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Evolution equations for Taylor vortices in the small-gap limit

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We consider the centrifugal instability of the viscous fluid flow between concentric circular cylinders in the small-gap limit. The amplitude of the Taylor vortex is allowed to depend on a slow time variable, a slow axial variable, and the polar angle θ. It is shown that the amplitude of the vortex cannot, in general, be described by a single amplitude equation. In the absence of any slow axial variations it is shown that a Taylor vortex remains stable to wavy vortex perturbations.

In two recent papers Tabeling[1] and Brand and Cross[2] have independently proposed an amplitude equation which governs the slow azimuthal and axial evolution of a Taylor vortex in the small-gap limit. In this Brief Report, we show that this amplitude equation corresponds to a velocity field which necessarily violates the no-slip condition at one of the cylinders. The remedy for this difficulty is well known in hydrodynamic stability theory following the work of Davey, Hocking, and Stewartson,[3] and requires the insertion of an eigenfunction in the expansion of the disturbance pressure field. The presence of this eigenfunction means that the evolution of a Taylor vortex cannot be described by a single amplitude equation.

We shall see that if axial variations are ignored then it is possible to describe the azimuthal evolution of a Taylor vortex by a single amplitude equation. However, even this reduced equation differs from the reduced form of the equation of Tabeling, Brand, and Cross. The appropriate amplitude equation is discussed in some detail and it is shown that in the small-gap limit a Taylor vortex is stable to wavy vortex perturbations. Thus the evolution equation approach to describe the azimuthal evaluation of a Taylor vortex gives results which are not consistent with the classical results of Davey, DiPrima, and Stuart[4] and the available experimental results. The implications of this situation will be discussed later.

We consider then the stability of the flow of a viscous fluid of kinematic viscosity ν between cylinders of radii R_1, R_1+d. The outer cylinder is held fixed while the inner one rotates with angular velocity Ω_1. We define the Reynolds number R and the parameter δ by

$$R=\frac{U_0 d}{\nu},\quad \delta=\frac{d}{R_1} \quad . \tag{1}$$

In the limit $\delta \to 0$ it is known that instability occurs when the Taylor number

$$T=R^2\delta$$

is $O(\delta^0)$. Following Krueger, Gross, and DiPrima[5] and Davey *et al.*[4] it has been customary in the small-gap limit to consider disturbances with azimuthal wave numbers $O(\delta^{-1/2})$ even though all the available experimental results suggest that only azimuthal wave numbers of order δ^0 are important in the transition from Taylor vortex flow to wavy vortex flow. Hence we shall take $\partial/\partial\theta \sim O(\delta^0)$, but the scalings of Davey *et al.*[4] which were used by Tabeling, Brand, and Cross can be recovered at a later stage by considering a further limiting process.

At this stage we restrict our attention to Taylor vortices of fixed axial wavelength with amplitude dependent on the polar angle θ and time. We consider the limit $\delta \to 0$ with

$$2R^2=\frac{1}{\delta}(T_0+\delta T_1+\delta^2 T_2+\dots)=\frac{T}{\delta} \quad ,$$

where $T_0=3390$ is the critical Taylor number in the small-gap limit. The velocity components in the radial, azimuthal, and axial directions are scaled on ν/d, $\Omega_1 R_1$, and ν/d, respectively, while the pressure is scaled on $\rho\nu^2/d^2$.

In the small-gap limit the basic flow $(0, \Omega_1 R_1 \bar{v}, 0)$ has the asymptotic form

$$\bar{v}=1-x+O(\delta) \quad ,$$

where x is a radial variable scaled on d. The equations governing the stability of this basic flow can be written in the form

$$Lu=-\frac{\partial p}{\partial x}+Q_1+T\bar{v}v+O(\delta) \quad , \tag{2a}$$

$$Lv=-\left(\frac{2}{T}\right)^{1/2}\delta^{3/2}\frac{\partial p}{\partial\theta}+Q_2+u\frac{d\bar{v}}{dx}+O(\delta) \quad , \tag{2b}$$

$$Lw=-\frac{\partial p}{\partial z}+Q_3+O(\delta) \quad , \tag{2c}$$

$$\frac{\partial u}{\partial x}+\left(\frac{T}{2}\right)^{1/2}\delta^{1/2}\frac{\partial v}{\partial\theta}+\frac{\partial w}{\partial z}=O(\delta) \quad , \tag{2d}$$

where z and t have been scaled on d and ν/d^2, respectively. The nonlinear functions Q_1, Q_2, and Q_3 are $O(\delta^0)$ while the operator L is defined by

$$L\equiv\frac{\partial}{\partial t}+\delta^{1/2}\left(\frac{T}{2}\right)^{1/2}\bar{v}\frac{\partial}{\partial\theta}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial z^2} \quad . \tag{3}$$

