Anomalous diffusion of charged particles in a strong magnetic field

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A self-consistent mode-coupling theory is used to calculate the coefficient of self-diffusion in a three-dimensional classical one-component plasma subjected to an external magnetic field. For asymptotically large fields a Bohm-like behavior is found for diffusion in the plane perpendicular to the magnetic field. The experimental consequences of these results are discussed.

In this paper we consider transport processes in a one-component plasma with a uniform neutralizing background and outline a computation of the dependence of transport coefficients on the strength of an applied uniform magnetic field $\vec{B}$. As an example, we will consider here the case of a test particle diffusing in an equilibrium plasma and evaluate the transport coefficient $D_\perp$ for diffusion in a plane perpendicular to the magnetic field. Our calculations are based upon the application of mode-coupling theory to the evaluation of the Green-Kubo time correlation function expressions for the transport coefficients. A more detailed presentation of this and related transport processes as well as a derivation of the mode-coupling theory from a microscopic point of view will be presented elsewhere.[1]

This work is motivated by the observations that in computer simulations[2–4] and in solid-state plasmas[5] transport properties of a dilute classical plasma in a strong magnetic field deviate appreciably from the predictions of the Balescu-Lenard-Guernsey (BGL) kinetic equation.[6,7] For sufficiently weak magnetic fields $D_\perp$ is larger than the value predicted by the BGL equation and is independent of the field strength, while for very strong fields $D_\perp$ exhibits a B^{-1} Bohm-like behavior.[2] Here we show that the behavior of $D_\perp$ in these various regions can be accounted for on the basis of mode-coupling theory. The relationship of the present work to previous ones[2–4,8,9] will be discussed below.

We consider a one-component plasma with plasma parameter $\epsilon_p=(4\pi n\lambda_D^3)^{-1}<1$, with n the number density, $\lambda_D=[(4\pi ne^2)/k_BT]^{-1/2}$, the Debye length where e is the electronic charge, k_B is Boltzmann's constant, and T the absolute temperature. We will also need the plasma frequency $\omega_p=(4\pi ne^2/m)^{1/2}$, the cyclotron frequency $\omega_B=eB/mc$, and the Larmor radius $r_L=(k_BT/m)^{1/2}\omega_B^{-1}$. The coefficient of diffusion of a test electron in the plane perpendicular to the magnetic field $\vec{B}$ can be written in terms of the velocity autocorrelation function through the Green-Kubo formula:

$$D_\perp=\int_0^\infty dt\,\langle v_{1x}(0)v_{1x}(t)\rangle \quad . \tag{1}$$

Here we take the magnetic field to be $\vec{B}=B\hat{z}$, where $\hat{z}$ is a unit vector in the z direction, $v_{1x}(t)$ the x component of the velocity of the test particle at time t, and the angular brackets denote an equilibrium grand canonical ensemble average.

For very short times the decay of the velocity autocorrelation function is described by the solution of the linearized BGL equation. The corresponding contribution to $D_\perp$ denoted by $D_\perp^{(0)}$ is easily computed,[6] and for large enough magnetic fields, $\nu_c/\omega_B \ll 1$, where ν_c is the BGL collision rate $\nu_c=\omega_p\epsilon_p\ln\epsilon_p^{-1}$, $D_\perp^{(0)}=(3\pi^{1/2})^{-1}r_L^2\nu_c\approx O(B^{-2})$. The behavior of the velocity autocorrelation function for intermediate times is at present unknown. However, we expect that the additional contribution to $D_\perp$ for this time interval can be described by a regular, if not analytic, expression in powers of ϵ_p and $\nu_c\omega_B^{-1}$ about $D_\perp^{(0)}$, since the velocity autocorrelation function will continue to decay through collisional processes, which can be described in terms of higher-order density (i.e., ϵ_p) corrections to the BGL equation. However, for longer times it can be shown by using either kinetic theory or a more general approach that another type of processes govern the decay of the velocity autocorrelation function. These are the hydrodynamiclike decays of long-lived collective excitations in the plasma. The contribution of these collective excitations, or hydrodynamic modes, to $D_\perp$ is computed by means of the mode-coupling theory.[10] Thus we write $D_\perp=D_{\perp,\mathrm{reg}}+\delta D_\perp$, with $D_{\perp,\mathrm{reg}}$ being the contribution to $D_\perp$ in (1) from the short and intermediate time regions[11] while the contribution to $D_\perp$ from mode-coupling effects is given by

$$\delta D_\perp=\frac{1}{\Omega}\sum_{\vec{k}}{}'\sum_{\alpha=1}^{5}\frac{\langle v_{1x}\Theta_{\alpha\vec{k}}C_{-\vec{k}}\rangle\langle\Theta_{\alpha\vec{k}}^\dagger C_{\vec{k}}v_{1x}\rangle}{\omega_D(\vec{k})+\omega_\alpha(\vec{k})} \quad . \tag{2}$$

