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Adiabatic elimination in stochastic systems. III. Application to renormalization-group transformations of the time-dependent Ginsburg-Landau model

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We present a stochastic theory of the renormalization-group transformations of the time-dependent Ginzburg-Landau model, describing a system which exhibits the characteristics of a second-order phase transition. We eliminate a certain range of the Fourier components of the magnetic spin variables via a projection-operator method. This effectively changes the scale of the system and transforms the coupling constants. The procedure is shown to be equivalent to the integration over short-wavelength modes as in the renormalization-group transformation performed by Wilson and Kogut. This equivalence shows that the projection-operator method is a valid procedure for scaling critical systems, and in particular it indicates that the treatment of fluctuations is systematic. We suggest that such a method should also provide a straightforward approach to the dynamic renormalization group.

I. INTRODUCTION

In this third paper on adiabatic elimination we wish to extend the range of problems considered to those associated with the renormalization-group theory of critical phenomena. In our second paper[1] (to be referred to as II) we treated reaction-diffusion and hydrodynamic-like systems, and showed that adiabatic elimination in the latter was equivalent to Graham's[2] method of multiple scales. However, in this hydrodynamic-like system, it was essential to introduce a factor which made the added noise small in the adiabatic limit.

Here we will attempt to remove that restriction, by methods which are closely related to the blocking process in the theory of the renormalization group.[3] We will consider the simplest model available, the time-dependent Ginzburg-Landau model for a ferromagnetic system.[3]

In Sec. II we construct a Fokker-Planck equation for the system, in the framework of the cell model, in which the system is divided into $(2n+1)^d$ cells, each of volume l^d (Ref. 4). In Sec. III we derive the renormalization-group (RG) transformations of Wilson and Kogut[3] by eliminating the short-wavelength components of the magnetic spin. Such an elimination maps the original system, defined in terms of a lattice of $(2n+1)^d$ cells with spacing l, to a coarser-grained system defined on a lattice of $(2N+1)^d$ cells, with lattice spacing bl, where $b=(2n+1)/(2N+1)$, and $n>N$. This is essentially the same procedure as in II. However, in this case we show that it is possible to derive a limit in which the blocking ratio b is large *and* the coefficient u of the nonlinearity is small. This latter condition is normally required in renormalization-group theory.

In Sec. IV we summarize and compare our work with conventional renormalization-group theory. Appendix A contains a description of the Ornstein-Uhlenbeck processes in terms of creation and destruction operators, which form a useful calculational tool, and Appendix B contains the details of the calculations needed to derive the blocking transformations. We also note that in all the following discussions, the paper of Ref. 5 will be referred to as I.

II. MODEL

We develop a stochastic theory of a system which exhibits the properties of a second-order phase transition, via an essentially phenomenological approach. One of the most commonly studied examples of a system which displays such critical behavior is the ferromagnet; for which the free energy is given by the Ginzburg-Landau functional,[6]

$$F\{m(\vec{x});T\}=F(0,T)+\int d^dx\{\tfrac{1}{2}r[m(\vec{x})]^2+\tfrac{1}{4}u[m(\vec{x})]^4+\tfrac{1}{2}[\vec{\nabla}m(\vec{x})]^2\}/\Gamma_0\,, \quad (2.1)$$

where $m(\vec{x})$ is the order parameter of the ferromagnetic system, i.e., the magnetic spin component, and T is the temperature. r,u represent coupling constants which are proportional to T; in particular, $r\propto T-T_c$, where T_c is the critical temperature. d is the dimension.

Following the phenomenological method of Haken,[6] we obtain an equation of motion for the variable $m(x)$ through

$$\dot{m}(\vec{x})=-\frac{\partial F}{\partial m(\vec{x})}\,. \quad (2.2)$$

To take account of statistical fluctuations in the system, we add a stochastic fluctuating term in an *ad hoc* manner and thus obtain the time-dependent Ginzburg-Landau (TDGL) equation,

$$\dot{m}(\vec{x})=-r[m(\vec{x})]-u[m(\vec{x})]^3+\nabla^2[m(\vec{x})]+\sqrt{\Gamma_0}\xi(\vec{x},t)\ , \quad (2.3)$$

where $\xi(\vec{x},t)$ is a Gaussian stochastic term, having the properties

$$\langle \xi(\vec{x},t)\rangle=0\ ,$$
$$\langle \xi(\vec{x},t)\xi(\vec{x}^{\,1},t^1)\rangle=\delta(\vec{x}-\vec{x}^{\,1})\delta(t-t^1)\ .$$

Γ_0 represents the size of the thermal fluctuations in the system, which for the moment is just an arbitrary constant.

We note that Eq. (2.3) is comprised of two distinct parts: deterministic terms and a fluctuating term. We assume that this combination accurately describes the statistical behavior of the system. This phenomenological description is equivalent to the method of Wilson and Kogut[3] which includes statistical fluctuations into the system via a functional-integral approach.

In order to perform the RG transformation, we wish to eliminate short-wavelength spin components, using an adiabatic elimination method. Our model of the system, Eq. (2.3), is of the form of a stochastic differential equation and such an elimination procedure is not well defined for these equations.

However, the projection-operator method formulated in I and II is defined for Fokker-Planck equations; thus, we define the equivalent Fokker-Planck equation to Eq. (2.3) in the framework of the cell model, in which we divide the system into $(2n+1)^d$ cells, each of volume l^d, and find

$$\frac{\partial p}{\partial t}=\left[-\sum_{\vec{j},\vec{k}}\frac{\partial}{\partial m_{\vec{j}}}D_{\vec{j}\,\vec{k}}m_{\vec{k}}+\sum_{\vec{j}}\frac{\partial}{\partial m_{\vec{j}}}(rm_{\vec{j}}+um_{\vec{j}}^3)+\frac{\Gamma_0}{2l^d}\sum_{\vec{j}}\frac{\partial^2}{\partial m_{\vec{j}}^2}\right]p\ , \quad (2.4)$$

where p is the probability distribution of the system, i.e., $p=p(m_{\vec{i}},\ldots,m_{\vec{j}},\ldots,t)$; m_i represents the spin of cell i ($\vec{j}$ is d dimensional).

In order to obtain a similar equation after projection, it is essential to consider the nonlinear term $m_{\vec{j}}^3$ to be a shorthand notation for the *band-limited* function $[m_{\vec{j}}]^3$ as defined in II, Sec. III.

The first term on the right-hand side of Eq. (2.4) describes the effects of spatial variation, i.e., which is equivalent to the term $\nabla^2 m$ in Eq. (2.3). In such a discretized model, $D_{\vec{j}\,\vec{k}}$ takes the form

$$D_{\vec{j}\,\vec{k}}=\sum_{r=1}^{d}\frac{1}{l^2}(\delta_{k_r,j_r+1}+\delta_{k_r,j_r-1}-2\delta_{k_r,j_r})\ . \quad (2.5)$$

A. Transformation to Fourier space

Our aim is to eliminate the short-wavelength spin components. Equivalently, we may eliminate high-q modes in reciprocal Fourier (q) space; in practice we find the latter to be more convenient and thus will formulate the procedure in q space.

