (but not all) solutions of the equations of motion as time approaches positive or negative infinity. This type of instability is found to be more severe for larger values of the orbital radius. Even so, it still appears to be significant for small radii, of the order of $\frac{1}{100}$ of the wiggler wavelength.

I. INTRODUCTION

The study of trajectories of relativistic electrons in periodic magnetic fields forms an important source of input data for the theory of free-electron lasers. In the present paper we will be interested in magnetic fields which are combinations of uniform guide fields and wiggler fields generated by a bifilar helical current winding:[1–3]

$$\begin{aligned} B_r &= 2B_w I_1'(kr)\cos(kz-\phi) , \\ B_\phi &= 2B_w \frac{I_1(kr)}{kr}\sin(kz-\phi) , \\ B_z &= B_0 - 2B_w I_1(kr)\sin(kz-\phi) , \end{aligned} \tag{1.1}$$

where we have used cylindrical coordinates (r,ϕ,z) and I_1 denotes the Bessel function of imaginary argument of order one with the prime indicating differentiation with respect to its argument. B_0 and B_w denote the (constant) magnitudes of the guide and wiggler fields, respectively, and the constant k denotes the wave number of the axially periodic wiggler field. In the limit $r \to 0$ (1.1) reduces to

$$\vec{B} = (B_w \cos(kz), B_w \sin(kz), B_0) , \tag{1.2}$$

where we now use Cartesian coordinates. This is a guide-wiggler combination commonly used in investigations of properties of free-electron lasers.

The classical equations of motion have helical trajectories as steady-state solutions for each field (1.1) and (1.2) for suitable ranges of the parameters B_0, B_w, and electron energy.[3] Indeed, in the case of (1.2) it can be shown that such trajectories are the *only* steady-state solutions. This does not mean, however, that these trajectories are followed by electrons in free-electron lasers because there usually exist inhomogeneous regions in these devices which are not taken into account in (1.1) and (1.2). Therefore, studies of the stability of electron helical trajectories are very important for the theory of free-electron lasers, especially in view of the fact that these trajectories are commonly used in calculations of the radiative properties of these devices.

Several studies of the linear stability of helical trajectories corresponding to the expression (1.2) have been done.[4–7] Also, in an interesting paper on the effect of replacing the field (1.1) by (1.2) in the equations of motion of the electron, Diament[3] uses what one might call a hybrid approach which consists of a linearization of the equations of motion corresponding to (1.1) about the helical trajectories corresponding to (1.2). As we shall see in more detail later, the validity of his procedure is restricted to the case of helical trajectories with sufficiently small radius. We are able to remove this restriction by the use of a more general method.

In the present paper we investigate the stability of electron helical trajectories in the magnetic field (1.1) without making the assumption of replacing this expression by (1.2). This is the stability problem that should be studied because the trajectories followed by real electrons are obtained by solving the equations of motion corresponding to the field (1.1), at least to the extent that this field is a good approximation to the actual magnetic field in a free-electron laser. If these trajectories are perturbations of a helical trajectory, the latter will be a solution of these same equations rather than of the equations corresponding to the expression (1.2).

We are able to bypass the approximation of passing from (1.1) to (1.2) by applying some results of the qualitative theory of ordinary differential equations to Hamilton's equations describing electron motion in the magnetic field (1.1). By using these powerful techniques, we are able to generalize Diament's results and also to obtain some additional ones.

We find that helical trajectories corresponding to (1.1) are unstable with the possible exception of one value $|a|$ of the orbital radius. The instabilities are of two types, called I and II herein (corresponding to Diament's strong and weak instabilities, respectively), associated with the respective cases when the matrix $\underline{A}$ associated with the linearization of the equations of motion does or does not have an eigenvalue with positive real part. For the exceptional value of $|a|$, the trajectories can be either unstable of type II or linearly stable depending upon the values of the parameters B_0, B_w, and electron energy. For the linearly stable situations it is not known whether the trajectories are nonlinearly stable or unstable.

We also prove that the helical trajectories of instability type I belong to a particular class called conditionally stable.[8–10] Roughly speaking, a solution $\psi(t)$ of a system of ordinary differential equations is called *stable* if every solution of the system which starts sufficiently close to $\psi(t)$ at $t=0$ remains close at all other times. A solution $\psi(t)$ is called *unstable* if there exists at least one solution of the system which starts near $\psi(t)$ at $t=0$ but does not remain close at all other times. A solution $\psi(t)$ is called *conditionally stable* when some (but generally not all) solutions of the system approach $\psi(t)$ in the limits[11] $t\to\pm\infty$.

In order to facilitate our discussion of the instability of the helical trajectories corresponding to the magnetic field (1.1), we find it convenient to divide this topic into two parts. In Sec. II we first briefly discuss these helical solutions and then give a qualitative analysis of their instability in terms of some naturally occurring auxiliary parameters. Then, in Sec. III we proceed to a quantitative discussion in terms of normalized versions of the physical parameters B_0, B_w, and electron energy. We summarize our results in Sec. IV and an Appendix is devoted to the derivation of some estimates needed to establish the conditional stability of the type-I unstable trajectories.

II. HELICAL TRAJECTORIES AND THEIR INSTABILITY

We consider Hamilton's equations for the Hamiltonian

$$\mathrm{H}=\left[c^2\left[\vec{\mathrm{P}}-\frac{e}{c}\vec{\mathrm{A}}\right]^2+m^2c^4\right]^{1/2},$$

where $e<0$ denotes the electronic charge, m the electron mass, and c the speed of light in vacuum. $\vec{\mathrm{A}}$ denotes the vector potential corresponding to the magnetic field (1.1):

$$A_r=-\frac{2B_w}{k}\frac{I_1(kr)}{kr}\cos(kz-\phi),$$

$$A_\phi=\frac{B_0 r}{2}-\frac{2B_w}{k}I_1'(kr)\sin(kz-\phi),$$

$$A_z=0$$

expressed in the cylindrical coordinates (r,ϕ,z). After changing from momentum to velocity variables,