Here Ω is the volume of the system and the prime on the $\vec{k}$ summation indicates the restriction to $\vec{k}\leq\vec{k}_0$ where $\vec{k}_0$ represents the largest wave vector at which a hydrodynamic description of the system may be applied. The summation in Eq. (2) is over the five hydrodynamic modes of the system. These hydrodynamic modes have been obtained in two ways: (a) from the Liouville equation by using formal projection operator methods, and (b) for small plasma parameters, by looking for the hydrodynamic modes of the Landau form of the BGL equation. In addition $C_{\vec{k}}=e^{-i\vec{k}\cdot\vec{r}_1}$ is the eigenfunction corresponding to the diffusive mode for the tagged particle whose position is denoted by $\vec{r}_1$ and $\omega_D(\vec{k})$ is the corresponding eigenvalue; $\Theta_{\alpha\vec{k}}$ and $\Theta_{\alpha\vec{k}}^\dagger$ are the right and left eigenfunctions of the hydrodynamic mode labeled by α, and $\omega_\alpha(\vec{k})$ is the corresponding eigenvalue. We consider now some specific details of the five modes labeled by α. For a one-component plasma in the presence of a magnetic field there are (in three dimensions) four propagating modes and one diffusionlike mode. In contrast to the case for neutral fluids, the four propagating modes for a one-component plasma are also finite frequency modes. That is, the imaginary part of the eigenvector $\omega_\alpha(\vec{k})$, describing propagation, approaches a nonzero limit as $k\to 0$. This is a consequence both of the long-range character of the Coulomb potential as well as of the presence of the magnetic field. The modes α are as follows.

(1) Two high-frequency modes, known in the Vlasov limit as the first Bernstein modes, or the upper hybrid modes, whose dispersion relation is given, for $\omega_p/\omega_B \ll 1$, by

$$\omega_{p\sigma}(\vec{k}) = i\sigma|\omega_B|\left(1+\frac{1}{2}\frac{\omega_p^2}{\omega_B^2}\hat{k}_\perp^2+\frac{\gamma}{2\rho\chi_T}\frac{k_\perp^2}{\omega_B^2}\right) + k_z^2\nu_\parallel(i\sigma|\omega_B|)+k_\perp^2\nu_\perp(i\sigma|\omega_B|)+O(k^3) \quad , \tag{3a}$$

with $\sigma=\pm 1$. When $\vec{B}=0$, these modes reduce to the plasma modes.

(2) Two finite frequency modes, known in the Vlasov limit as the propagating plasma modes, with frequency $\omega_{\nu\sigma}(\vec{k})$, for $\omega_p/\omega_B \ll 1$

$$\omega_{\nu\sigma}(\vec{k}) = i\sigma\omega_p|\hat{k}_z|\left(1+\frac{\gamma}{2\rho\chi_T}\frac{k^2}{\omega_p^2}\right)+k_z^2\nu'_\parallel(i\sigma\omega_p|\hat{k}_z|) + k_\perp^2\nu'_\perp(i\sigma\omega_p|\hat{k}_z|)+O(k^3) \quad . \tag{3b}$$

When $\vec{B}=0$, these reduce to the shear modes.