To this end we define the eigenfunctions and eigenvalues of $D_{\vec{j}\,\vec{k}}$ through the equation

$$\sum_{\vec{k}}D_{\vec{j}\,\vec{k}}f_{\vec{k}}(\vec{q})=-\lambda(\vec{q})f_{\vec{j}}(\vec{q})\ , \quad (2.6a)$$

where

$$f_{\vec{k}}(\vec{q})=(2n+1)^{-d/2}\exp(i\vec{k}\cdot\vec{q}l)\ , \quad (2.6b)$$

$$\vec{q}=\{q_i\},\quad \vec{k}=\{k_i\};\quad q_i=\frac{2r_i\pi}{(2n+1)l},$$
$$r_i=-n,-n+1,\ldots,n\ ,\quad k_i=-n,\ldots,n \quad (2.6c)$$

and

$$\lambda(\vec{q})=\sum_i(4/l^2)\sin^2(lq_i/2)\ . \quad (2.6d)$$

We note the orthogonality relations

$$\sum_{\vec{j}}f_{\vec{j}}^*(\vec{q})f_{\vec{j}}(\vec{q}\,')=\delta_{\vec{q},\vec{q}\,'}\ , \quad (2.6e)$$

$$\sum_{\vec{q}}f_{\vec{i}}^*(\vec{q})f_{\vec{j}}(\vec{q})=\delta_{\vec{i},\vec{j}}\ . \quad (2.6f)$$

We thus transform Eq. (2.4) to q space by expanding variables as

$$m_{\vec{j}}=\sum_{\vec{q}}f_{\vec{j}}(\vec{q})m(\vec{q})\ ,$$
$$m(\vec{q})=\sum_{\vec{j}}f_{\vec{j}}^*(\vec{q})m_{\vec{j}}\ . \quad (2.7)$$

As we wish to eliminate the high-q modes, we make the following identification:

$$\sum_{\vec{q}}m(\vec{q})=\sum_{\vec{Q}\in\mathscr{R}(N)}m(\vec{Q})+\sum_{\vec{q}\notin\mathscr{R}(N)}m(\vec{q}), \quad (2.8a)$$

where

$$Q_i\in\mathscr{R}(N)=\frac{\pi}{(2n+1)l}\{-N,-N+1,\ldots,N\}\ . \quad (2.8b)$$

Thus by eliminating the $m(\vec{q})$ we will obtain a Fokker-Planck equation for the $m(\vec{Q})$ alone and will have equivalently changed the scale of our system. The effects of such a scale change will be apparent in the presence of correction terms to the coupling constants. These transformed coupling constants then define the RG transformation.

So, using Eq. (2.7), with the identification of (2.8), we transform the Fokker-Planck equation (2.4) to q space, and use relations (2.6) to find that Eq. (2.4) gives

$$\frac{\partial p}{\partial t}=(L_1+L_2+L_3)p\ , \quad (2.9)$$

where

$$L_1=\sum_{\vec{q}}[\lambda(\vec{q})+r]\frac{\partial}{\partial m(\vec{q})}m(\vec{q})+\frac{1}{2}\frac{\Gamma_0}{l^d}\sum_{\vec{q}}\frac{\partial^2}{\partial m(\vec{q})\partial m(-\vec{q})}, \tag{2.10a}$$

$$L_2=\frac{u}{(2n+1)^d}\sum_{\{\vec{q},\vec{Q}\}}\left[\frac{\partial}{\partial m(\vec{q})}m(\vec{Q}_1)m(\vec{Q}_2)m(\vec{Q}_3)\delta_{\vec{q},\vec{Q}_1+\vec{Q}_2+\vec{Q}_3}+3\frac{\partial}{\partial m(\vec{q})}m(\vec{q}_1)m(\vec{Q}_1)m(\vec{Q}_2)\delta_{\vec{q},\vec{q}_1+\vec{Q}_1+\vec{Q}_2}\right.$$
$$\left.+3\frac{\partial}{\partial m(\vec{q})}m(\vec{q}_1)m(\vec{q}_2)m(\vec{Q}_1)\delta_{\vec{q},\vec{q}_1+\vec{q}_2+\vec{Q}_1}+\frac{\partial}{\partial m(\vec{q})}m(\vec{q}_1)m(\vec{q}_2)m(\vec{q}_3)\delta_{\vec{q},\vec{q}_1+\vec{q}_2+\vec{q}_3}\right] \tag{2.10b}$$

and

$$L_3=\sum_{\vec{Q}}[\lambda(\vec{Q})+r]\frac{\partial}{\partial m(\vec{Q})}m(\vec{Q})+\frac{1}{2}\frac{\Gamma_0}{l^d}\sum_{\vec{Q}}\frac{\partial^2}{\partial m(\vec{Q})\partial m(-\vec{Q})}$$
$$+\frac{u}{(2n+1)^d}\sum_{\{\vec{Q},\vec{q}\}}\left[\frac{\partial}{\partial m(\vec{Q})}m(\vec{Q}_1)m(\vec{Q}_2)m(\vec{Q}_3)\delta_{\vec{Q},\vec{Q}_1+\vec{Q}_2+\vec{Q}_3}+3\frac{\partial}{\partial m(\vec{Q})}m(\vec{Q}_1)m(\vec{Q}_2)m(\vec{q}_1)\delta_{\vec{Q},\vec{Q}_1+\vec{Q}_2+\vec{q}_1}\right.$$
$$\left.+3\frac{\partial}{\partial m(\vec{Q})}m(\vec{Q}_1)m(\vec{q}_1)m(q_2)\delta_{\vec{Q},\vec{Q}_1+\vec{q}_1+\vec{q}_2}+\frac{\partial}{\partial m(\vec{Q})}m(\vec{q}_1)m(\vec{q}_2)m(\vec{q}_3)\delta_{\vec{Q},\vec{q}_1+\vec{q}_2+\vec{q}_3}\right]. \tag{2.10c}$$

In the preceding equations, $\sum_{\{\vec{q},\vec{Q}\}}$ represents a sum over all the $\vec{q}$, $\vec{Q}$ appearing in the terms, where $\vec{q}\notin\mathscr{R}(N)$ and $\vec{Q}\in\mathscr{R}(N)$.

We have defined the Fokker-Planck equation as in Eqs. (2.9) and (2.10) in order to perform the projection-operator method of adiabatic elimination, as will become apparent in Sec. III.

III. ELIMINATION OF q MODES: PROJECTION-OPERATOR METHOD

The details of the projection-operator elimination procedure have been described in I and II and we refer the reader to the work of Gardiner[7] for a thorough discussion of the method. In essence, the method consists of constructing a projection operator in terms of the stationary distribution function (p_s) of the variables to be eliminated. Such an operator projects variables into a subspace in which all variables are expressed as a product of the stationary distribution p_s, and an arbitrary function of the remaining variables. This new subspace thus represents a reduced system in which the effect of the eliminated variables is seen in correction terms to coupling constants.

A. Exact projection equations using Laplace transforms

Consider now the Fokker-Planck equation (2.9) and associated definitions (2.10). We note that the operator L_1 concerns only the $m(\vec{q})$, i.e., the variables we wish to eliminate. Thus we construct a projection operator P in terms of the stationary distribution function of L_1, and define it through its action on an arbitrary function $u(\vec{q},\vec{Q})$,

$$Pu(\vec{q},\vec{Q})=p_s\int u(\vec{q},\vec{Q})dm(\vec{q}), \tag{3.1}$$

where p_s satisfies

$$L_1p_s=0. \tag{3.2}$$

We note the following:

$$L_1P=PL_1=0 \tag{3.3}$$

and

$$PL_2P=0. \tag{3.4}$$

Indeed, since by (2.10b) L_2 contains only terms which begin with $\partial/\partial m(\vec{q})$, it is clear that integrating these always gives boundary terms, which vanish, so that we have the stronger result,

$$PL_2=0. \tag{3.5}$$

The relations (3.3) and (3.4) are essential requirements of the elimination procedure as shown in I.

The method follows by applying P to the Fokker-Planck equation (2.9). Defining

$$v=Pp, \tag{3.6a}$$

$$w=(1-P)p, \tag{3.6b}$$

we find

$$P\frac{\partial p}{\partial t}=\frac{\partial}{\partial t}v=PL_2(v+w) \tag{3.7}$$

and

$$(1-P)\frac{\partial}{\partial t}p=\frac{\partial}{\partial t}w=L_1w+(1-P)L_2w+L_2v$$
$$+(1-P)L_3(v+w), \tag{3.8}$$

where we have used the relations (3.3), (3.4), and (3.5).