$$\vec{\mathrm{V}}=\frac{1}{m}\left[\vec{\mathrm{P}}-\frac{e}{c}\vec{\mathrm{A}}\right],$$

we write Hamilton's equations in the following form:

$$\begin{aligned}
&\dot r=\frac{V_r}{\gamma},\quad \dot\phi=\frac{V_\phi}{r\gamma},\quad \dot z=\frac{V_z}{\gamma},\\
&\dot V_r=[-\Omega_0+2\Omega_w I_1(kr)\sin(kz-\phi)]V_\phi\\
&\qquad+2\Omega_w V_z\frac{I_1(kr)}{kr}\sin(kz-\phi)+(r\gamma)^{-1}V_\phi^2,\\
&\dot V_\phi=-[-\Omega_0+2\Omega_w I_1(kr)\sin(kz-\phi)]V_r\\
&\qquad-2\Omega_w V_z I_1'(kr)\cos(kz-\phi)-(r\gamma)^{-1}V_\phi V_r,\\
&\dot V_z=-2\Omega_w\left[V_r\frac{I_1(kr)}{kr}\sin(kz-\phi)\right.\\
&\qquad\left.-V_\phi I_1'(kr)\cos(kz-\phi)\right],
\end{aligned}\tag{2.1}$$

where the overdot denotes differentiation with respect to time, and we have introduced the frequencies

$$\Omega_0=\frac{|e|B_0}{mc\gamma},\quad \Omega_w=\frac{|e|B_w}{mc\gamma},$$

where γ denotes the usual relativistic factor.

We will consider the following helical solutions of (2.1):

$$\begin{aligned}
&r=|a|,\quad \phi=\frac{kut}{\gamma}\pm\frac{\pi}{2},\quad z=\frac{ut}{\gamma},\\
&V_r=0,\quad V_\phi=kau,\quad V_z=u,
\end{aligned}\tag{2.2}$$

where a and u are constants and we have, without loss in generality, chosen z to vanish at $t=0$. The two signs in (2.2) correspond to the fact that two types of solutions to (2.1) are possible,[3] the negative sign corresponding to negative values of ka.

We will confine our attention to the study of the instability of the trajectories (2.2). Although these admittedly form a very restricted class of solutions of (2.1), with their guiding centers along the wiggler axis,[3] they are, nevertheless, commonly used in investigations of properties of free-electron lasers.[12] Since one of our objects in the present work is the study of the validity of the procedure of replacing (1.1) by (1.2), generalizing some aspects of the work of Diament,[3] we want to investigate the instability of solutions of (2.1) that reduce to appropriate solutions of the equations of motion corresponding to (1.2) in the limit $ka\to 0$. As we stated in the Introduction, these analogs of (2.2) are the *only* steady-state solutions of the equations of motion corresponding to (1.2). Thus (2.2) are the appropriate solutions of (2.1) to study in order to delineate the domain of validity of the replacement $(1.1)\to(1.2)$.

In the case of (1.2), it is possible to obtain additional solutions of the equations of motion.[7] This is possible because in this special case these equations have an additional invariant besides the electron energy which allows one to reduce the system to a single nonlinear differential equation which can be solved by means of Jacobi elliptic functions. This procedure is not possible in the case of the system (2.1). One could, of course, follow Diament's procedure and study the instability of one or more of these non-steady-state solutions of the equations of motion corresponding to (1.2) by using them as zero-order terms in an expansion of solutions of (2.1). However, this procedure would only give results for a limited range of values of the parameters. We do not believe that this is an appropriate course of action and so have restricted ourselves to a study of the instability of the solutions (2.2).

Substitution of (2.2) into (2.1) yields the following relations between u and a:

$$\frac{ku}{\gamma}=\Omega_0\pm 2\Omega_w I_1(ka)[1+(ka)^{-2}] \;. \tag{2.3}$$

One obtains another relation between these parameters by noting that the quantity $V_r^2+V_\phi^2+V_z^2$ is an invariant equal to $\gamma^2\beta^2c^2$ where βc denotes the speed of the electron. Combination of these relations readily yields the following expression:

$$\frac{g}{\sqrt{1+(ka)^2}}=1+2pI_1(\,|ka|\,)[1+(ka)^{-2}] \;, \tag{2.4}$$

which is valid for either sign in (2.3). The parameter $g=k\beta c/\Omega_0$ can be thought of as an energy parameter normalized to the cyclotron frequency of the uniform axial guide field and $p=\Omega_w/\Omega_0=B_w/B_0$ denotes the ratio of the wiggler and guide-field strengths. The above equations for the helical trajectories have been discussed in greater detail by Diament.[3]

We study the instability of the trajectories (2.2) by passing from (2.1) to the corresponding variational system of equations.[13] Thus, suppose that (2.1) is written in the form

$$\dot{x}_i=y_i(x_1,\ldots,x_6)\;,\quad i=1,2,\ldots,6 \;.$$

Let $x_0=(x_{01},\ldots,x_{06})$ denote the solutions (2.2) and write $x_i=x_{0i}+w_i$ $(i=1,2,\ldots,6)$ so that $x_0=(\,|a|\,,kut/\gamma\pm\pi/2,ut/\gamma,0,kua,u)$ and the w_i $(i=1,2,\ldots,6)$ denote perturbed values of the variables $(r,\phi,z,V_r,V_\phi,V_z)$, respectively. The variational system is then $(i=1,2,\ldots,6)$

$$\dot{w}_i=y_i(x_{01}+w_1,\ldots,x_{06}+w_6)-y_i(x_{01},\ldots,x_{06}) \tag{2.5}$$

and the solutions (2.2) of (2.1) correspond to the solution $w_i=0$ $(i=1,2,\ldots,6)$ of (2.5). We now rearrange (2.5) in the form

$$\dot{w}_i=(Aw)_i+f_i(w_1,\ldots,w_6)\;,\quad i=1,2,\ldots,6\;, \tag{2.6}$$

where $\underline{A}$ is a constant matrix depending upon the parameters g, p, and ka, and the f_i $(i=1,2,\ldots,6)$ contain only terms nonlinear in the w_i. Explicitly,