(3) One diffusive heat mode which is not needed here. In Eqs. (3a) and (3b) terms of $O(\omega_p^3/\omega_B^3)$ have been neglected. Also $\hat{k}_\perp=\vec{k}_\perp/k$, $\hat{k}_z=k_z/k$, where $k^2=k_z^2+k_\perp^2$; $D_\parallel^T$ and $D_\perp^T$ are the thermal diffusivities in the direction of $\vec{B}$ and in the plane orthogonal to $\vec{B}$, respectively, $\gamma=c_p/c_v$ is the ratio of specific heats, $\rho=nm$, and χ_T is the isothermal compressibility. The quantities $\nu_\parallel$, $\nu_\perp$, $\nu'_\parallel$ and $\nu'_\perp$ are linear combinations of the five kinematic viscosities that appear in the magnetohydrodynamic equations, with one important—and for our work here—crucial difference. The kinematic viscosities used here are complex, frequency-dependent transport coefficients appearing in a set of "generalized" magnetohydrodynamic equations and they are each to be evaluated at the frequency indicated by their arguments. When the kinematic viscosities are evaluated at the indicated frequencies, by using the BGL kinetic equation[12] one finds that $\mathrm{Re}\nu_\parallel$ and $\mathrm{Re}\nu'_\parallel$ are $O(\omega_B^0)$, $\mathrm{Re}\nu_\perp$ and $\mathrm{Re}\nu'_\perp$ are $O(\omega_B^{-2})$, and $\mathrm{Im}\nu_\perp$ as well as $\mathrm{Im}\nu'_\perp$ are $O(\omega_B^{-1})$. It is worth noting that the presence of the frequency-dependent viscosities in the dispersion relations [(3a),(3b)] means that these relations cannot be obtained from the usual magnetohydrodynamic equations with constant transport coefficients. This point has been discussed in detail by Baus[13] for the case $\vec{B}=0$. Finally we will need the eigenvalue for the diffusive mode

$$\omega_D(\vec{k})=k_z^2D_\parallel+k_\perp^2D_\perp \quad , \tag{4}$$

where $D_\parallel$ is the coefficient of self-diffusion in the z direction.

In order to complete the preliminary discussion prior to the actual computation of the mode-coupling effects, we must estimate the values of the cutoffs on the $\vec{k}$ summations in Eq. (2). Because the magnetic field introduces an anisotropy in the system, cutoffs in the k_z sum may be different from those in the k_x, k_y sums. Further the hydrodynamic mode description may extend over a larger region of k space for one mode than another, so the cut-off values may differ for the different modes. To estimate the order of magnitude of the k cutoffs we require that the $\omega_\alpha(\vec{k})$ dispersion relation be well ordered in powers of k for all $\vec{k}$ such that $|\vec{k}|<|\vec{k}_0|$. We find that for the upper hybrid modes the cutoffs are $k_{z,0}\sim(\nu_c/\mathrm{Re}\nu_\parallel)^{1/2}$ and $k_{\perp,0}\sim r_L^{-1}$. For the low-frequency mode extension of the shear modes we find that $k_{z,0}$ and $k_{\perp,0}\sim l^{-1}$, where $l=\lambda_D\omega_p/\nu_c$ is the mean free path of an electron in the plasma. This result is consistent with the fact that, in the Vlasov limit, the propagating plasma waves are well-defined excitations only for $k<\lambda_D^{-1}$.[14]

The mode-coupling contribution to $D_\perp$ can now be evaluated. We find that the heat mode does not contribute to $\delta D_\perp$, and that $\delta D_\perp=\delta D_\perp^{(\nu)}+\delta D_\perp^{(p)}$ with

$$\delta D_\perp^{(\nu)}=\frac{k_BT}{2nm}\frac{\omega_p^2}{\omega_B^2}\mathrm{Re}\frac{1}{\Omega}\sum_{\vec{k}}{}'\,\hat{k}_\perp^2[\omega_D(\vec{k})+\omega_{\nu+}(\vec{k})]^{-1} \quad , \tag{5a}$$

and

$$\delta D_\perp^{(p)}=\frac{k_BT}{nm}\mathrm{Re}\sum_{\vec{k}}{}'\,[\omega_D(\vec{k})+\omega_{p+}(\vec{k})]^{-1} \quad . \tag{5b}$$