Central to this method is the idea of projecting out information about $m(\vec{q})$ from the probability distribution of the system $p(m(\vec{q}),m(\vec{Q}),t)$, i.e., we wish to obtain an equation of motion (a Fokker-Planck equation) for the reduced probability distribution $\hat{p}(m(\vec{Q}))$. Clearly, this follows by constructing an equation of motion for v $(=Pp)$, by solving Eqs. (3.7) and (3.8) simultaneously.

Such a solution follows most easily through use of Laplace transforms, and has been explained in papers I and II, i.e., defining the Laplace transforms $L(v(t))=v(s) = \int_0^\infty e^{-st}v(t)$ Eqs. (3.7) and (3.8) become, respectively,

$$sv(s)-v(0)=PL_3w(s)+PL_3v(s)\ , \tag{3.9}$$

$$sw(s)=[L_1+(1-P)(L_2+L_3)]w(s) + [L_2+(1-P)L_3]v(s)\ , \tag{3.10}$$

where $v(0)$ means v at $t=0$; and we have set the initial condition $w=0$ at $t=0$.

Solving (3.9) and (3.10) for $v(s)$ we find

$$sv(s)-v(0)=PL_3[s-L_1-(1-P)(L_2+L_3)]^{-1} \times[L_2+(1-P)L_3]v(s)+PL_3v(s)\ . \tag{3.11}$$

B. Perturbation expansion for small u

We note that we can write L_3, as defined in (2.10c), in the form

$$L_3=G+uF\ , \tag{3.12a}$$

where F is the coefficient of u in (2.10c) and G is the remainder which commutes with L_1.

Further, L_2, as defined in (2.10b), is also proportional to u and can be written

$$L_2=uH\ . \tag{3.12b}$$

We first note that $PL_3v(s)$ contains nonvanishing terms of zero and first order in u, that $L_2v(s)$ is $O(u)$, and that

$$(1-P)L_3v(s)=(1-P)(G+uF)Pp\ . \tag{3.13}$$

Furthermore, by definition it is clear that G commutes with L_1 and hence with P, so

$$(1-P)L_3v(s)=u(1-P)Fv(s)=O(u)\ . \tag{3.14}$$

Thus, to get a perturbation expansion in u, we may neglect all terms of order u in $[\cdots]^{-1}$ in (3.11). We thus reduce to

$$PL_3[s-L_1-(1-P)G]^{-1}[uH+u(1-P)F]v(s) +PL_3v(s)\ , \tag{3.15}$$

which is approximately equal to Eq. (3.11).

We note that $(1-P)$ commutes with G and L_1, and so may be omitted in $[\cdots]^{-1}$. Now everything in $[s-L_1-G]^{-1}$ commutes with L_1 and P so we can replace PL_3 by $PL_3(1-P)$, and

$$PL_3(1-P)=P(G+uF)(1-P)=uPF(1-P) \tag{3.16}$$

so that to second order in u

$$u^2PF[s-L_1-G]^{-1}[H+(1-P)F]v(s)+P(G+uF)v(s)\ , \tag{3.17}$$

which is approximately equal to (3.11).

C. Asymptotic expansion for large b

We now want to show that for $b=(2n+1)/(2N+1)$ sufficiently large and l sufficiently small, we may approximate $[s-L_1-G]^{-1}$ by $[s-L_1]^{-1}$. The essence of the argument is quite simple: L_1 is much larger than G.

For the eigenvalues of L_1 are

$$\Lambda(\vec{q})\simeq r+\vec{q}^{\,2}\simeq l^{-2}+r \tag{3.18}$$

and the eigenvlaues of G are

$$\Lambda(\vec{Q})\simeq r+\vec{Q}^{2}\simeq(bl)^{-2}+r\ . \tag{3.19}$$

Now we must constrain l so that

$$l^{-2}\gg r \tag{3.20a}$$

and b must satisfy

$$b\gg 1\ . \tag{3.20b}$$

The first of these requirements is essentially that the time constant of the process generated by the term $rm(\vec{q})$ is much less than the typical time for a diffusive jump from one lattice point to another. This requirement can obviously be physically made. It then follows that large b implies all the eigenvalues of G are of order b^{-2} times those of L_1, and we can neglect G.

We may also neglect the term s, providing $s\ll l^{-2}$ which effectively defines a coarse-grained time scale $\tau\sim l^2$ at which the equation is valid.

Hence we reduce (3.11) to

$$\frac{\partial v}{\partial t}=P(G+uF)v-u^2PFL_1^{-1}[H+(1-P)F]v\ . \tag{3.21}$$

IV. EXPLICIT PROJECTED FOKKER-PLANCK EQUATION

Equation (3.21) represents the Fokker-Planck equation for the transformed system in which the short-wavelength spin components have been eliminated.

However, in order to obtain explicit expressions for the transformed coupling constants (and thus define the RG transformation) we need to evaluate Eq. (3.21) and invert from Fourier space to coordinate space.

The details of this procedure can be found in Appendix B; and we obtain a new Fokker-Planck equation describing the rescaled system [i.e., see Eq. (B15)],

$$\frac{\partial\hat{p}}{\partial t}=\Bigg\{-\sum_{\vec{J},\vec{K}}\frac{\partial}{\partial\tilde{m}_{\vec{J}}}D_{\vec{J},\vec{K}}\tilde{m}_{\vec{K}} +\sum_{\vec{J}}\frac{\partial}{\partial\tilde{m}_{\vec{J}}}\Bigg[\Bigg[r+\frac{3u\Gamma_0c}{8(\Omega^2+r)^2}-18u^2\Gamma_0^2a_1-\frac{18u^2\Gamma_0c^2}{64}\frac{1}{(\Omega^2+r)^3}\Bigg]\tilde{m}_{\vec{J}}+\Bigg[u-\frac{9u^2\Gamma_0c}{8(\Omega^2+r)^2}\Bigg]\tilde{m}_{\vec{J}}^{\,3} - 3u^2(2n+1)^dl^da_2(\tilde{m}_{\vec{J}})^5\Bigg] + \frac{\Gamma_0}{(bl)^d}\sum_{\vec{J}}\frac{\partial^2}{\partial\tilde{m}_{\vec{J}}^{\,2}}\Bigg[\frac{1}{2}+u^2\Gamma_0^2a_4+\frac{9u^2c}{16(\Omega^2+r)^3}(\tilde{m}_{\vec{J}})^2+\tfrac{9}{2}u^2(2n+1)^dl^da_3(\tilde{m}_{\vec{J}})^4\Bigg]\Bigg\}\hat{p}\ . \tag{4.1}$$

In expression (4.1), $\hat{p}$ is the probability distribution function for the $\{\tilde{m}_{\vec{J}}\}$; where $\tilde{m}_{\vec{J}}$ represents the macroscopic magnetization of cell J.

These new J cells describe a coarser-grained lattice than that of the original system [Eq. (2.4)]. That is, our original system was described in terms of a lattice of $(2n+1)^d$ cells, whereas Eq. (4.1) describes a system of $(2N+1)^d$ cells each of volume $(bl)^d$ where $N<n$. We have thus effectively blocked the system and the blocking ratio, b is $(2n+1)/(2N+1)$.

The operator $D_{\vec{J},\vec{K}}$ in Eq. (4.1) represents the diffusion operator in the new coarser-grained system [cf. Eq. (2.5)]. We define the eigenvalues and eigenfunctions of $D_{\vec{J},\vec{K}}$ through the equation

$$\sum_K D_{\vec{J},\vec{K}} \tilde{f}_{\vec{K}}(\vec{Q}) = -\lambda(\vec{Q})\tilde{f}_{\vec{J}}(\vec{Q}) , \tag{4.2a}$$

where

$$\tilde{f}_K(\vec{Q}) = (2N+1)^{-d/2}\exp(i\vec{K}\cdot\vec{Q}L) \tag{4.2b}$$

and

$$\vec{Q} \in R(N) ,$$

see (2.8b).