Since the Taylor number differs from its critical value by $O(\delta)$ we expect a finite amplitude motion of $O(\delta^{1/2})$ and therefore expand $\vec{u}=(U,V,W)$ in the form

$$\vec{u}=\delta^{1/2}\vec{u}_0+\delta\vec{u}_1+\delta^{3/2}\vec{u}_2+\cdots \quad , \tag{4}$$

together with a similar expansion for the pressure. We then define the slow time scales $\tau=\delta^{1/2}t$, $\bar{t}=\delta t$. The details of such an expansion procedure follow closely those of Tabeling, Brand, and Cross and at order $\delta^{1/2}$ it is found that

$$\vec{u}_0=\frac{A(\theta,\tau,\bar{t})e^{iaz}}{2}[U_0(x),V_0(x),W_0(x)]+\text{c.c.} ,$$

where a is the critical axial wave number while (U_0,V_0,W_0) is the velocity eigenfunction corresponding to the critical point on the neutral curve. At order δ it is found that A must satisfy the equation

$$\left(\frac{T_0}{2}\right)^{1/2} s_0\frac{\partial A}{\partial\theta}+\frac{\partial A}{\partial\tau}=0 ,$$

and Tabeling has calculated s_0 numerically and found that $s_0=0.5261$. This equation had been derived previously by Weinstein.[6] It follows from the above equation that $A=A(\Phi,\bar{t})$, where

$$\Phi=\theta-\left(\frac{T_0}{2}\right)^{1/2} s_0\tau .$$

At order δ the first harmonic and mean flow correction are determined, and it is at this stage that the difficulty overlooked by Tabeling, Brand, and Cross arises. The mean flow correction at this order is in the θ direction and we denote it by v_M. However, we see from the equation of continuity that this mean flow drives a radial mean velocity field of $O(\delta^{3/2})$, which we denote by u_M. The equation which determines u_M is

$$\frac{\partial u_M}{\partial x}=-\left(\frac{T_0}{2}\right)^{1/2}\frac{\partial v_M}{\partial\theta} ,$$

and this equation must be integrated to satisfy $u_M=0$ at $x=0,1$. This cannot be achieved unless v_M contains some arbitrary function of θ and $\bar{t}$. It is for this reason that the solution given by Tabeling, Brand, and Cross does not satisfy the no-slip condition everywhere. The remedy is to allow for a pressure eigenfunction in the manner discussed by Davey *et al.*[3] and DiPrima and Stuart.[7] Thus the perturbation pressure must be expanded in the form

$$p=\delta^{1/2}p_0+\delta p_1+\delta^{3/2}p_1+\cdots+\delta^{-1/2}\bar{p}_0(\theta,\bar{t},\tau)+\cdots , \tag{5a}$$

where the relatively large size of the induced mean pressure field is, of course, a lubrication effect. The equation for v_M now becomes

$$\frac{\partial^2 v_M}{\partial x^2}=-\left(\frac{2}{T_0}\right)^{1/2}\frac{\partial\bar{p}_0}{\partial\theta}-\frac{1}{2}|A|^2\frac{d}{dx}(U_0V_0) , \tag{5b}$$

which can be integrated subject to $v_M=0$ at $x=0,1$. We can then substitute for v_M into the equation of continuity to find u_M. The condition that u_M should vanish at both $x=0$ and $x=1$ gives

$$\left(\frac{2}{T_0}\right)^{1/2}\frac{\partial^2\bar{p}_0}{\partial\theta^2}=6\frac{\partial}{\partial\theta}|A|^2Q_0 , \tag{5c}$$

where

$$Q_0=\int_0^1 F_0(x)dx$$

with

$$F_0(x)=\int_0^x U_0V_0\,dx-x\int_0^1 U_0V_0\,dx .$$

The equation for $\bar{p}_0$ is now integrated once and the arbitrary constant which appears in the resulting expression for $\partial\bar{p}_0/\partial\theta$ is fixed by insisting that $\bar{p}_0$ be periodic in θ. The function v_M is then completely determined and we find

