Method for calculating a Lyapunov exponent

Jon Wright
Center for Studies of Nonlinear Dynamics, La Jolla Institute
(affiliated with the University of California, San Diego),
Suite 2150, 8950 Villa La Jolla Drive,
La Jolla, California 92037
(Received 17 January 1984)

A new method is presented for calculating the leading Lyapunov exponent directly from experimental data for systems having a strange attractor with dimensionality near 2. The method is exact for one-dimensional maps and gives good results for systems that have approximate one-dimensional maps associated with them even in the presence of some noise. Numerical examples are given.

I. INTRODUCTION

The experimental determination of Lyapunov characteristic exponents is a problem of current concern. In this paper we will describe a method when there is only one positive exponent and the dimension of the strange attractor is near 2. In a recent report[1] Wolf and Swift reviewed some other possible methods.

A few years ago Chang and the present author[2] derived a formula for the Lyaponuv exponent of a one-dimensional map. The advantage of that formula is that it involves certain invariant densities associated with the underlying map which are the experimental observables. In this paper, I will show how to apply the formula to experimental data and show how to accelerate convergence if only limited data points are available. I will also demonstrate how to treat maps that are multivalued, but still almost one dimensional. Finally, we note that the method is robust in the presence of noise. In Sec. II we discuss the method. In Sec. III we treat in detail the one-dimensional logistic map. There we will present the numerical details of our method and also will show results in the presence of noise. In Sec. IV we will give results for the Lorenz equations, and in Sec. V we will show how to treat a one-dimensional map associated with a multivalued function. This latter situation is typical of maps derived from physical situations or from differential equations.

II. ENTROPY OF MIXING

The Lyapunov exponent measures a rate of expansion and if the volume is bounded the expansion implies some folding. For a one-dimensional map this folding becomes a two or more to one mapping. So instead of looking directly at the expansion we look at the folding. For a complete discussion and derivation we refer to the paper by Chang and Wright;[2] here we will only present the relevant formula.

Consider a one-dimensional map:

$$x_{n+1}=f(x_n) \ . \tag{1}$$

Further suppose for simplicity that it has only one extremum inside the mapping region. Associated with the map will be an invariant measure $\phi(x)$. Now if x_a and x_b both map to x, we can define the invariant measure ϕ as made up of two pieces corresponding to points that map from region dx_a around x_a and from dx_b around x_b.

$$\phi(x)=\phi_a(x)+\phi_b(x) \ , \tag{2}$$

where

$$\begin{aligned}&\phi(x_a)dx_a\equiv\phi_a(x)dx \ ,\\ &\phi(x_b)dx_b\equiv\phi_b(x)dx \ .\end{aligned} \tag{3}$$

This says that the points in dx came from either dx_a or dx_b, and no points are lost in mapping. This looks like the mixing of two independent densities, and in fact the Lyapunov exponent is just the entropy of mixing

$$\mu=-\int dx\,\phi(x)\frac{\frac{\phi_a(x)}{\phi(x)}\ln\frac{\phi_a(x)}{\phi(x)}+\frac{\phi_b(x)}{\phi(x)}\ln\frac{\phi_b(x)}{\phi(x)}}{\int\phi(x)dx} \ . \tag{4}$$

Note that there is only a contribution when there is folding. For regions of x where the mapping is one to one then $\phi_a(x)=\phi(x)$ and $\phi_b(x)=0$. This is a very nice result as the densities are directly observable experimentally. Also it is easy to show[2] that $\mu\leq\ln 2$ for a one-humped map.

The simplest way to evaluate this is to collect a number of points $x_1,x_2,\ldots,x_N$ and divided the x axis into a number of bins N_{bins}. For a case where there is no more than a 2 to 1 mapping, we examine the points in each bin and ask what fraction came from each of the two regions a and b. Then if $\eta_i(a)$ is the fraction of points coming from region a in the ith bin, and $\eta_i(b)$ from b and N_i is the number of points in the ith bin,

$$\mu=-\sum_{i=1}^{N_{\text{bins}}}\frac{N_i}{N}[\eta_i(a)\ln\eta_i(a)+\eta_i(b)\ln\eta_i(b)] \ . \tag{5}$$

The generalization to 3 to 1 (or more) foldings where x_a, x_b, and x_c all map to x is obvious.