We also note the inversion relations,

$$m_{\vec{J}} = \sum_{\vec{Q}} \tilde{f}_{\vec{J}}(\vec{Q}) m(\vec{Q}) ,$$
$$m(\vec{Q}) = \sum_{\vec{J}} \tilde{f}^{*}_{\vec{J}}(\vec{Q}) m_{\vec{J}} , \tag{4.3}$$

and the $m_{\vec{J}}$ are related to $\tilde{m}_{\vec{J}}$ by

$$\tilde{m}_{\vec{J}} = b^{-d/2} m_{\vec{J}} . \tag{4.4}$$

The scaling (4.4) ensures that both $\tilde{m}_{\vec{J}}$ and $m_{\vec{J}}$ describe the magnetization per unit volume, in a manner similar to that in II, Eq. (4.5).

The constants a_1, a_2, a_3, a_4, and c are defined in Appendix B, i.e.,

$$a_1 \sim O(l^{4-d}(1-b^{4-d}))^2 \tag{4.5a}$$

is a finite constant,

$$a_4 \sim O(l^{8-2d}(1-b^{8-2d})) \tag{4.5b}$$

is also a finite constant,

$$a_3 = a_2 \sim \sum_{q \simeq (bl)^{-1}}^{3(bl)^{-1}} \Lambda^{-1}(q) \tag{4.5c}$$

is a parameter which is negligible in comparison to others, and

$$c = 4\int_{\Omega b^{-1}}^{\Omega} d^d q . \tag{4.5d}$$

The constant c is defined in a manner similar to the constant c appearing in the S^4 model of Wilson and Kogut.[3]

Equation (4.1) represents essentially a coarse-grained version of the system described by the Fokker-Planck equation (2.4). Central to the method was the assumption that the nonlinear coupling constant, u, is small.

The coarse-graining procedure has given rise to transformed coupling constants.

We note that the procedure has also generated a fifth-order drift term—not present in the original system. However, as discussed in Appendix B this term has a coefficient which is negligible in comparison to other system parameters; thus we do not expect this term to have a noticeable effect on the system's behavior.

Equation (4.1) shows that all correction terms (apart from the fifth-order term) contain the factor Γ_0, which represents the size of molecular fluctuations in the system. This indicates the importance of including finite fluctuations into our model of the system, i.e., the correction terms will only produce a significant contribution when fluctuations of order unity are considered.

We also see how the fluctuations couple through the nonlinearities of the system to give rise to correction terms.

All corrections to the noise term are of order u^2 and will thus be very small in comparison to the leading term, which is of order unity.

Finally, we should note that all the correction coefficients in (4.1) are evaluated only approximately. This is merely because the integrations are rather complicated—the relevant exact expressions are given later.

V. COMPARISON WITH RG RESULTS

We now compare the results of our transformation with that of the RG transformation derived by Wilson and Kogut[3] for the same model. Noting that the assumption of u small was essential to our method we will keep only terms to leading order in u in Eq. (4.1). Thus we neglect terms of order u^2 in the linear drift and all corrections to the noise term, which are of order u^2. We also omit the fifth-order drift term, as we expect it to be negligible.

In this regime, Eq. (4.1) becomes

$$\frac{\partial\hat{p}}{\partial t} = \left[-\sum_{\vec{J},\vec{K}} \frac{\partial}{\partial\tilde{m}_{\vec{J}}} D_{\vec{J},\vec{K}} \tilde{m}_{\vec{K}} + \sum_{\vec{J}} \frac{\partial}{\partial\tilde{m}_{\vec{J}}} (r^1 \tilde{m}_{\vec{J}} + u^1 \tilde{m}^3_{\vec{J}}) + \frac{\Gamma_0}{(bl)^d} \sum_{\vec{J}} \frac{\partial^2}{\partial\tilde{m}^2_{\vec{J}}} \right] \hat{p} , \tag{5.1}$$

where

$$r^1 = r + \frac{3c^1 u}{(\Omega^2 + r)} , \tag{5.2a}$$

$$u^1 = u - \frac{9u^2 c^1}{(\Omega^2+r)^2} , \tag{5.2b}$$

and

$$c^1 = \frac{\Gamma_0 c}{8}$$

is a finite constant.

The equations corresponding to (5.2) in which no approximations are made in the integrals are, in the limit of large b,

$$r^1 = r + \frac{3u\Gamma_0}{2}(2\pi)^{-d}\int_{\Omega b^{-1}}^{\Omega} \frac{d^d q}{\lambda(\vec{q})+r}\ , \qquad (5.3a)$$

$$u^1 = u - \tfrac{9}{2}u^2\Gamma_0(2\pi)^{-d}\int_{\Omega b^{-1}}^{\Omega} \frac{d^d q}{[\lambda(\vec{q})+r]^2}\ . \qquad (5.3b)$$

The free-energy functional corresponding to this Fokker-Planck equation is of exactly the same form as Eq. (2.1). The extra factor of b^{-d} multiplying Γ_0 exactly compensates for the increased cell size needed to convert a sum to an integral. The upper q cutoff, however, is explicitly Ωb^{-1} as opposed to Ω in the unblocked system.

Hence, Eqs. (5.2) are equivalent to the renormalization-group blocking equations of Wilson and Kogut;[3] i.e., apart from factors of 4 multiplying (5.2a) and 2^{4-d} multiplying (5.2b), which are introduced by Wilson and Kogut to ensure the rescaled Hamiltonian for the transformed system looks the same as the original one, they are the same as Eq. (4.26) of Ref. 3.

The fact that our derivation is valid only for large blocking ratio b is in concordance with the work of Bruce, Droz, and Aharony,[8] who have carried out the kind of blocking usually used in renormalization-group theory, in which variables are explicitly integrated out from the stationary distribution function, and have shown that many terms, which should be included when the blocking ratio is finite, can be ignored in the large-b limit. From our point of view, it is hard to see how one could justify any thing other than large b, though finite b is commonly used in such calculations. What we do suggest is that the adiabatic elimination technique provides a logical, justifiable, and essentially straightforward method of carrying out blocking. The method is also one which fits into a broad general framework, and thus unifies a range of phenomena whose understanding involves some kind of coarse-graining. Finally, the convergence of the method is under control.

VI. DISCUSSION AND CONCLUSIONS

In this paper we have shown the following main points.

(i) A simple formulation in terms of projectors can achieve the renormalization-group blocking equations, and can enable a quite precise limit to be given in which these equations are valid. This is the limit of small nonlinearity and large blocking ratio b.

(ii) Coarse-graining of the time-dependent Landau-Ginsburg model is achieved in a way which could have application to other systems, in a way similar to the derivation of the amplitude equations in hydrodynamics.

(iii) The blocking in renormalization-group theory is shown to be essentially the same kind of process used in derivation of the fluctuating amplitude equation in hydrodynamics and the treatment of diffusion homogenization in reaction-diffusion systems.

(iv) Since the blocking is carried out directly on a Fokker-Planck equation, dynamical data such as time correlation functions can also be, in principle, evaluated. We have not gone into this in detail.

Finally, we note that previous methods employed to obtain dynamic renormalization-group transformations on equations of motion have involved complicated perturbative analyses.[9] Our method, however, may be applied to equations of motion in a most straightforward manner and should produce the dynamic renormalization-group transformations in a much simpler fashion than previously.