$$\underline{A}=\begin{bmatrix} 0 & 0 & 0 & \gamma^{-1} & 0 & 0\\ \dfrac{-ku}{\gamma a} & 0 & 0 & 0 & (\gamma a)^{-1} & 0\\ 0 & 0 & 0 & 0 & 0 & \gamma^{-1}\\ \alpha & 0 & 0 & 0 & \mathfrak{z} & -\psi\\ 0 & \dfrac{-\kappa}{k} & \kappa & \dfrac{-\psi}{ka} & 0 & 0\\ 0 & a\kappa & -ka\kappa & \psi & 0 & 0 \end{bmatrix}, \tag{2.7}$$

where

$$\begin{aligned} \alpha&=-ku\{\Omega_0\pm 2\Omega_w[I_1(ka)+I_1'(ka)(ka+(ka)^{-1})]\}\;,\\ \mathfrak{z}&=\Omega_0\pm 2\Omega_w I_1(ka)[1+2(ka)^{-2}]\;,\\ \kappa&=\mp 2\Omega_w kuI_1'(ka)\;,\\ \psi&=\pm 2\Omega_w\frac{I_1(ka)}{ka}\;, \end{aligned} \tag{2.8}$$

and (2.3) has been used in obtaining these expressions for α and $\mathfrak{z}$. The functions f_i are as follows:

$$f_1=0=f_3\;, \tag{2.9a}$$

$$f_2=\frac{kau}{\gamma}[(a+w_1)^{-1}-a^{-1}+w_1a^{-2}]+\frac{w_5}{\gamma}[(a+w_1)^{-1}-a^{-1}]\;, \tag{2.9b}$$

$$\begin{aligned} f_4={}&\mp 2\Omega_w kau[I_1(k(a+w_1))\cos(kw_3-w_2)-I_1(ka)-kw_1I_1'(ka)]\\ &\mp 2\Omega_w w_5[I_1(k(a+w_1))\cos(kw_3-w_2)-I_1(ka)]\\ &\mp 2\Omega_w u\left[\frac{I_1(k(a+w_1))}{k(a+w_1)}\cos(kw_3-w_2)-\frac{I_1(ka)}{ka}-kw_1\left[\frac{I_1(x)}{x}\right]'\Bigg|_{x=ka}\right]\\ &\mp 2\Omega_w w_6\left[\frac{I_1(k(a+w_1))}{k(a+w_1)}\cos(kw_3-w_2)-\frac{I_1(ka)}{ka}\right]\\ &+\frac{(kau)^2}{\gamma}[(a+w_1)^{-1}-a^{-1}+w_1a^{-2}]+2kau\frac{w_5}{\gamma}[(a+w_1)^{-1}-a^{-1}]+\frac{w_5^2}{\gamma}(a+w_1)^{-1}\;, \end{aligned} \tag{2.9c}$$

$$f_5=\pm 2\Omega_w w_4[I_1(k(a+w_1))\cos(kw_3-w_2)-I_1(ka)]$$
$$\mp 2\Omega_w u[I_1'(k(a+w_1))\sin(kw_3-w_2)-I_1'(ka)(kw_3-w_2)]$$
$$\mp 2\Omega_w w_6 I_1'(k(a+w_1))\sin(kw_3-w_2)-kau\frac{w_4}{\gamma}[(a+w_1)^{-1}-a^{-1}]-\frac{w_4w_5}{\gamma}(a+w_1)^{-1}\,, \tag{2.9d}$$

$$f_6=\pm 2\Omega_w w_4\left[\frac{I_1(k(a+w_1))}{k(a+w_1)}\cos(kw_3-w_2)-\frac{I_1(ka)}{ka}\right]\pm 2\Omega_w w_5 I_1'(k(a+w_1))\sin(kw_3-w_2)$$
$$\pm 2\Omega_w kau\,[I_1'(k(a+w_1))\sin(kw_3-w_2)-I_1'(ka)(kw_3-w_2)]\,. \tag{2.9e}$$

We will use the well-known result that many aspects of the instability of the solution $w_i=0$ ($i=1,2,\ldots,6$) of (2.6) can be studied by considering the simpler problem of the instability of the solutions of the corresponding linearized problem

$$\dot{\xi}_i=(\underline{A}\,\underline{\xi})_i\,,\quad i=1,2,\ldots,6\,. \tag{2.10}$$

For an arbitrary n vector, y, we use the norm

$$||y||=\sum_{i=1}^{n}|y_i|\,. \tag{2.11}$$

Then the results that we want to use can be summarized by the following.[14]

Theorem 1. Suppose that the $f_i(w_1,\cdots,w_6)$ ($i=2,4,5,6$) are continuous functions of the w_j ($j=1,2,\ldots,6$) and that $||f(w)||/||w||\to 0$ as $||w||\to 0$. Then we have the following:

(i) The solution $w_i=0$ ($i=1,2,\ldots,6$) of (2.6) is unstable if at least one eigenvalue of $\underline{A}$ has positive real part.

(ii) Suppose that all eigenvalues of $\underline{A}$ have nonpositive real part and that l eigenvalues λ_j ($j=1,2,\ldots,l\le 6$) have zero real part. Then every solution of (2.10) is stable if each eigenvalue λ_j ($j=1,2,\ldots,l$) has equal algebraic and geometric multiplicities. Otherwise, every solution of (2.10) is unstable.

Estimates proving that the functions f_i defined in (2.9) satisfy the conditions of this theorem will be derived in the Appendix. We note that the instability of the solutions of the linear problem (2.10) implies the instability of the solution $w_i=0$ ($i=1,2,\ldots,6$) of (2.6). However, we cannot conclude that the latter solution is stable on the basis of the stability of the solutions of the corresponding linearized problem because we are dealing with a Hamiltonian system. In such a situation, it is possible that arbitrarily small nonlinear terms can destroy the stability of the solutions of the linearized problem.[15]

We also note that the instabilities of the solutions of (2.10) referred to in parts (i) and (ii) of Theorem 1 are of qualitatively different types. Such solutions described by (i) are exponentially increasing as a function of time, whereas such solutions covered by (ii) generally have a polynomial time dependence. For ease of reference, we will refer to these instabilities as types I and II, respectively.