This gives a reasonable estimate for μ but it is possible to improve upon it. We can vary either the number of bins or the number of total points. We can get a substantial improvement by varying the number of total points and a somewhat lesser improvement by varying the number of bins. We are interested in extrapolating from measurements at a few different values of N (for fixed number of bins) to $N=\infty$. We have to make an assumption about the

TABLE I. Lyapunov exponent for Eq. (6).

N_{bins}	N 200	400	600	800	1000	Infinity
20	0.307	0.326	0.350	0.355	0.353	0.367
21	0.292	0.340	0.356	0.361	0.360	0.382
22	0.310	0.339	0.350	0.355	0.354	0.368
23	0.317	0.332	0.339	0.349	0.349	0.356
24	0.313	0.336	0.343	0.349	0.347	0.358
25	0.296	0.323	0.335	0.336	0.341	0.352
26	0.282	0.317	0.331	0.338	0.342	0.356
27	0.277	0.319	0.337	0.343	0.344	0.363
28	0.271	0.316	0.334	0.342	0.347	0.365
29	0.293	0.330	0.340	0.348	0.351	0.365
30	0.289	0.318	0.336	0.337	0.336	0.352

asymptotic behavior as $N \to \infty$. We assume that

$$\mu(N) \to \mu(\infty) + \frac{A}{N} + \frac{B}{N^2} + \cdots \quad . \tag{6}$$

The data to be presented in Sec. III support this form. We note here that if this extrapolation is not done, the estimate is somewhat uncertain. We did a least-squares fit to data from five different values of N keeping only the first three terms in Eq. (6). Keeping only two terms does almost as well, and keeping more terms is not useful with limited data. We also point out that the measurements are of a statistical nature and there will be fluctuations.

The next improvement is to calculate $\mu(N=\infty, N_{bins})$ for several different values of N_{bins}. At this stage we average μ for several different N_{bins}.

III. NUMERICAL EXAMPLES

In this section we will consider in detail the map

$$x_{n+1} = a x_n (1 - x_n) \quad . \tag{7}$$

Since we know the form of the map, it is easy to compute μ by the conventional formula,

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \ln |f'(x_i)| \quad . \tag{8}$$

For $a = 3.7$ we used 10^6 points and obtained

$$\mu_{3.7} = 0.359 \quad . \tag{9}$$

To obtain the data for Table I, we iterated the map a number of times to get rid of transients and then collected a sequence of 1000 points. $\mu(N)$ was then calculated for $N = 200$, 400, 600, 800, and 1000 for several different numbers of bins. For each fixed N_{bins} a least-squares fit was done to the form given in Eq. (6). The results are shown in Table I. It should be noted that a calculation of μ from Eq. (8) using only 1000 points would have fluctuations with a standard deviation of 0.01. The average of the last column of Table I is 0.362 with a sample standard deviation of 0.009 which is comparable to that from Eq. (8). When the function $f(x)$ is unknown, our method does not require the estimation of $f'(x)$ which would introduce additional errors. Note that the extrapolation using Eq. (6) is a definite improvement over the result that would be obtained from column 5. Averaging μ for different N_{bins} also improved our estimate

We now turn to the question of noisy data. Because our calculation of the Lyapunov exponent has the form of an entropy we expect it to be relatively robust, with the errors related to the entropy change due to noise. We now consider the case where the dynamical system is an exact one-dimensional map, but the measurements are subject to error. As an example we work with the quantity with x_n generated from Eq. (7) and ϵ_n an error uniformly distributed

TABLE II. Lyapunov exponent for Eq. (11) and Fig. 1 using 4000 points.

N_{bins}	N 800	1600	2400	3200	4000	Infinity
10	0.676	0.678	0.677	0.678	0.679	0.679
20	0.608	0.641	0.644	0.648	0.652	0.663
30	0.607	0.639	0.644	0.651	0.651	0.664
40	0.565	0.620	0.632	0.638	0.642	0.664
50	0.559	0.622	0.635	0.642	0.644	0.670
60	0.545	0.608	0.626	0.635	0.638	0.665
70	0.529	0.606	0.623	0.636	0.638	0.670
80	0.485	0.598	0.617	0.626	0.629	0.675
90	0.468	0.580	0.613	0.628	0.631	0.679
100	0.460	0.574	0.607	0.622	0.628	0.675

TABLE III. Lyapunov exponent for Eq. (11) and Fig. 1 using 1000 points.