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APPENDIX A: ORNSTEIN-UHLENBECK PROCESSES IN TERMS OF OPERATORS

The operator L_1 of Eq. (2.10a) can be simplified by writing

$$m(\vec{q}) = \frac{\Gamma_0^{1/2} l^{-d/2}}{[\lambda(\vec{q})+r]^{1/2}} y(\vec{q}) \qquad (A1)$$

in terms of which

$$L_1 = \sum_{\vec{q}} \Lambda(\vec{q}) \left[\frac{\partial}{\partial y(\vec{q})} y(\vec{q}) + \frac{1}{2}\frac{\partial^2}{\partial y(\vec{q})\partial y(-\vec{q})} \right], \qquad (A2)$$

where

$$\Lambda(\vec{q}) = \lambda(\vec{q}) + r\ . \qquad (A3)$$

We introduce operators $A(\vec{q}), A^\dagger(\vec{q})$ by

$$y(\vec{q}) = \frac{A^\dagger(\vec{q})+A(-\vec{q})}{\sqrt{2}}\ , \qquad \frac{\partial}{\partial y(\vec{q})} = -\sqrt{2}A^\dagger(-\vec{q})\ , \qquad (A4)$$

and $A^\dagger(q), A(q)$ obey boson commutation relations

$$[A(\vec{q}), A^\dagger(\vec{q}\,')] = \delta_{\vec{q},\vec{q}\,'} \qquad (A5)$$

and all other commutators vanish. We can then further simplify the expression (A2) for L_1 to

$$L_1 = -\sum_{\vec{q}} \Lambda(\vec{q}) A^\dagger(\vec{q}) A(\vec{q})\ . \qquad (A6)$$

We can similarly define operators $A(Q), A^\dagger(Q)$, and in terms of which we can write the first part of L_3 [Eq. (2.10c)]. The creation and destruction operators $A^\dagger(\vec{q}), A(\vec{q})$ generate eigenfunctions of L_1 and p_s in exactly the same way as happens in the quantum-mechanical harmonic oscillator, for from (A6), it follows that p_s satisfies

$$A(\vec{q}) p_s = 0\ , \qquad (A7)$$

and this is simply a definition of the vacuum state $|0\rangle$ for these operators. Hence

$$\begin{aligned} P[A(\vec{q})A^\dagger(\vec{q}\,')A(\vec{q}\,'')A^\dagger(\vec{q}\,''')] &= p_s \int dy(\vec{q}) A(\vec{q}) A^\dagger(\vec{q}\,') A(\vec{q}\,'') A^\dagger(\vec{q}\,''') p_s \\ &= p_s \langle 0 | A(\vec{q}) A^\dagger(\vec{q}\,') A(\vec{q}\,'') A^\dagger(\vec{q}\,''') | 0 \rangle\ . \qquad (A8) \end{aligned}$$

Terms like this arise in the evaluation of Eq. (3.21), and this technique will be used extensively in Appendix B.

APPENDIX B: EVALUATION OF EXPRESSION (3.21)

Equation (3.21) implies

$$\frac{\partial v(t)}{\partial t}=PL_3 v(t)-PL_3 L_1^{-1}[L_2+(1-P)L_3]v(t)\ .$$

First let us relist the terms of L_2 and L_3 as follows:

$$L_2=-\frac{u l^{d/2}\Gamma_0^{-1/2}}{(2n+1)^d}\sum_{\{\vec{q},\vec{Q}\}}\Lambda^{1/2}(\vec{q})\sqrt{2}A^{\dagger}(-\vec{q})(L_2^{(1)}+L_2^{(2)}+L_2^{(3)}+L_2^{(4)})=uH \tag{B1}$$

[see Eq. (3.12b)] where

$$\begin{aligned}
L_2^{(1)}&=m(\vec{Q}_1)m(\vec{Q}_2)m(\vec{Q}_3)\delta_{\vec{q},\vec{Q}_1+\vec{Q}_2+\vec{Q}_3}\ ,\\
L_2^{(2)}&=\frac{3}{\sqrt{2}}\Gamma_0^{1/2}l^{-d/2}\Lambda^{-1/2}(\vec{q}\,')[A^{\dagger}(\vec{q}\,')+A(-\vec{q}\,')]m(\vec{Q}_1)m(\vec{Q}_2)\delta_{\vec{q},\vec{q}\,'+\vec{Q}_1+\vec{Q}_2}\ ,\\
L_2^{(3)}&=\tfrac{3}{2}\Gamma_0 l^{-d}[\Lambda(\vec{q}_1)\Lambda(\vec{q}_2)]^{-1/2}[A^{\dagger}(\vec{q}_1)+A(-\vec{q}_1)][A^{\dagger}(\vec{q}_2)+A(-\vec{q}_2)]m(\vec{Q}_1)\delta_{\vec{q},\vec{q}_1+\vec{q}_2+\vec{Q}_1}\ ,\\
L_2^{(4)}&=\frac{\Gamma_0^{3/2}l^{-3d/2}}{2\sqrt{2}}[\Lambda(\vec{q}_1)\Lambda(\vec{q}_2)\Lambda(\vec{q}_3)]^{-1/2}[A^{\dagger}(\vec{q}_1)+A(-\vec{q}_1)][A^{\dagger}(\vec{q}_2)+A(-\vec{q}_2)][A^{\dagger}(\vec{q}_3)+A(-\vec{q}_3)]\delta_{\vec{q},\vec{q}_1+\vec{q}_2+\vec{q}_3}\ ,
\end{aligned} \tag{B2}$$

$$L_3=L_3(\vec{Q})+\mathscr{L}_3\ ,$$

where

$$L_3(\vec{Q})=\sum_{\{\vec{Q}\}}\left[\Lambda_1(\vec{Q})\frac{\partial}{\partial m(\vec{Q})}m(\vec{Q})+\frac{u}{(2n+1)^d}\frac{\partial}{\partial m(\vec{Q})}m(\vec{Q}_1)m(\vec{Q}_2)m(\vec{Q}_3)\delta_{\vec{Q},\vec{Q}_1+\vec{Q}_2+\vec{Q}_3}+\frac{\Gamma_0}{l^d}\frac{\partial^2}{\partial m(\vec{Q})\partial m(-\vec{Q})}\right]$$

and

$$\mathscr{L}_3=\frac{u}{(2n+1)^d}\sum_{\{\vec{Q},\vec{q}\}}\frac{\partial}{\partial m(\vec{Q})}(\mathscr{L}_3^{(1)}+\mathscr{L}_3^{(2)}+\mathscr{L}_3^{(3)})\ ,$$

where

$$\begin{aligned}
\mathscr{L}_3^{(1)}&=\frac{3}{\sqrt{2}}\Gamma_0^{1/2}l^{-d/2}m(\vec{Q}_1)m(\vec{Q}_2)\Lambda^{-1/2}(\vec{q}\,')[A^{\dagger}(\vec{q}\,')+A(-\vec{q}\,')]\delta_{\vec{Q},\vec{Q}_1+\vec{Q}_2+\vec{q}\,'}\ ,\\
\mathscr{L}_3^{(2)}&=\tfrac{3}{2}\Gamma_0 l^{-d}m(\vec{Q}_1)[\Lambda(\vec{q}_1)\Lambda(\vec{q}_2)]^{-1/2}[A^{\dagger}(\vec{q}_1)+A(-\vec{q}_1)][A^{\dagger}(\vec{q}_2)+A(-\vec{q}_2)]\delta_{\vec{Q},\vec{Q}_1+\vec{q}_1+\vec{q}_2}\ ,\\
\mathscr{L}_3^{(3)}&=\frac{\Gamma_0^{3/2}}{2\sqrt{2}}l^{-3d/2}[\Lambda(\vec{q}_1)\Lambda(\vec{q}_2)\Lambda(\vec{q}_3)]^{-1/2}[A^{\dagger}(\vec{q}_1)+A(-\vec{q}_1)][A^{\dagger}(\vec{q}_2)+A(-\vec{q}_2)][A^{\dagger}(\vec{q}_3)+A(-\vec{q}_3)]\delta_{\vec{Q},\vec{q}_1+\vec{q}_2+\vec{q}_3}\ ,
\end{aligned}$$

and we note that [see Eq. (3.12a)]

$$F=\sum_{\{\vec{Q}\}}\frac{1}{(2n+1)^d}\frac{\partial}{\partial m(\vec{Q})}[m(\vec{Q}_1)m(\vec{Q}_2)m(\vec{Q}_3)\delta_{\vec{Q},\vec{Q}_1+\vec{Q}_2+\vec{Q}_3}+\mathscr{L}_3^{(1)}+\mathscr{L}_3^{(2)}+\mathscr{L}_3^{(3)}]\ ,$$

where we have also used the relation (A4) defined in Appendix A. We now consider each of the contributions to expression (3.21).