The matrix $\underline{A}$ has two zero eigenvalues with the remaining four being solutions of the equation

$$\lambda^4+b\lambda^2+d=0\,,$$

where

$$b=\Omega_0^2\left[\{1\pm 2pI_1(ka)[1+(ka)^{-2}]\}^2 +4p^2\frac{I_1^2(ka)}{(ka)^4}[1+(ka)^2]\right] \tag{2.12}$$

and

$$d=\mp 2\Omega_0^2 p\left[\frac{ku}{\gamma}\right]^2\frac{I_1'(ka)}{(ka)^2}[(ka)^3\mp pZ(ka)] \tag{2.13}$$

with

$$Z(ka)=2[1+(ka)^2]\left[3\frac{I_1(ka)}{ka}-I_0(ka)[1+(ka)^2]\right]. \tag{2.14}$$

In obtaining these expressions we have used (2.3) and the identity

$$I_1'(ka)=I_0(ka)-\frac{I_1(ka)}{ka}\,.$$

We now discuss the two types of instabilities indicated in Theorem 1 as a function of the parameters g, p, and ka. In order to facilitate this discussion, it is convenient to divide it into two parts. In the remainder of the present section we give a qualitative discussion of the various cases in terms of the quantities b and d. Then, in Sec. III we discuss the quantitative dependence of each case in terms of the parameters g, p, and ka.

In general, both situations listed in Theorem 1 occur. We see from (2.12) that $b>0$ for all values of the parameters g, p, and ka. Thus, the various possibilities which may occur relative to the qualitative behavior of the eigenvalues of $\underline{A}$ reduce to the determination of the signs of d and b^2-4d as functions of the three parameters.

When $d<0$ (and hence also $b^2-4d>0$), $\underline{A}$ has one positive, one negative, two imaginary, and two zero eigenvalues so that Theorem 1(i) applies and the solutions (2.2) of (2.1) are unstable of type I. In addition, as a consequence of the fact that $\underline{A}$ has a negative eigenvalue, we can prove that the helical trajectories are conditionally stable in the sense that some (but not all) solutions of (2.1) approach the trajectories (2.2) as $t\to\pm\infty$. There is a one-parameter family of such solutions. In order to prove this conditional stability property, we must show that the nonlinear terms (2.9) in (2.6) satisfy a Lipschitz condition in addition to the conditions stated in Theorem 1. The estimates required for this will be derived in the Appendix. In this discussion we are referring to a result on conditional stability proved by Perron in Satz 10 of Ref. 8 and further discussed by Bellman.[9]

When $d>0$ and $b^2>4d$, all four nonzero eigenvalues of $\underline{A}$ are imaginary and distinct so that we must consider the eigenvalue problem for the zero eigenvalues to determine which possibility in case (ii) of Theorem 1 applies. Similarly, when $d=0$, $\underline{A}$ has four zero eigenvalues and two distinct imaginary nonzero eigenvalues so that we must consider the eigenvalue problem for the zero eigenvalues also in this case.

Let $\mathcal{X}$ denote an eigenvector for this problem with components $\mathcal{X}_i$ $(i=1,2,\ldots,6)$. From the form of $\underline{A}$ in (2.7) and the equations defining $\mathcal{X}$ as an eigenvector of $\underline{A}$ corresponding to eigenvalue zero, we find $\mathcal{X}_4=0=\mathcal{X}_6$, $\mathcal{X}_5=ku\mathcal{X}_1$, $\mathcal{X}_2=k\mathcal{X}_3$, and

$$(\alpha+\tfrac{1}{2}ku)\mathcal{X}_1=0 \qquad (2.15)$$

using the fact that $\kappa\neq 0$ since we assume that Ω_w, u, and $|a|$ are nonzero. From (2.15) we obtain $\mathcal{X}_1=0$ provided that

$$\alpha+\tfrac{1}{2}ku\neq 0 \qquad (2.16)$$

so that $\mathcal{X}$ has the form

$$\mathcal{X}=\mathcal{X}_3(0,k,1,0,0,0)$$

which we write as a row vector for typographical convenience.

Thus, the zero eigenvalues have only one eigenvector up to a normalization constant provided that (2.16) holds. Subject to this proviso, it follows from Theorem 1(ii) that, in the cases $d\geq 0$ and $b^2-4d>0$, all solutions of (2.10) are unstable and so, in particular, the trajectories (2.2) are unstable of type II.

From (2.8), we see that (2.16) can be written in the form

$$\pm 2\Omega_w ku\left[\frac{3+(ka)^2}{ka}\right]I_0(ka)\left[\frac{I_1(ka)}{kaI_0(ka)}-\frac{1+(ka)^2}{3+(ka)^2}\right]\neq 0\ . \qquad (2.17)$$

In accordance with the discussion following (2.15), the only factor in (2.17) that can vanish is the one in the large parentheses on the right. Note that the function $(1+x^2)/(3+x^2)$ has the value $\frac{1}{3}$ at $x=0$, increases monotonically on the interval $[0,\infty)$, and approaches unity as $x\to+\infty$. On the other hand, the function $I_1(x)/xI_0(x)$ is monotonically decreasing on $[0,\infty)$, has the value $\frac{1}{2}$ at $x=0$, and approaches zero as $x\to+\infty$. Since these functions are continuous, it follows that the quantity in the large parentheses on the right in (2.17) vanishes at one (and only one) value $ka>0$. By graphical methods we determine this value to be approximately $ka=(ka)_0\cong 0.850$.