N_{bins}	N 200	400	600	800	1000	Infinity
10	0.539	0.622	0.653	0.659	0.667	0.694
20	0.402	0.520	0.560	0.581	0.597	0.649
30	0.304	0.420	0.518	0.544	0.578	0.715
40	0.259	0.346	0.429	0.499	0.541	0.732
50	0.217	0.331	0.435	0.477	0.510	0.674
60	0.198	0.293	0.379	0.418	0.469	0.623
70	0.145	0.269	0.359	0.431	0.473	0.663
80	0.175	0.257	0.325	0.391	0.433	0.601
90	0.131	0.215	0.283	0.345	0.391	0.556
100	0.151	0.207	0.292	0.358	0.395	0.588

on $(-1, 1)$ and Δ is the level of error. We report on three different values of Δ.

$$\Delta = 0.1, \quad \mu = 0.351 \ , \qquad \Delta = 0.04, \quad \mu = 0.41 \ , \qquad \Delta = 0.02, \quad \mu = 0.4 \ . \tag{10}$$

Note that there is not a dramatic effect on μ.

IV. LORENZ MODEL

We now consider a dynamical system with a strange attractor with one positive Lyapunov exponent. The equations, which are simple rescalings of those due to Lorenz,[3] are

$$\dot{x} = y - \sigma\epsilon x \ , \qquad \dot{y} = -xz + x - \epsilon y \ , \qquad \dot{z} = xy - \epsilon b z \ . \tag{11}$$

The parameters are taken to be

$$\epsilon = 0.02216 \ , \qquad b = 8/3 \ , \qquad \sigma = 10 \ . \tag{12}$$

These parameters are in a regime very near the period doubling regime. There are actually two strange attractors as the symmetry of the equations is broken by the solution and hence there are two attractors. We plot successive maxima of x in Fig. 1. The curve is of course an infinity of curves, but we are supposing our resolution is not good enough to resolve them. We have calculated the Lyapunov exponent using the prescription of Ref. 4, and obtained $\mu = 0.0274$. The average period is $T_{av} = 23.15$ and the exponent of the map is related to that of the curve by

$$\mu_{map} = T_{av}\mu_{diff.\ eq.} \ , \qquad \mu_{map} \approx 0.634 \ . \tag{13}$$

Note that this number is near ln2; that is, it is almost as large as it can be. We now calculate this number from Eqs. (5) and (6). Using 4000 points we obtain the numbers shown in Table II. Averaging the numbers in the last column, we obtain

$$\mu = 0.67 \ , \tag{14}$$

which is slightly larger than the correct value. After we discuss how to treat multivalued cases, it will be apparent that because the bound on μ is nearly attained, we should expect the exact answer to be smaller than the estimate given by Eq. (14).

As experiments may have many fewer points, we now consider 1000 points rather than 4000. There are barely enough points to divide amongst 100 bins and we expect the answers using larger bin numbers to be of questionable vali-

TABLE IV. Lyapunov exponent for Eq. (11) and Fig. 2.

N_{bins}	N 200	400	600	800	1000	Infinity
10	0.539	0.622	0.653	0.659	0.667	0.702
12	0.546	0.610	0.640	0.645	0.653	0.680
14	0.516	0.593	0.626	0.632	0.640	0.673
16	0.482	0.596	0.628	0.634	0.641	0.689
18	0.454	0.576	0.609	0.611	0.618	0.670
20	0.402	0.520	0.560	0.581	0.597	0.642
22	0.357	0.508	0.556	0.574	0.591	0.651
24	0.369	0.497	0.552	0.572	0.589	0.641
26	0.363	0.486	0.541	0.578	0.591	0.641
28	0.366	0.487	0.530	0.560	0.566	0.617
30	0.304	0.420	0.518	0.544	0.578	0.626

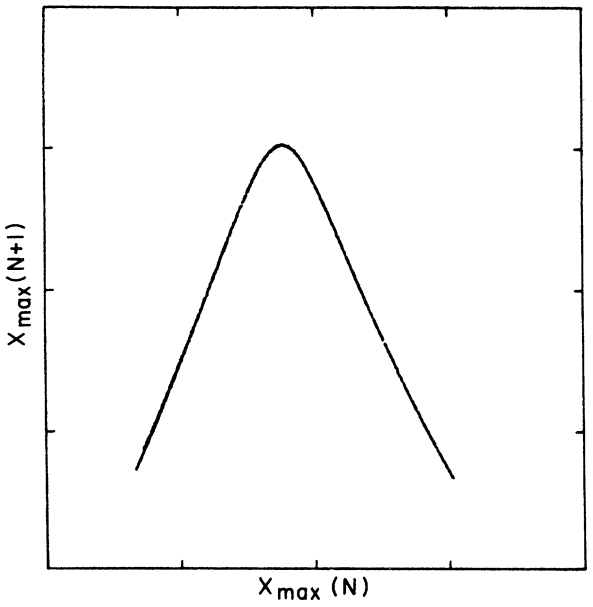

FIG. 1. Plot of successive maxima of x from Eq. (11) $X_{\max}(N+1)$ vs $X_{\max}(N)$.