(1) $PL_3 v(t)$ is the first contribution we will discuss. Note

$$v(t)=Pp=p_s\hat{p}(m(\vec{Q}))$$

which implies

$$\begin{aligned}
PL_3 v(t)&=p_s\int L_3 dy(\vec{q})p_s\hat{p}\\
&=p_s\langle\mathscr{L}_3\rangle\hat{p}+p_s\langle L_3(\vec{Q})\rangle\hat{p}\ ,
\end{aligned} \tag{B3}$$

where $\langle\ \rangle$ denotes the average over the stationary distribution of the $y(q)$.

Consider the first expression on the right-hand side of Eq. (B3), $p_s\langle\mathscr{L}_3\rangle\hat{p}$. We note from Eq. (2.10a) that the $y(q)$ comprise an Ornstein-Uhlenbeck process. As all odd moments of such a process are zero, the only contribution to the first term will arise from $\mathscr{L}_3^{(2)}$, i.e.,

$$\langle \mathscr{L}_3 \rangle = \frac{u}{(2n+1)^d} \sum_{\{\vec{q},\vec{Q}\}} \frac{\partial}{\partial m(\vec{Q})} \langle \mathscr{L}_3^{(2)} \rangle$$
$$= \frac{3u\Gamma_0 l^{-d}}{2(2n+1)^d} \sum_{\{\vec{q},\vec{Q}\}} \frac{\partial}{\partial m(\vec{Q})} m(\vec{Q})[\Lambda(\vec{q}_1)\Lambda(\vec{q}_2)]^{-1/2} \langle [A^\dagger(\vec{q}_1)+A(-\vec{q}_1)][A^\dagger(\vec{q}_2)+A(-\vec{q}_2)] \rangle \delta_{\vec{Q},\vec{Q}_1+\vec{q}_1+\vec{q}_2}$$

as

$$\langle [A^\dagger(\vec{q}_1)+A(-\vec{q}_1)][A^\dagger(\vec{q}_2)+A(-\vec{q}_2)] \rangle = \langle 0 | [A^\dagger(\vec{q}_1)+A(-\vec{q}_1)][A^\dagger(\vec{q}_2)+A(-\vec{q}_2) | 0 \rangle$$
$$= \delta_{-\vec{q}_1,\vec{q}_2} ,$$

we find

$$\langle \mathscr{L}_3^{(2)} \rangle = \frac{3}{2} \frac{u\Gamma_0 l^{-d}}{(2n+1)^d} \sum_{\{\vec{Q}\}} \frac{\partial}{\partial m(\vec{Q})} m(\vec{Q}_1) \delta_{\vec{Q},\vec{Q}_1} \sum_{\{\vec{q}\}} \Lambda^{-1}(\vec{q}) . \tag{B4}$$

We may also set

$$\sum_{\{\vec{q}\}} \Lambda^{-1}(\vec{q}) \Longrightarrow \left[\frac{(2n+1)l}{2\pi} \right]^d \int_{\Omega b^{-1}}^{\Omega} \Lambda^{-1}(\vec{q}) d^d q , \tag{B5}$$

where $\Omega = 2\pi/l$,

$$\Lambda(\vec{q}) = \lambda(\vec{q}) + r \simeq q^2 + r$$

for small q. An integral similar to (B5) was evaluated by Wilson and Kogut,[3] and they set

$$(2\pi)^{-d} \int_{\Omega b^{-1}}^{\Omega} \frac{d^d q}{q^2+r} \Longrightarrow \frac{c}{4} \frac{1}{\Omega^2+r}, \quad c = 4 \int_{\Omega b^{-1}}^{\Omega} d^d q . \tag{B6}$$

(Note Wilson and Kogut set the upper cutoff of the integral equal to 1; we set the upper cutoff equal to Ω.) Thus, using (B6) and (B5), Eq. (B4) becomes

$$\langle \mathscr{L}_3^{(2)} \rangle = \tfrac{3}{2} u \Gamma_0 \left[\sum_{\{\vec{Q}\}} \frac{\partial}{\partial m(\vec{Q})} m(\vec{Q}_1) \delta_{\vec{Q},\vec{Q}_1} \right] \frac{c}{4(\Omega^2+r)} . \tag{B7}$$

The expression (B7), however, is still expressed in Q space, and to compare this transformed expression with the original Fokker-Planck equation (2.4) we need to invert (B7) to an equivalent coordinate or J space.

Rather than invert to the original j space via transformations (2.7), we introduce a new coarser-grained J space, in terms of the Q modes. (Such a transformation has been previously discussed by the authors.[1,5])

Transformation to coarser-grained space

We now define the eigenfunctions,[1]

$$\tilde{f}_{\vec{J}}(\vec{Q}) = (2N+1)^{-d/2} \exp(i\vec{J}\cdot\vec{Q}L) , \tag{B8}$$

where

$$Q_i \in \mathscr{R}(N) ,$$
$$\mathscr{R}(N) \equiv \{-N, \ldots, N\} \pi/(2n+1)l ,$$
$$J_i \in \{-N, \ldots, N\}$$

of the operator $D_{\vec{J},\vec{K}}$

$$\sum_{\vec{J}} D_{\vec{J},\vec{K}} \tilde{f}_{\vec{J}}(\vec{Q}) = -\lambda(\vec{Q}) \tilde{f}_{\vec{K}}(\vec{Q}) , \tag{B9}$$

where the $\lambda(\vec{Q})$ are as defined in (2.6d). The eigenfunctions are orthogonal as in Eqs. (2.6e) and (2.6f).

We invert (B7) using the relations

$$m_{\vec{J}} = \sum_{\vec{Q}} \tilde{f}_{\vec{J}}(\vec{Q}) m(\vec{Q}) ,$$
$$m(\vec{Q}) = \sum_{J} \tilde{f}^*_{\vec{J}}(\vec{Q}) m_{\vec{J}} . \tag{B10}$$

Using (B10) in (B7) we find

$$\langle \mathscr{L}_3^{(2)} \rangle = \frac{3u\Gamma_0 c}{8(\Omega^2+r)} \sum_{\vec{J}} \frac{\partial}{\partial m_{\vec{J}}} m_{\vec{J}} . \tag{B11}$$

Also, to ensure the variable m_J describes the macroscopic magnetization per J cell in the same fashion as m_j describes magnetization per j cell in the original system, we set the scaling[1]

$$\tilde{m}_{\vec{J}} = m_{\vec{J}} \left[\frac{2N+1}{2n+1} \right]^{d/2} = b^{-d/2} m_{\vec{J}} , \tag{B12}$$

where $b = (2n+1)/(2N+1)$ is the blocking ratio. Thus, using B(12) in (B11), we find

$$P\mathscr{L}_3 v(t) = p_s \langle \mathscr{L}_3^{(2)} \rangle \hat{p}$$
$$= \frac{3p_s u \Gamma_0 c}{8(\Omega^2+r)} \sum_{J} \frac{\partial}{\partial \tilde{m}_{\vec{J}}} \tilde{m}_{\vec{J}} \hat{p} . \tag{B13}$$