Hence, for the cases $ka=\pm(ka)_0$ the condition (2.15) does not require that $\mathcal{X}_1$ be zero. If $\mathcal{X}_1\neq 0$, the eigenvector $\mathcal{X}$ becomes

$$\mathcal{X}=\mathcal{X}_3\left[\frac{\mathcal{X}_1}{\mathcal{X}_3},k,1,0,0,0\right]. \qquad (2.18)$$

One can easily find two normalized (to unity) eigenfunctions of the form (2.18) that are linearly independent. A simple example is the pair

$$\mathcal{X}_a=(2+k^2)^{-1/2}(1,k,1,0,0,0)\ ,$$

$$\mathcal{X}_b=(2+k^2)^{-1/2}(-1,k,1,0,0,0)\ .$$

It follows from Theorem 1(ii) that every solution of (2.10) is stable when $d>0$, $b^2-4d>0$, and $ka=\pm(ka)_0$.

In the situation $d=0$ we still have instability for the cases $ka=\pm(ka)_0$ because it is not possible to find four linearly independent eigenvectors of the form (2.18). When $d=\frac{1}{4}b^2$ the nonzero eigenvalues are all imaginary and occur in doubly degenerate complex conjugate pairs. However, since this equality is not satisfied when $ka=\pm(ka)_0$ so that the above analysis of the zero eigenvalue problem shows that the trajectories (2.2) are unstable of type II, an analysis of the problems for the degenerate eigenvalues is not necessary.

The remaining cases to be considered are those for which $d>0$ and $b^2-4d<0$. In this situation all nonzero eigenvalues of $\underline{A}$ are complex, two having positive real part and two having negative real part. Thus, we have a case of type-I instability and we can prove that the relevant helical trajectories are conditionally stable. There is now a difference, however, compared with the type-I unstable conditionally stable trajectories in the case $d<0$ mentioned previously in that we now have, due to the existence of two eigenvalues with negative real part, a two-parameter family of solutions of (2.1) which approach the trajectories (2.2) as $t\to\pm\infty$.

III. FUNCTIONAL DEPENDENCE OF THE EIGENVALUES OF A ON THE PARAMETERS g, p, AND ka

In order to complete our discussion of the instability of the helical trajectories (2.2) we need to determine the signs of d and b^2-4d as functions of the parameters g, p, and ka when $ka\neq\pm(ka)_0$. We first derive conditions on the three parameters such that $d<0$. We find from (2.13) that this is true if and only if

$$p[\,|ka|^3-pZ(ka)]>0\ . \qquad (3.1)$$

It is seen from (2.14) that $Z(ka)$ approaches unity in the limit $ka\to 0$ so that (3.1) agrees with Diament's condition in the cases that he considers. We note from Fig. 1 that the curves of $|ka|^3$ and $|ka|^3/Z(ka)$ begin to separate at approximately $|ka|\cong 0.36$. For $p>0$ our condition $p<|ka|^3/Z(ka)$ obtained from (3.1) gives a larger range of p values at a given value of $|ka|$ for which the type-I instability occurs compared with Diament's condition $p<|ka|^3$. The function $Z(ka)$ is negative for $|ka|\gtrsim 0.652$ so that, for this range of values of ka, instability occurs for all $p>0$ which satisfy (2.4). This is a striking contrast to the corresponding conclusion obtained from Diament's condition.

When $p<0$ we find from (3.1)

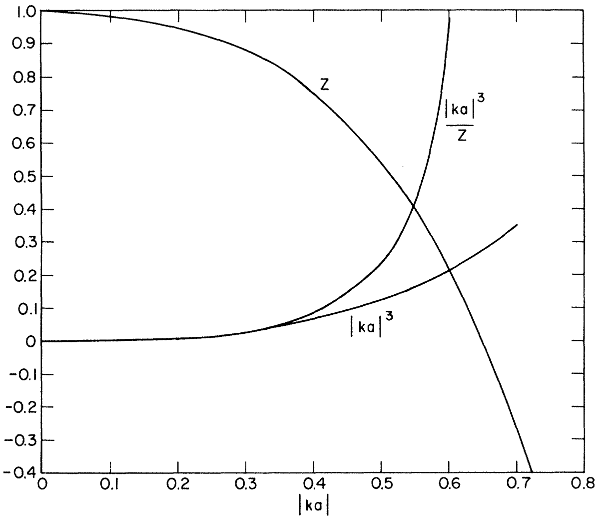

FIG. 1. $|ka|^3$, $Z(ka)$, and $|ka|^3/Z(ka)$ vs $|ka|$.

$$|ka|^3+|p|Z(ka)<0$$

which cannot be satisfied for values of ka such that $Z(ka)\geq 0$. However, when $Z(ka)<0$ (i.e., when $|ka|\gtrsim 0.652$) we have

$$\frac{|ka|^3}{|Z(ka)|}<|p| \ .$$

Thus, we can summarize the conditions for $d<0$ in the form

$$p<\frac{|ka|^3}{Z(ka)} \qquad (3.2)$$

for $p>0$, or $p<0$ and $Z(ka)<0$.

These inequalities only give conditions for the type-I instability with $d<0$ when (2.4) is satisfied. We see by comparison of Fig. 1 with results obtained from the inversion of (2.4) (see Ref. 3) that this is most likely to occur, in the cases $g>1$, for relatively small g and relatively large $|ka|$. For example, one finds that the type-I instability with $d<0$ begins at $|ka|\cong 0.65$ for $g=4$, $|ka|\cong 0.57$ for $g=3$, $|ka|\cong 0.525$ for $g=2$, and at $|ka|\cong 0.35$ for $g=1.25$.

In the case of the type-I instability defined by the condition $d<0$, let λ denote the single positive eigenvalue of $\underline{A}$. Then λ^{-1} is the e-folding time for the unstable trajectory, i.e., the time required for a solution of (2.1) to differ from a helical solution[16] (2.2) by a factor of e in the sense of the norm (2.11). These times are so short (of the order of nanoseconds or smaller) that we find it more convenient to present the results in terms of distances. We will use the quantity c/λ which can be interpreted as an approximate e-folding distance, i.e., the approximate distance required for the magnitude of a trajectory satisfying (2.1) to be greater than the corresponding magnitude of a helical trajectory (2.2) by a factor of e in the sense of the norm (2.11).

In Figs. 2–4 we have plotted c/λ versus p for various values of ka. In each case the permissible values of p are determined by the quantity $|ka|^3/Z(ka)$ as indicated in (3.2).