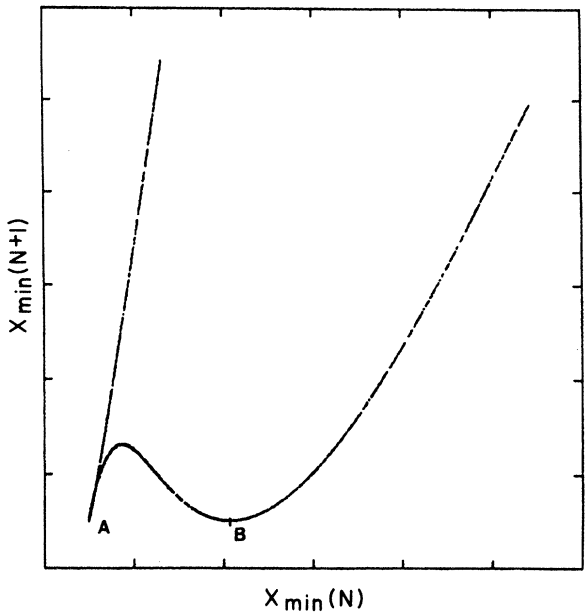

FIG. 2. Plot of successive minima of X.

dity. That feature can be seen in Table III. To get reliable estimates it is necessary to restrict the number of bins. Table IV shows the result for between 10 and 30 bins. The average is 0.67 ± 0.006.

V. MULTIVALUEDNESS

It is quite common to plot successive maxima (or some other quantity) and obtain a multivalued function. In fact if we plot minima of the example treated in Sec. IV we obtain the curve shown in Fig. 2. There are two kinds of multivaluedness present—the obvious one which we call macroscopic, and a microscopic multivaluedness that we cannot resolve. Of course we must get the same answer for the Lyapunov exponent as that given in Eq. (14).

We first explain the way to handle such multivalued maps. For the region of overlap define two separate x axes. Call one a and the other b. If the x value has label a use the lower curve and if label b use the upper. To determine when to use a and when to use b, follow an iteration sequence. For this particular case it will turn out that the first time a point enerts the region between the points labeled A and B the next point is determined from curve a. Thereafter there is an alteration between a and b until the iterate leaves the region. Then the process is repeated. Other situations can have slightly different rules, although just as simple. It is necessary to know the rule in order to obtain correct results.

Now the generalization of Eq. (5) is obvious. Each bin has an x label and is either a or b as appropriate. In this case there are still no more than two points mapping to the same point. Applying Eq. (5) suitably modified and using 1000 points, we obtain

$$\mu = 0.62 \ .$$

We have also tried adding measurement noise and the results were even less sensitive than those reported in Sec. III.

If we could resolve the curve in Figs. 1 or 2 to the next scale we would see instead two curves. Our prescription could then be applied to it. We expect the calculated μ to now be slightly smaller. The reason is that there will still be only 2 to 1 mixing and for μ to be near its maximum value the two densities being mixed must be equal. Any change will only decrease μ. This may not be the case for μ much less than $\ln 2$.

IV. CONCLUSION

We have demonstrated a method for calculating Lyapunov exponents for maps that are approximately one-dimensional. The method is not exact because all scales cannot be resolved; however, it gives answers that are very nearly correct even in the presence of noise. It has the additional benefit of being very easy to implement.

[1]A. Wolf and J. Swift, in *Statistical Physics and Chaos in Fusion Plasma,* edited by W. Horton and L. Reichl (Wiley Interscience, New York, 1984).

[2]S. J. Chang and J. Wright, Phys. Rev. A 23, 1419 (1981).

[3]E. N. Lorenz, J. Atom. Sci. 20, 130 (1963).

[4]I. Shimada and T. Nagashima, Prog. Theor. Phys. 61, 1605 (1979).