We have analyzed the contribution of the propagating plasma modes $\delta D_\perp^{(\nu)}$ to $D_\perp$ and can show that in the thermodynamic limit, in three dimensions, and for small ϵ_p, $\delta D_\perp^{(\nu)}$ provides a correction of order ϵ_p^3/B^2 to the BGL value of $D_\perp$. The contribution from the upper hybrid modes is given in the thermodynamic limit $\Omega\to\infty$ by

$$\delta D_\perp^{(p)}=\frac{k_BT}{2\pi^2nm}\int_0^{(\nu_c/D_\parallel)^{1/2}}dk_z\int_0^{1/r_L}dk_\perp k_\perp[\omega_D(\vec{k})+\omega_{p+}(\vec{k})]^{-1}$$
$$\cong\frac{\nu_c}{4\pi^2n}\left(\frac{\nu_c}{D_\parallel}\right)^{1/2}\left[\frac{1}{3}\left(1+\frac{\mathrm{Re}\nu_\parallel}{D_\parallel}\right)+\frac{m\omega_B^2}{2k_BT\nu_c}(D_\perp+\mathrm{Re}\nu_\perp)\right]+O(\omega_B^{-2}) \quad . \tag{6}$$

For small values of the plasma parameter, the transport coefficients appearing on the right-hand side of Eq. (6) can be evaluated using the BGL equation. One then finds

$$\frac{\delta D_\perp^{(p)}}{D_\perp^{(0)}}\approx 0.6\frac{\omega_B^2}{\omega_p^2}\epsilon_p\frac{\nu_c}{\omega_p} \quad . \tag{7}$$

The quantities $\delta D_\perp^{(p)}$ and $D_\perp^{(0)}$ are of the same order for large enough magnetic fields that $\omega_B/\omega_p\approx(\omega_p/\epsilon_p\nu_c)^{1/2}$. For larger fields the two-mode approximation for $\delta D_\perp$ is no longer adequate and more complicated mode-coupling effects need to be taken into account. An approximate way of doing this is to use a "self-consistent" mode-coupling theory, whereby all the transport coefficients appearing in the mode-coupling equations are replaced by their mode-coupling values and the entire set of equations is solved simultaneously. In our case it is essential to note that only the kinematic viscosities $\nu_\perp$ and $\nu'_\perp$ differ considerably from their BGL values at the magnetic fields of interest.

When we take into account the cutoffs on the various k integrands, we obtain the following self-consistent equations for $\delta\nu_{\perp}^{sc}$ and $\delta D_{\perp}^{sc}$, where $D_{\perp}=D_{\perp}^{(0)}+\delta D_{\perp}^{sc}$, $\nu_{\perp}=\nu_{\perp}^{(0)}+\delta\nu_{\perp}^{sc}$:

$$\delta D_{\perp}^{sc} \cong \alpha_1 \left[\frac{k_B T}{m\omega_B}\right]^2 \epsilon_p \frac{\nu_c^3}{\omega_p^3}\left[\frac{20}{9}+\frac{\mathrm{Re}\delta\nu_{\perp}^{sc}}{\delta D_{\perp}^{sc}}\right]\frac{1}{\delta D_{\perp}^{sc}} \quad , \tag{8a}$$

and

$$\mathrm{Re}\nu_{\perp}^{sc} \cong \alpha_2 \left[\frac{k_B T}{m\omega_B}\right]^2 \epsilon_p \frac{\nu_c^3}{\omega_p^3}\frac{1}{\mathrm{Re}\delta\nu_{\perp}^{sc}} \quad , \tag{8b}$$

where α_1 and α_2 are numerical constants and we used the BGL values $\nu_{\parallel}$ and $D_{\parallel}$ since their mode-coupling contribution is much smaller than their bare values for all values of the magnetic field used here. Equations (8a) and (8b) are easily solved with the result that $\mathrm{Re}\nu_{\perp}^{sc}$ and $\delta D_{\perp}^{sc} \sim B^{-1}$. In particular,

$$\delta D_{\perp}^{sc} \cong \alpha \frac{k_B T}{m\omega_B}\epsilon_p^{1/2}(\epsilon_p \ln\epsilon_p^{-1})^{3/2} \quad , \tag{9}$$

where $\alpha \approx 0.5$. For large enough magnetic fields, the contribution $\delta D_{\perp}^{sc}$ dominates the other contributions of $O(B^{-2})$, and the leading magnetic field dependence of $D_{\perp}$ is of order B^{-1}. A crude estimate of the value of B where the Bohm-like diffusion is dominant is easily found to be given by $\omega_B/\omega_p \sim 4(\epsilon_p \nu_c/\omega_p)^{-1/2}$.