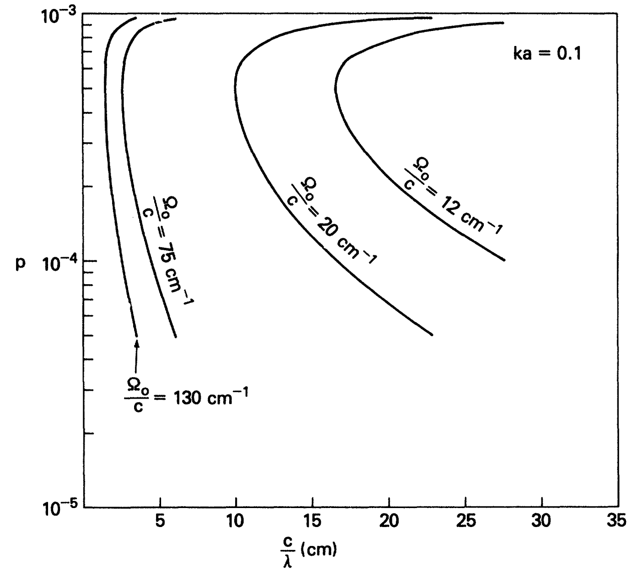

FIG. 2. p vs c/λ. $ka=0.1$, $\Omega_0/c=12,20,75,130$ cm^{-1}.

We now make several observations concerning the curves in these figures.

(i) The curves for the relatively large value $ka=1.0$ in Figs. 3 and 4 are monotonic for both positive and negative values of p whereas this is not the case for the plots of c/λ versus $p>0$ for the smaller value of ka shown in Fig. 2.

(ii) The e-folding distance c/λ has its largest values for smaller values of $|ka|$. On the other hand, the required values of p in order that the $d<0$ type-I instability exist are much smaller for these cases than for larger ka values.

(iii) It is seen from Fig. 4 that the slopes of the curves of c/λ versus p for negative p is much steeper near the maximum allowed value of p (minimum allowed value of $|p|$) than for the other portions of these curves or for the curves corresponding to positive values of p shown in

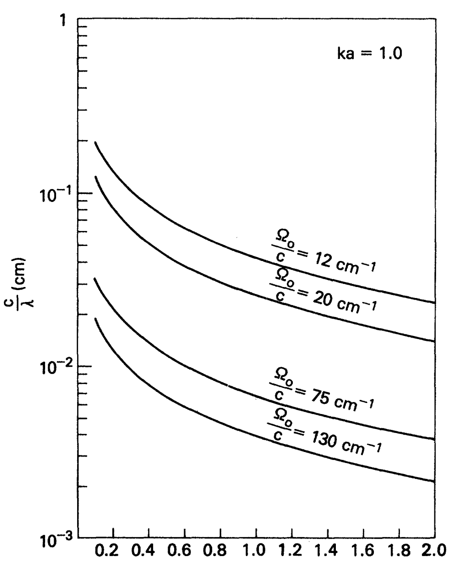

FIG. 3. c/λ vs p. $ka=1.0$, $p>0$, $\Omega_0/c=12,20,75,130$ cm^{-1}.

Figs. 2 and 3. Thus, in Fig. 4, c/λ is relatively large over a significant range of values of p near the maximum allowed value of this quantity.

The conditions for the existence of type-II instabilities are that $b^2-4d \geq 0$ and that the inequalities (3.2) are violated, coupled with the additional conditions $ka \neq \pm (ka)_0$ when the parameters correspond to the case $d>0$. It will be seen that we can make some general statements concerning the validity of the inequality $b^2-4d>0$ when $p>0$, but the cases $p<0$ generally require detailed numerical investigation. From (2.12) and (2.13) we have

$$b^2-4d=\frac{\Omega_0^4}{[1+(ka)^2]^2}\left[16\frac{I_1^4(ka)}{(ka)^8}[1+(ka)^2]^4p^4+8g^2[1+(ka)^{-2}]\left\{\frac{I_1^2(ka)}{(ka)^2}[1+(ka)^2]-I_1'(ka)Z(ka)\right\}p^2 + 8g^2[1+(ka)^2]I_1'(ka)\,|\,ka\,|\,p+g^4\right] . \tag{3.3}$$

It follows from numerical considerations that the part of the coefficient of p^2 in brackets is positive when $|\,ka\,| \gtrsim 0.478$ so that $b^2-4d>0$ for this range of ka when $p>0$ and the pair (g,p) satisfies (2.4).

For further investigation of the positivity of (3.3) it is advantageous to use the constraint (2.4) to eliminate g. Some tedious algebraic manipulations yield the following expression:

$$b^2-4d=\frac{16\Omega_0^4 I_1^2(ka)I_0^2(ka)}{(ka)^6}[1+(ka)^2]^2D\left\{p^4+\frac{2(ka)^2}{I_1(\,|\,ka\,|\,)D}\{2[1+(ka)^2]-\delta[8+(ka)^2]+8\delta^2\}p^3 + \frac{(ka)^4\{2[1+(ka)^2]-2\delta[4-(ka)^2]+\delta^2[10-(ka)^2]\}}{2I_1^2(ka)[1+(ka)^2]D}p^2 + \frac{|\,ka\,|^7\left\{1+\frac{\delta}{(ka)^2}\right\}p}{2I_0(ka)I_1^2(ka)[1+(ka)^2]^2D}+\frac{(ka)^6}{16I_0^2(ka)I_1^2(ka)[1+(ka)^2]^2D}\right\} , \tag{3.4}$$

where

$$\delta=\frac{I_1(ka)}{kaI_0(ka)}$$

and

$$D=4[1+(ka)^2]^2-4\delta[1+(ka)^2][4+(ka)^2] +\delta^2\{16[1+(ka)^2]+(ka)^4\} .$$

