Adiabatic elimination in stochastic systems. II. Application to reaction diffusion and hydrodynamic-like systems

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We develop a stochastic theory of rapidly diffusing spatially distributed systems. Discussion is within the framework of the cell model, in which the system is described in terms of a lattice of n cells. Utilizing projector-operator techniques, we formalize the method of homogenization of such systems. That is, by projecting out high-q Fourier modes in the adiabatic limit of large diffusion, we map the system to one defined on a coarser-grained lattice. We thus demonstrate a "blocking" procedure in the cell model. Finally we consider a simple hydrodynamic model and show that near the point of convective instability projection-operation methods predict the same amplitude equations for the slow hydrodynamic modes as does the method of multiple scales.

I. INTRODUCTION

The description of spatially distributed fluctuating systems by means of stochastic partial-differential equations is a deceptively simple way of representing extremely complex behavior. It is usually considered reasonable to divide the system into many small cells, and to model the system by means of some linear transport operator, which connects adjacent cells, to which is added the nonlinear part of the system, which occurs locally. Such a description gives rise to a set of many coupled stochastic differential equations, to which can be written a corresponding many-variable Fokker-Planck equation. In this form there are no difficulties of interpretation, a many variable Fokker-Planck equation being a well-understood concept.

The problem which immediately arises is the influence of the cell size, for the cell is an arbitrary construct, and its size should not have any effect on measurable results. It is tempting to let the cell size go to zero, but whenever any degree of nonlinearity exists, this gives rise to infinities, and the theory thus requires renormalization. It is not entirely clear that this renormalization has any physical meaning, since in all physical systems the cell size cannot be made as small as the size of a molecule; at that stage noncontinuum effects—those of the very atomistic nature of matter—become important.

Physically, one expects the transport operator to enforce a degree of homogenization in the system: The cell size should be chosen small enough so that one can guarantee that the concentration (which we shall, for simplicity, call the fluctuating variable) is the same across a number of cells. The main function of Secs. II and III of this paper is to formulate and prove the validity of this concept.

Section II shows that for D sufficiently large, any finite system can be regarded as homogeneous; that we can consider the whole system as one large cell. This result was first noted by Kuramoto,[1] and has already been shown with complete rigor by Arnold[2] in a slightly narrower context—but it is included here for completeness.

Section III gives the more complete result: It shows how to go from a small-cell description to a description in terms of larger cells by eliminating the modes which are rapidly damped to zero by the transport operator. It shows that provided a certain combination of transport operator, cell size, and local reaction rates is chosen sufficiently large, the description is independent of cell size. That is, the cells are indeed artificial constructs, having no genuine influence on the observable properties of the system being described.

The methods used in both of these sections are similar, based on stochastic adiabatic elimination of the small-wavelength modes, according to methods developed in Ref. 3, referred to as I in this paper.

Finally, in Sec. IV we extend the same method to the case where the eigenfunction of the transport operator with eigenvalue zero corresponds to nonzero wavelength. The most famous example of this occurs at the Bénard instability in hydrodynamics, which has been treated deterministically by Newell and Whitehead[4] and others,[5–9] and stochastically by Graham.[10] The limit taken in this case is not that of large transport operator; rather it is a limit of close approach to an instability point, and there are, in principle, substantial technical differences between the deterministic treatment and the stochastic treatment.

Both Newell and Whitehead, and Graham, used multiple space scales to derive their amplitude equations and Graham's amplitude equation is a stochastic version of that of Newell and Whitehead. We use the same technique in Secs. II and III to derive for a simple model the same results as Graham, which turn out to be applicable under the same conditions as he assumes. (The adiabatic elimination technique corresponds to that used by Cross[5] deterministically.) Thus we show the equivalence of stochastic adiabatic elimination with the method of multiple space scales.

The fact that the stochastic result corresponds to a fluctuating version of the deterministic result is the result of an assumption that the fluctuations vanish as $\epsilon^{3/4}$ as ϵ, the deviation from the instability point, goes to zero. Strictly speaking, this is an unjustified assumption—and it will be attended to in paper III, which connects adiabatic elimination to the renormalization-group theory of critical phenomena.

II. COMPLETE HOMOGENIZATION OF A FINITE-VOLUME SYSTEM: REACTION-DIFFUSION MODEL

We consider the following model system. We assume d-dimensional space is divided into n cells of side length l. Cells are labeled by indices (j,k) and x_j represents the number of molecules of a chemical X in the cell j.

The system is then described by the following Fokker-Planck equation:[11]

$$\frac{\partial P}{\partial t}=\left[\sum_{k,j}\left[-\frac{\partial}{\partial x_k}D_{kj}x_j\right]+\sum_j\left[\frac{\partial}{\partial x_j}a(x_j)+l^{-d}\frac{\partial^2}{\partial x_j^2}b'(x_j)\right]\right]P\ . \tag{1}$$

The last two terms on the right-hand side of Eq. (1) concern the chemical reaction process. The drift term $a(x_j)$ describes the deterministic behavior of the chemical system—its explicit form depends on the specific nature of the reaction we are considering. The term $b'(x_j)$ describes noise in the system; that is, it concerns the stochastic behavior of the system. The terms $a(x_j)$ and $b'(x_j)$ are proportional to a rate constant, which we will denote by κ, characterizing the chemical reaction in question.