Consider now the second expression on the right-hand side of Eq. (B3),

$$p_s \langle L_3(\vec{Q}) \rangle \hat{p} .$$

Equation (B2) shows that we need only invert this above expression to J space, as it involves only Q modes. Using Eqs. (B8), (B9), and (B10) we find

$$PL_3(\vec{Q})v(t)=p_s\langle L_3(\vec{Q})\rangle\hat{p}$$
$$=p_s\left[\sum_{\vec{J},\vec{K}}\frac{\partial}{\partial\tilde{m}_{\vec{J}}}D_{\vec{J},\vec{K}}\tilde{m}_{\vec{K}}+r\sum_{\vec{J}}\frac{\partial}{\partial\tilde{m}_{\vec{J}}}\tilde{m}_{\vec{J}}+u\sum_{\vec{J}}\frac{\partial}{\partial\tilde{m}_{\vec{J}}}\tilde{m}_{\vec{J}}^3+\frac{\Gamma_0}{(bl)^d}\sum_{\vec{J}}\frac{\partial^2}{\partial\tilde{m}_{\vec{J}}^2}\right]\hat{p}\,, \tag{B14}$$

where we have used the scaling relation (B12).

(2) Finally, we evaluate the second expression of Eq. (3.21)

$$-P[\mathscr{L}_3+L_3(\vec{Q})]L_1^{-1}\{L_2+(1-P)[\mathscr{L}_3+L_3(\vec{Q})]\}v(t)\,.$$

Noting that

$$-PL_3(\vec{Q})L_1^{-1}\{L_2+(1-P)[\mathscr{L}_3+L_3(\vec{Q})]\}v(t)=0\,,$$

i.e., P commutes with both $L_3(Q)$ and L_1^{-1} and gives zero when applied to the L_2 and $(1-P)$ terms and

$$(1-P)L_3(\vec{Q})v(t)=L_3(\vec{Q})(1-P)Pp$$
$$=0\quad(P^2=P)\,.$$

We find that the expression discussed in (2) implies

$$-P\mathscr{L}_3L_1^{-1}[L_2+(1-P)\mathscr{L}_3]v(t)\,.$$

As previously discussed, expression (2) involves expectation values of boson operators. As all odd moments are zero we find the only nonzero contributions arising from (2) are discussed in the following.

Linear drift terms

The linear drift terms are as follows:

$$\text{(i)}\quad\frac{Pu}{(2n+1)^d}\sum_{\{\vec{Q},\vec{q}\}}\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(2)}L_1^{-1}\frac{ul^{d/2}\Gamma_0^{-1/2}}{(2n+1)^d}\Lambda^{1/2}(\vec{q}\,')\sqrt{2}A^{\dagger}(-\vec{q}\,')L_2^{(4)}v(t)$$
$$\Longrightarrow-\tfrac{18}{4}p_su^2\Gamma_0^2\frac{c^2}{16}\frac{1}{(\Omega^2+r)^3}\sum_{\vec{J}}\frac{\partial}{\partial\tilde{m}_{\vec{J}}}\tilde{m}_{\vec{J}}\hat{p}\,.$$

$$\text{(ii)}\quad\frac{Pu}{(2n+1)^d}\sum_{\{\vec{Q},\vec{q}\}}\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(3)}L_1^{-1}\frac{ul^{d/2}\Gamma_0^{-1/2}}{(2n+1)^d}\Lambda^{1/2}(\vec{q}\,)\sqrt{2}A^{\dagger}(-\vec{q}\,)L_2^{(3)}v(t)\Longrightarrow-18p_su^2\Gamma_0^2a_1\sum_{\vec{J}}\frac{\partial}{\partial\tilde{m}_{\vec{J}}}\tilde{m}_{\vec{J}}\hat{p}\,,$$

where

$$a_1=\sum_{\{\vec{q}\}}[\Lambda(\vec{q}_1)+\Lambda(\vec{q}_2)+\Lambda(\vec{q}_1+\vec{q}_2)]^{-1}[\Lambda(\vec{q}_1)\Lambda(\vec{q}_2)]^{-1}\sim O(l^{4-d}(1-b^{4-d}))^2\,.$$

Cubic drift terms

The cubic drift terms are as follows:

$$\text{(iii)}\quad P\frac{u}{(2n+1)^d}\sum_{\{\vec{Q},\vec{q}\}}\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(1)}L_1^{-1}\frac{ul^{d/2}\Gamma_0^{-1/2}}{(2n+1)^d}\Lambda^{1/2}(\vec{q}\,)\sqrt{2}A^{\dagger}(-\vec{q}\,)L_2^{(3)}v(t)=0\,.$$

$$\text{(iv)}\quad P\frac{u}{(2n+1)^d}\sum_{\{\vec{Q},\vec{q}\}}\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(3)}L_1^{-1}\frac{ul^{d/2}\Gamma_0^{-1/2}}{(2n+1)^d}\Lambda^{1/2}(\vec{q}\,)\sqrt{2}A^{\dagger}(-\vec{q}\,)L_2^{(1)}v(t)=0\,.$$

Terms (iii) and (iv) are both zero as the expectation values generated give rise to δ functions of the type $\delta_{\vec{q},\vec{Q}}$—which are clearly not allowed.

The only contribution to the cubic drift term is

$$\text{(v)}\quad P\frac{u^2l^{d/2}\Gamma_0^{-1/2}}{(2n+1)^{2d}}\sum_{\{\vec{Q},\vec{q}\}}\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(2)}L_1^{-1}\Lambda^{1/2}(\vec{q}\,)\sqrt{2}A^{\dagger}(-\vec{q}\,)L_2^{(2)}v(t)$$
$$=\frac{u^29p_s\Gamma_0}{(2n+1)^{2d}2l^d}\sum_{\{\vec{q},\vec{Q}\}}\frac{\partial}{\partial m(\vec{Q})}m(\vec{Q}_1)[\Lambda(\vec{q}_1)\Lambda(\vec{q}_2)]^{-1/2}\delta_{\vec{Q},\vec{Q}_1+\vec{q}_1+\vec{q}_2}m(\vec{Q}_1')m(\vec{Q}_2')\Lambda^{1/2}(\vec{q}\,)\Lambda^{-1/2}(\vec{q}\,')$$
$$\times\langle[A^{\dagger}(\vec{q}_1)+A(-\vec{q}_1)][A^{\dagger}(\vec{q}_2)+A(-\vec{q}_2)]L_1^{-1}$$
$$\times A^{\dagger}(-\vec{q}\,)[A^{\dagger}(\vec{q}\,')+A(-\vec{q}\,')]\rangle\delta_{\vec{q},\vec{q}\,'+\vec{Q}_1'+\vec{Q}_2'}\hat{p}\,.$$

Evaluating the expectation value, (v) becomes

$$-\frac{9u^2p_s\Gamma_0}{2(2n+1)^d}\sum_{\{\vec{Q}\}}\frac{\partial}{\partial m(\vec{Q})}m(\vec{Q}_1)m(\vec{Q}_1')m(\vec{Q}_2')\delta_{\vec{Q},\vec{Q}_1+\vec{Q}_1'+\vec{Q}_2'}[(2n+1)l]^{-d}\sum_{q'}2\Lambda^{-1}(\vec{q}\,')[\Lambda(\vec{q}\,')+\Lambda(\vec{q}\,'+\vec{Q}_1'+\vec{Q}_2')]^{-1}.$$