$$v_M=-\tfrac{1}{2}|A|^2F_0(x)+(\tfrac{1}{2}\sigma_0-3Q_0|A|^2)(x^2-x) , \tag{6}$$

where

$$\sigma_0=3\frac{Q_0}{\pi}\int_0^{2\pi}|A|^2d\theta .$$

The amplitude equation found by Tabeling, Brand, and Cross corresponds to $v_M=-\frac{1}{2}|A|^2F_0(x)$ so that the radial mean flow induced at higher order in their expansions cannot satisfy the no-slip condition at both cylinders. At order $\delta^{3/2}$ we find that A satisfies the equation

$$\frac{\partial A}{\partial\bar{t}}=T_1'A+c_3\frac{\partial^2 A}{\partial\Phi^2}-c_4A|A|^2+c_5(\tfrac{1}{2}\sigma_0-3Q_0|A|^2)A , \tag{7}$$

where $T_1'=(c_0/2T_0)(T_1-\bar{T}_1)$, with $\bar{T}_1$ the order δ correction to the axisymmetric critical Taylor number. The constants c_0, c_3, and c_4 are given by Tabeling while c_5 is given by DiPrima and Stuart.[7] The amplitude equation given by Tabeling, Brandt, and Cross corresponds to setting $c_5=0$ in (7). The linearized form of (7) shows that the nonaxisymmetric mode with wave number M is linearly unstable for

$$T_1'>T_{1c}'=c_3M^2 \tag{8}$$

and the finite amplitude mode which bifurcates from T_{1c}' is

$$A=A_e=\left(\frac{T_1'-c_3M^2}{c_4}\right)^{1/2}e^{iM\Phi} , \tag{9}$$

and of course only integer values of M have any physical relevance. The first mode to bifurcate is the Taylor vortex solution which has $M=0$. In order to investigate the stability of (9) we write $A=A_e+b$ and linearize to give

$$\frac{\partial b}{\partial\bar{t}}=c_3\frac{\partial^2 b}{\partial\Phi^2}-(T_1'-c_3M^2)(b+\bar{b}e^{2iM\Phi})(1+\epsilon)+c_3M^2b+\frac{e^{iM\Phi}\epsilon}{2\pi}(T_1'-c_3M^2)\int_0^{2\pi}(\bar{b}e^{iM\phi}+b\bar{e}^{iM\phi})d\phi , \tag{10}$$

where

$$\epsilon=\frac{3Q_0c_5}{c_4} .$$

If we set $M=0$ in (10) we can study the stability of a Taylor vortex to Φ-dependent perturbations. We can then see from (10) that the growth rate of a disturbance proportional to $i\cos M\Phi$ or $i\sin M\Phi$ is $-c_3M^2$ so that, in the small-gap limit, there is no bifurcation from a Taylor vortex to a wavy vortex solution. The nonaxisymmetric modes with $M\neq 0$ are susceptible to the Eckhaus-Benjamin-Feir sideband instability mechanism. Following Stuart and DiPrima[8] it can be shown from (10) that the nonaxisymmetric mode is unstable to sidebands with integer wave numbers $\gamma M,(2-\gamma)M$ for $-1<\gamma<3$, $\gamma\neq 1$. The nonaxisymmetric mode with wave number M which bifurcates from T_{1c}' is found to be unstable to such a sideband for

$$T_{1c}'<T_1'<\left(1+\frac{(\gamma+1)(3-\gamma)}{2(1+\epsilon)}\right)T_{1c}' ,$$

which reduces to Eckhaus's result $T_1' < 3T_{1c}'$ in the limit $\gamma \rightarrow 1$ with $\epsilon = 0$. We see then that the Eckhaus criterion is altered if $\epsilon \neq 0$ so that the pressure eigenfunction decreases the unstable regime. In the present problem only integer values of γM are physically acceptable so that the nonaxisymmetric mode is unstable for

$$T_{1c}' < T_1' < \left[1 + \frac{(4 - 1/M^2)}{2(1+\epsilon)} \right] T_{1c}' \ .$$