As a result of this analysis we can describe the magnetic field dependence of $D_{\perp}$ for $\omega_B/\omega_p > 1$:

(a) a classical region where $D_{\perp} \sim B^{-2}$ for

$$\nu_c/\omega_p < \omega_B/\omega_p < 0.4(\epsilon_p\nu_c/\omega_p)^{-1/2} \quad ;$$

(b) a plateau region where $D_{\perp} \sim B^0$ for

$$0.4(\epsilon_p\nu_c/\omega_p)^{-1/2} < \omega_B/\omega_p < (\epsilon_p\nu_c/\omega_p)^{-1/2} \quad ;$$

(c) a Bohm region where $D_{\perp} \sim B^{-1}$ for

$$\omega_B/\omega_p > 4(\epsilon_p\nu_c/\omega_p)^{-1/2} \quad .$$

We conclude our discussion with a number of remarks.

(1) The results given by Eqs. (6)–(9) are cut-off dependent and since we only have estimates of these cutoffs the resulting numerical coefficients in these equations are not precisely known. In principle the exact coefficients can be obtained by using kinetic theory.

(2) A related calculation for two-dimensional (2D) systems was carried out by Krommes and Oberman.[4] In two-dimensional systems only the two upper hybrid modes are propagating modes. The two propagating plasma modes are replaced with a single diffusive mode, usually referred to as the "convective cells mode." The dominant mode-coupling effect is then due to the coupling of the convective cells with the diffusion mode. In contrast, we find that for 3D systems the dominant effect is due to the coupling with the upper hybrid modes.

(3) The 3D problem has also been considered by Okuda and Dawson[3] and by Montgomery, Liu, and Vahala.[9] However, these authors effectively reduce the 3D problem to a 2D one by considering the case where the size of the system in the direction of the field is small. As a consequence of this, the contribution they calculate vanishes in the thermodynamic limit.

(4) For bounded, finite systems both kinds of modes contribute to $\delta D_{\perp}$ and we estimate that $\delta D_{\perp}^{(p)}$ will dominate only if the size of the system in the direction of $\vec{B}$, L_z, satisfies the inequality

$$L_z/l > \alpha'[(\nu_c/\omega_p)^2 \ln(L_{\perp}/k_{\perp,0})]^{-1} \quad ,$$

where $L_{\perp}$ is the size of the system in the direction perpendicular to the field. This restriction is important for a comparison of the theoretical predictions with computer or experimental results.

(5) The size-dependent contribution $\delta D_{\perp}^{(\nu)}$ has been considered by us also. This term exhibits three regions with similar behavior or $\delta D_{\perp}^{(p)}$. However there is a substantial difference between the qualitative features of the two effects. In particular, $\delta D_{\perp}^{(p)}$ is proportional to ϵ_p^3 for small ϵ_p while $\delta D_{\perp}^{(\nu)}$ is independent of ϵ_p. For laboratory plasmas ϵ_p is very small and the finite size of the system usually guarantees that $\delta D_{\perp}^{(\nu)}$ will provide the dominant effect. Thus our calculations are not entirely relevant for the description of such systems. A similar conclusion also applies to the computer experiments of Okuda and Dawson.[2–4] For these experiments the system size is only on the order of a few mean free paths while the estimate in point (3) above gives for the largest plasma parameter used in the computer simulations, corresponding to $\nu_c/\omega^p \sim 0.4\times10^{-2}$, $L_z/l \geq 10^3$ for $\delta D_{\perp}^{(p)}$ to dominate.

(6) The bulk behavior should be seen in solid-state plasmas. The anomalous diffusion associated with $\delta D_{\perp}^{(p)}$ should be observable for systems with $L_z > 10^{-2}$ cm, typically. One might even hope to see the transition from the 2D-like behavior given by $\delta D_{\perp}^{(\nu)}$ to the 3D behavior of $\delta D_{\perp}^{(p)}$.

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[11]For sufficient small plasma parameters $D_{\perp,\mathrm{reg}}$ can be well approximated by $D_{\perp}^{(0)}$.

[12]In the limit $\omega_p/\omega_B < 1$, a $\vec{B}$-dependent collision operator should be used. However, this only leads to logarithmic corrections to the leading magnetic field dependence of the transport coefficients (at least in the range of ω_p/ω_B of interest here). Such corrections have been neglected here. For a discussion of the collision operator in a strong magnetic field, see, for example, Ref. 7, Chap. 5.

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