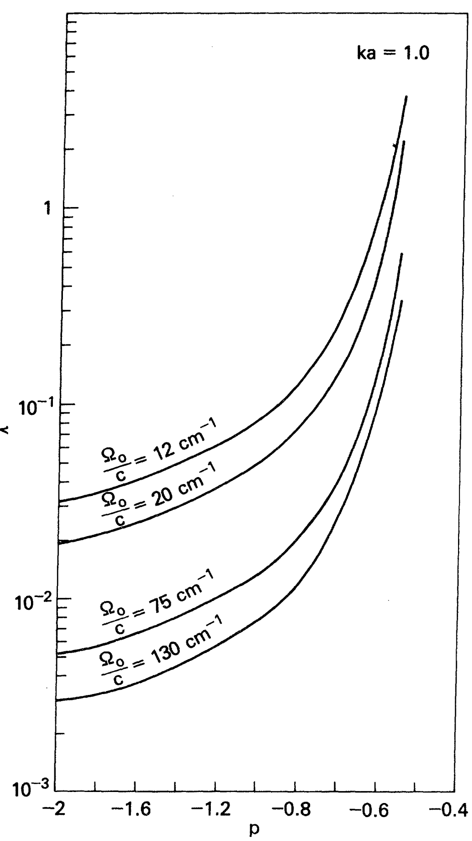

FIG. 4. c/λ vs p. $ka=1.0$, $p<0$, $\Omega_0/c=12,20,75,130$ cm^{-1}.

The coefficients of the quartic expression in (3.4) are positive when $0<|\,ka\,|<0.50$ so that, by combining this result with that stated following (3.3), we conclude that $b^2-4d>0$ when $p\geq 0$ and the inequality (3.2) is violated.

Finally, we consider the cases when $p<0$. These are best analyzed by direct numerical investigation of (3.4). In the range of values $0<|\,ka\,|\leq 10$ we find that $b^2-4d>0$ when $0.059\leq |\,ka\,|<8.25$ and $9.80 \leq |\,ka\,| \leq 10.0$, but that there are subintervals of the intervals J_1, $0<|\,ka\,|<0.059$, and J_2, $8.25\leq |\,ka\,| <9.80$, in which the inequality $b^2-4d<0$ is valid. The situation when $|\,ka\,|$ belongs to either of these intervals is very complicated. Generally, both cases $b^2-4d \lesseqgtr 0$ occur as one passes to finer subintervals of J_1 and J_2. Thus, these results show that when $p<0$ and $|\,ka\,|$ belongs to either of the intervals J_1 or J_2, the helical trajectories can change from type-I unstable to type-II unstable (or vice versa) as a result of a very small change in the value of ka.

IV. SUMMARY

In the present paper we have investigated the instability of electron helical trajectories in the magnetic field (1.1) without replacing this expression by (1.2). This is the first analysis of such trajectories which is free of this approximation. Our results are of two kinds.

First, we have shown that helical trajectories corresponding to (1.1) are unstable with the possible exception of one value $|a|$ of the orbital radius. The instabilities are of two types, called I and II, corresponding to the respective cases when the matrix $\underline{A}$ associated with the linearization of the equations of motion does or does not have an eigenvalue with positive real part. For the exceptional value of $|a|$, the trajectories can be either unstable of type II or linearly stable depending upon the values of the parameters B_0, B_w, and electron energy. As one expects, these results are in qualitative agreement with those of Diament[3] when $|ka|$ is small. However, there are significant quantitative differences between his results and ours even for values of $|ka|$ that are not exceedingly large—of the order of 0.4. This is surprising because *a priori* one does not expect Diament's approximations to break down until $|ka|$ is of the order of unity. In general, the parameter ranges over which the two types of instabilities occur are quite different in the two analyses. We may conclude that the approximation of replacing (1.1) by (1.2) is worse than that stated by Diament. The exceptional value of $|a|$ for which some of the helical trajectories are linearly stable does not show up in Diament's analysis because the relatively large value $|a| \cong 0.850/k$, at which this situation occurs is outside the domain of applicability of his approximations.

Second, we have shown that the unstable trajectories of type I belong to the special instability type called conditionally stable.[8-10] This means that, even though these helical trajectories are unstable so that not all solutions of (2.1) which are close to them at $t=0$ remain close at all other times, there are some solutions which remain close to them at all other times and in fact approach them in the limits $t \rightarrow \pm\infty$.

In Figs. 2—4 we have plotted some typical values of e-folding distances as a function of p for several values of ka for type-I unstable trajectories corresponding to the case $d<0$ when the matrix $\underline{A}$ has a single eigenvalue with positive real part. The order of magnitude of these distances decreases as $|ka|$ increases, indicating that the instability is more severe for larger values of $|ka|$. Even so, this type-I instability appears to be significant also for relatively small values of $|ka|$, being of the order of 2—27 cm for typical values of the guide and wiggler field magnitudes when $ka=0.1$, as shown in Fig. 2. We have discussed some numerical results which indicate that, when $p<0$ and $|ka|$ belongs to certain intervals, the helical trajectories can change character between the instability types I and II as a result of a very small change in the value of ka.

ACKNOWLEDGMENTS

I wish to thank Mrs. Linda Tsui of the Naval Research Laboratory and Dr. Anton Nagl of Catholic University, Washington, D.C., for performing the numerical computations reported in this paper.

APPENDIX: CONDITIONAL STABILITY

In this appendix we show that the helical trajectories (2.2) are conditionally stable in the case of the type-I instability, a circumstance made possible by the fact that $\underline{A}$ has at least one eigenvalue with negative real part in this case. The result on conditional stability that we use is due to Perron,[8] and a nice discussion of it has been given by Bellman.[9] We must show that the functions f_i $(i=2,4,5,6)$ defined in (2.9) satisfy the conditions of Theorem I and also the following Lipschitz condition:

$$||f(w^{(1)})-f(w^{(2)})|| \le s||w^{(1)}-w^{(2)}|| ,\quad \text{for } ||w^{(1)}|| \le h,\ ||w^{(2)}|| \le h \tag{A1}$$

where $s \rightarrow 0$ as $h \rightarrow 0$. It is clear from (2.9) that the f_i $(i=2,4,5,6)$ are continuous functions of the variables w_j $(j=1,2,\ldots,6)$ so that we can proceed directly to a verification of the conditions in Theorem 1 and (A1). To obtain the required estimates, we use the fact that the functions occurring in the f_i (Bessel, trigonometric, and rational functions) have series expansions that converge absolutely. Our method is to expand these functions in their series representations, obtain estimates for the summands, and then sum the resulting series. This procedure, although straightforward in principle, is quite tedious in practice due to the complicated nature of the f_i functions. Therefore, in order not to encumber the paper with excessive complicated expressions, we will only state the final results for the estimates obtained by the procedure outlined above.