Noting that $\vec{Q}_1'+\vec{Q}_2' \ll \vec{q}\,' \Longrightarrow \Lambda(\vec{q}\,'+\vec{Q}_1'+\vec{Q}_2')\approx\Lambda(\vec{q}\,')$, we see that

$$[(2n+1)l]^{-d}\sum_{\vec{q}\,'}2\Lambda^{-1}(\vec{q}\,')[\Lambda(\vec{q}\,')+\Lambda(\vec{q}\,'+\vec{Q}_1'+\vec{Q}_2')]^{-1}\approx[(2n+1)l]^{-d}\sum_{\vec{q}}\Lambda^{-2}(\vec{q}\,)\Longrightarrow(2\pi)^{-d}\int_{\Omega b^{-1}}^{\Omega}\frac{1}{(q^2+r)^2}d^dq\ .$$

Again, following Wilson and Kogut,[1] we set

$$(2\pi)^{-d}\int_{\Omega b^{-1}}^{\Omega}\frac{1}{(\vec{q}\,^2+r)}d^dq\Longrightarrow\frac{c}{4}\frac{1}{(\Omega^2+r)^2}\ ,$$

where $c=4\int_{\Omega b^{-1}}^{\Omega}d^dq$ and inverting to J space, we find that (v) becomes

$$-\frac{9p_su^2\Gamma_0c}{8(\Omega^2+r)^2}\sum_{\vec{J}}\frac{\partial}{\partial\tilde{m}_{\vec{J}}}\tilde{m}\,^3_{\vec{J}}\hat{p}\ .$$

Fifth-order drift term

The fifth-order drift term is given as follows:

$$\text{(vi)}\quad P\frac{u^2\Gamma_0^{-1/2}}{(2n+1)^{2d}}l^{d/2}\sum_{\{\vec{q},\vec{Q}\}}\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(1)}L_1^{-1}\Lambda^{1/2}(\vec{q}\,)\sqrt{2}A^{\dagger}(-\vec{q}\,)L_2^{(1)}v(t)=-3p_su^2(2n+1)^dl^da_2\sum_{\vec{J}}\frac{\partial}{\partial\tilde{m}_{\vec{J}}}\tilde{m}\,^5_{\vec{J}}\hat{p}\ ,$$

where

$$a_2=[(2n+1)l]^{-d}\sum_{\vec{q}\approx\vec{Q}}^{3\vec{Q}}\Lambda^{-1}(\vec{q}\,)\simeq(2\pi)^{-d}\int_{\Omega b^{-1}}^{3\Omega b^{-1}}\frac{d^d\vec{q}}{(\vec{q}\,^2+r)}\ ,$$

i.e., the sum over the $\vec{q}$ is extremely restricted in this case and for large b we expect the sum a_2 to be very small in comparison to other terms. Finally we obtain the noise terms from expression (2).

Noise terms

The noise terms are given in the following:

$$\text{(vii)}\quad -P\frac{u^2}{(2n+1)^{2d}}\sum_{\{\vec{Q},\vec{q}\}}\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(1)}L_1^{-1}(1-P)\frac{\partial}{\partial m(\vec{Q}\,')}\mathscr{L}_3^{(3)}v(t)=0\ .$$

$$\text{(viii)}\quad -P\frac{u^2}{(2n+1)^{2d}}\sum_{\{\vec{Q},\vec{q}\}}\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(3)}L_1^{-1}(1-P)\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(1)}v(t)=0\ .$$

These terms are zero for the same reason as given for terms (iii) and (iv).

$$\text{(ix)}\quad -P\frac{u^2}{(2n+1)^{2d}}\sum_{\{\vec{Q},\vec{q}\}}\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(1)}L_1^{-1}(1-P)\frac{\partial}{\partial m(\vec{Q}\,')}\mathscr{L}_3^{(1)}v(t)\Longrightarrow\tfrac{9}{2}p_su^2\Gamma_0(2n+1)^{+d}b^{-d}a_3\sum_{\vec{J}}\frac{\partial^2}{\partial\tilde{m}_J^2}\tilde{m}_J^4\hat{p}\ ,$$

where $a_3=[(2n+1)l]^{-d}\sum_{\vec{q}\approx\vec{Q}}^{3\vec{Q}}\Lambda^{-2}(\vec{q}\,)$, i.e., a_3 is negligible in comparison to other parameters [see (vi)].

$$\text{(x)}\quad -P\frac{u^2}{(2n+1)^{2d}}\sum_{\{\vec{Q},\vec{q}\}}\frac{\partial}{\partial m(\vec{Q})}\mathscr{L}_3^{(2)}L_1^{-1}\frac{\partial}{\partial m(\vec{Q}\,')}\mathscr{L}_3^{(2)}v(t)\Longrightarrow\frac{9u^2p_s\Gamma_0c}{16l^db^d(\Omega^2+r)^3}\sum_{\vec{J}}\frac{\partial^2}{\partial\tilde{m}_{\vec{J}}^2}\tilde{m}\,^2_{\vec{J}}\hat{p}\ ,$$

where

$$c = 4\int_{\Omega b^{-1}}^{\Omega} d^d q \; .$$

(xi) $$-P\frac{u^2}{(2n+1)^{2d}} \sum_{\{\vec{Q},\vec{q}\}} \frac{\partial}{\partial m(\vec{Q})} \mathscr{L}_3^{(3)} L_1^{-1} \frac{\partial}{\partial m(\vec{Q}')} \mathscr{L}_3^{(3)} v(t) \Longrightarrow u^2 p_s \Gamma_0^3 (lb)^{-d} a_4 \sum_{\vec{J}} \frac{\partial^2}{\partial \tilde{m}_{\vec{J}}^2} \hat{p} \; ,$$

where

$$a_4 = [(2n+1)l]^{-2d} \sum_{\{\vec{q}\}} [\Lambda(\vec{q}_1)\Lambda(\vec{q}_2)\Lambda(\vec{q}_1+\vec{q}_2)]^{-1} [\Lambda(\vec{q}_1)+\Lambda(\vec{q}_2)+\Lambda(\vec{q}_1+\vec{q}_2)]^{-1}$$
$$\sim O(l^{-2d+8}(1-b^{8-2d})) \; .$$

Combining all these results, we find expression (3.21) becomes

$$\frac{\partial v}{\partial t} = p_s \frac{\partial \hat{p}}{\partial t} = p_s \Bigg\{ -\sum_{\vec{J},\vec{K}} \frac{\partial}{\partial \tilde{m}_{\vec{J}}} D_{\vec{J},\vec{K}} \tilde{m}_{\vec{K}} + \sum_{\vec{J}} \frac{\partial}{\partial \tilde{m}_{\vec{J}}} \Bigg[\left[r + \frac{3u\Gamma_0 c}{8(\Omega^2+r)} - 18u^2\Gamma_0^2 a_1 - \frac{18u^2\Gamma_0^2 c^2}{64(\Omega^2+r)^3} \right] \tilde{m}_{\vec{J}}$$
$$+ \left[u - \frac{9u^2\Gamma_0 c}{8(\Omega^2+r)^2} \right] \tilde{m}_{\vec{J}}^3 - 3u^2(2n+1)^d l^d a_2 \tilde{m}_{\vec{J}}^5 \Bigg]$$
$$+ \frac{\Gamma_0}{(bl)^d} \sum_{\vec{J}} \frac{\partial^2}{\partial \tilde{m}_{\vec{J}}^2} \left[\tfrac{1}{2} + u^2\Gamma_0^2 a_4 + \frac{9u^2 c}{16(\Omega^2+r)^3} \tilde{m}_{\vec{J}}^2 + \tfrac{9}{2} u^2 (2n+1)^d l^d a_3 \tilde{m}_{\vec{J}}^4 \right] \Bigg\} \hat{p} \; , \qquad \text{(B15)}$$

where $D_{\vec{J},\vec{K}}$ is the difference operator in the coarse-grained J space, defined by

$$\sum_{\vec{J}} D_{\vec{J},\vec{K}} \tilde{f}_{\vec{J}}(\vec{Q}) = -\lambda(\vec{Q}) \tilde{f}_{\vec{K}}(\vec{Q}) \; .$$

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