We shall now derive a generalized form of (7) which takes account of slow variations of the vortex amplitude in both the axial and azimuthal directions. Such an equation has been given by Tabeling, Brand, and Cross but the velocity field associated with that equation does not satisfy the no-slip condition at one of the cylinders. We again assume that $\partial/\partial\theta \sim 0(1)$ and now define $\zeta = \delta^{1/2} z$. We retain the expansion (4) but allow for a slow dependence of the amplitude function on ζ. In the absence of any pressure eigenfunction the radial mean flow at order $\delta^{3/2}$ cannot satisfy the no-slip condition at both cylinders. We therefore retain (5a), the expansion of the pressure field, but allow for a slow dependence of $\bar{p}_0$, $\bar{p}_1$, etc. on ζ. However, it follows from (2c) that $\bar{p}_0$ will drive an axial mean flow of order δ^0 if $\partial\bar{p}_0/\partial\zeta \neq 0$. Thus we set $\partial\bar{p}_0/\partial\zeta = 0$ and $\partial\bar{p}_1/\partial\zeta = 0$ in order that the azimuthal and axial mean flows induced by the disturbance should be comparable.

The radial mean flow at order $\delta^{3/2}$ is now driven by both the axial and azimuthal mean flows and satisfies the no-slip conditon at the outer cylinder if

$$\frac{\partial^2 \bar{p}_0}{\partial\theta^2} + \frac{\partial^2 \bar{p}_2}{\partial\zeta^2} = 6 \frac{\partial}{\partial\theta} |A|^2 Q_0 \left(\frac{T_0}{2} \right)^{1/2} \ . \tag{11}$$

We see that if the disturbance has no θ dependence the pressure eigenfunctions $\bar{p}_0$ and $\bar{p}_2$ both vanish, while if there is no ζ dependence we recover (5c). The amplitude function A is now found to satisfy

$$\frac{\partial A}{\partial \bar{t}} = T_1' A + c_3 \frac{\partial^2 A}{\partial\theta^2} + c_6 \frac{\partial^2 A}{\partial\zeta^2} + i c_7 \frac{\partial^2 A}{\partial\zeta\partial\theta} - c_4 A |A|^2 - c_5 \frac{\partial \bar{p}_0}{\partial\theta} A - i c_8 \frac{\partial \bar{p}_2}{\partial\zeta} A \ , \tag{12}$$

where c_3, c_4, and c_5 are as defined previously while Tabeling has calculated c_6, c_7. The constant c_8 is purely real and determined in terms of the first-order eigenfunction and its adjoint. Equations (11) and (12) are coupled and cannot, in general, be reduced to a single evolution equation. However, if the disturbance is, for example, assumed periodic in ζ it is a simple matter (see Hall[9]) to express $\bar{p}_0$ and $\bar{p}_2$ in terms of A.

CONCLUSION

The evolution equations which describe Taylor vortices in the small-gap limit have been derived in a self-consistent manner. An alternative single evolution equation had been previously proposed as a possible means for studying the possible flow configurations in the supercritical regime in much the same way that has been so successful in Benard convection. We have shown that a second equation is needed and that the evolution equation approach does not predict the transition from Taylor vortex flow to a wavy vortex flow which is well known to occur at quite small values of δ.

The reason for this surprising result was recognized some time ago by Davey *et al.*[3] and can be traced to the fact that any nonaxisymmetric mode with azimuthal wave number M has an eigenfunction coincident at first order with the Taylor vortex eigenfunction in the limit $\delta \rightarrow 0$. Thus in order to produce a secondary bifurcation to a wavy vortex flow it is necessary to distort the axisymmetric and nonaxisymmetric eigenfunctions sufficiently by splitting apart the corresponding eigenvalues. This is essentially the aim of the expansion procedure of Davey *et al.*[3] who take $M \sim O(\delta^{-1/2})$ when $\delta \rightarrow 0$. The results obtained by Davey *et al.*[3] were applied at finite values of M at quite small values of δ and agree well with experimental observations and with the subsequent higher-order theory of Eagles.[10] It seems, therefore, that their expansion procedure gives a good approximation to the solution of the appropriate partial differential system even for finite values of M with δ not too small.

It is possible that at sufficiently small values of δ no transition to a wavy vortex flow would be observed experimentally, at least with small wave numbers, and the evolution equation approach would be useful. It does, however, seem that in order to understand the experimental results which are presently available, the failure of the evolution equation approach to predict the crucial bifurcation from Taylor vortex to wavy vortex flow means that this approach should not be used.

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