Consider first the condition in Theorem 1. Since we are interested in the limit $||w|| \rightarrow 0$ we can assume, without loss in generality, that $||w|| < |a|$. We then find

$$\frac{||f||}{||w||} \le \frac{||w||}{\gamma|a|}(1+|ku|)\left[1-\frac{||w||}{|a|}\right]^{-1}(2+|ku|+|a|^{-1})$$
$$+2|\Omega_w|\left\{2I_1(|ka|+|k|\,||w||)\cos\{h[(1+|k|)||w||]\}\right.$$
$$\times[1+(|ka|+|k|\,||w||)^{-1}]-2I_1(|ka|)(1+|ka|^{-1})$$
$$+2I_1'(|ka|+|k|\,||w||)\sin\{h[(1+|k|)||w||]\}$$

$$+\frac{|u|}{||w||}[(1+|ka|)(I_1'(|ka|+|k|\,||w||)\sin\{h[(1+|k|)||w||]\}$$

$$-I_1'(|ka|)(1+|k|)||w||)$$

$$+I_1(|ka|+|k|\,||w||)\cos\{h[(1+|k|)||w||]\}[|ka|+(|ka|+|k|\,||w||)^{-1}]$$

$$-I_1(|ka|)(|ka|+|ka|^{-1})]-|ku|\left[|ka|I_1'(|ka|)+\left[\frac{I_1(x)}{x}\right]'\Bigg|_{x=|ka|}\right]\Bigg\}.$$

The validity of the condition in Theorem 1 follows easily from this estimate.

In order to verify (A1), we consider the situation $||w^{(1)}||\le h$ and $||w^{(2)}||\le h$ in the limit $h\to 0$ so that we can, without loss in generality, assume that $h<|a|$. Then, using the identity

$$a^n-b^n=(a-b)\sum_{i=0}^{n-1}a^{n-1-l}b^l$$

and the triangle inequality, we obtain the first inequality in (A1) with

$$s=(1+|ku|)\frac{h}{\gamma|a|}\left[2-\frac{h}{|a|}\right]\left[1-\frac{h}{|a|}\right]^{-2}(2+|ku|+|a|^{-1})$$

$$+4|\Omega_w|\Bigg[I_1(|ka|+|k|h)[1+(|ka|+|k|h)^{-1}]$$

$$\times((1+|k|)h\sin\{h[(1+|k|)h]\}+\cos\{h[(1+|k|)h]\})-I_1(|ka|)(1+|ka|^{-1})$$

$$+I_1'(|ka|+|k|h)(\sin\{h[(1+|k|)h]\}+(1+|k|)h\cos\{h[(1+|k|)h]\})$$

$$+|k|h\Bigg\{\cos\{h[(1+|k|)h]\}\left[I_1'(|ka|+|k|h)+\left[\frac{I_1(x)}{x}\right]'\Bigg|_{x=|ka|+|k|h}\right]$$

$$+\sin\{h[(1+|k|)h]\}I_1''(|ka|+|k|h)\Bigg\}$$

$$+\frac{|u|}{2}\Bigg\{[(1+|k|)(1+|ka|)+k^2|a|](\cos\{h[(1+|k|)h]\}I_1'(|ka|+|k|h)-I_1'(|ka|))$$

$$+\sin\{h[(1+|k|)h]\}\{|k|(1+|ka|)I_1''(|ka|+|k|h)$$

$$+(1+|k|)[|ka|+(|ka|+|k|h)^{-1}]I_1(|ka|+|k|h)\}$$

$$+|k|\left[\cos\{h[(1+|k|)h]\}\left[\frac{I_1(x)}{x}\right]'\Bigg|_{x=|ka|+|k|h}-\left[\frac{I_1(|ka|)}{|ka|}\right]'\right]\Bigg\}\Bigg].$$

One easily shows that $s\to 0$ as $h\to 0$ thereby verifying condition (A1).

[1]J. P. Blewett and R. Chasman, J. Appl. Phys. 48, 2692 (1977).
[2]L. A. Vainshtein, Zh. Tekh. Fiz. 49, 1129 (1979) [Sov. Phys.—Tech. Phys. 24, 625 (1979)].
[3]P. Diament, Phys. Rev. A 23, 2537 (1981).
[4]T. Kwan and J. M. Dawson, Phys. Fluids 22, 1089 (1979).
[5]L. Friedland and J. L. Hirshfield, Phys. Rev. Lett. 44, 1456 (1980).
[6]L. Friedland, Phys. Fluids 23, 2376 (1980).

[7]H. P. Freund and A. T. Drobot, Phys. Fluids 25, 736 (1982).

[8]O. Perron, Math. Z. 29, 129 (1929).

[9]R. Bellman, *Stability Theory of Differential Equations* (McGraw-Hill, New York, 1953), theorem 4, p. 90.

[10]E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955), theorem 4.1, p. 330.

[11]This intuitive description of stability is taken from Ref. 14. We have chosen to give these definitions in lieu of more precise ones of ϵ-δ type.

[12]Actually, one usually uses the analog of (2.2) corresponding to (1.2).

[13]See for example, L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations* (Springer, Berlin, 1963), p. 10.

[14]M. Braun, *Differential Equations and Their Applications,* 2nd ed. (Springer, New York, 1978).

[15]V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1978), p. 385.

[16]This interpretation arises because the representation of the helical trajectories (2.2) in the variational system (2.5) is given by $w_i = 0$ ($i = 1, 2, \ldots, 6$).