As it stands, this model can be considered as the Fokker-Planck equation corresponding to a Poisson-representation formulation of a chemical master equation.[10] To correspond to the more usual chemical Langevin equation, the second-derivative term should be modified to take account of diffusion noise.[12,13]

Molecular diffusion, i.e., transfer between cells, is described by the matrix D_{kj} in the first term of Eq. (1). In the simplest case, we consider a one-dimensional system and assume that only transfer between adjacent cells occurs. Thus D_{kj} is a simple difference operator,

$$D_{kj}=\frac{D}{l^2}(\delta_{k,j+1}+\delta_{k,j-1}-2\delta_{k,j})\ , \tag{2}$$

where D is the diffusion constant. (Rapid diffusion implies D is very large in comparison to other system parameters.)

As mentioned in the Sec. I, to facilitate homogenization we must eliminate the large-q modes. We thus need to transform Eq. (1) to q space. To this end we introduce the eigenfunctions $f_k(q)$, such that

$$\sum_j D_{kj}f_j(q)=-D\lambda(q)f_k(q)\ . \tag{3}$$

For D_{kj} as defined in Eq. (2) and with reflecting boundaries at the end walls of the system, the eigenfunctions have the form

$$f_j(q)\propto\cos(qlj)\ , \tag{4}$$

where l is the cell length and $q=r\pi/nl$ $(r=0,1,\ldots,n)$. Thus

$$\lambda(q)=4\sin^2\left[\frac{ql}{2}\right]\Big/l^2\ . \tag{5}$$

Clearly,

$$\lambda(q)>0,\quad q\neq0 \tag{6}$$

and

$$\lambda(0)=0\ .$$

The $f_j(q)$ form a complete, orthogonal set,

$$\sum_j f_j(q)f_j(q')=\delta_{q'q}\ , \tag{7}$$

$$\sum_q f_k(q)f_j(q)=\delta_{k,j}\ . \tag{8}$$

Relations (7) imply $f_k(0)=n^{-1/2}$. These properties [(6)—(8)] are valid in any number of dimensions, and are all that are required in the remainder of this section.

Equation (1) is thus transformed to q space via the expansion

$$x(q)=\sum_j f_j(q)x_j \tag{9}$$

and thus

$$x_j=\sum_q f_j(q)x(q)\ .$$

To completely homogenize the system, we must adiabatically eliminate all q modes except one, namely, $x(0)$, i.e.,

$$x(0)=n^{-1/2}\sum_j x_j \tag{10}$$

which is proportional to the total amount of chemical present.

We adiabatically eliminate all other q modes by utilizing the projection-operator technique of paper I. This requires a certain scaling of variables, and as will be later apparent, elimination of q modes is most advantageously carried out in terms of the scaled variables $y(q)$, defined by

$$y(q)=x(q)\sqrt{D},\quad q\neq0\ . \tag{11}$$

Thus, Eqs. (8)—(11) imply

$$x_j=\frac{x}{n}+\frac{z_j}{\sqrt{D}}\ , \tag{12}$$

where $x=\sum_j x_j$ total amount of chemical substance in the system and

$$z_j=\sum_{q\ (\neq0)} f_j(q)y(q)\ .$$

We use Eqs. (9) and (12) in Eq. (1) to write the Fokker-Planck equation in q space as

$$\partial_t P(x,y)=[DL_1+\sqrt{D}L_2(D)+L_3(D)]P(x,y)\ , \tag{13}$$

where

$$L_1=-\sum_{q\,(\neq 0)}\lambda(q)\frac{\partial}{\partial y(q)}y(q)+\sum_j\sum_{\substack{q'\,(\neq 0),\\ q\,(\neq 0)}}\frac{\partial^2}{\partial y(q)\partial y(q')}f_j(q)f_j(q')b\left[\frac{x}{n}\right], \tag{14}$$

$$L_2(D)=\sum_{q\,(\neq 0)}\frac{\partial}{\partial y(q)}\sum f_j(q)a\left[\frac{x}{n}+\frac{z_j}{\sqrt{D}}\right]+\frac{\partial}{\partial x}\sum_j\sum_{q\,(\neq 0)}\frac{\partial}{\partial y(q)}f_j(q)b\left[\frac{x}{n}+\frac{z_j}{\sqrt{D}}\right]$$
$$+\sum_j\sum_{q\,(\neq 0)}\frac{\partial}{\partial y(q)\partial y(q')}f_j(q)f_j(q')b_2\left[\frac{x}{n},\frac{z_j}{\sqrt{D}}\right], \tag{15}$$

where

$$l^{-d}b'(x_j)=b\left[\frac{x}{n}+\frac{z_j}{\sqrt{D}}\right]=b\left[\frac{x}{n}\right]+D^{-1/2}b_2\left[\frac{x}{n},z_j\right]+O(D^{-1})$$

and

$$L_3(D)=\sum_j\left[\frac{\partial}{\partial x}a\left[\frac{x}{n}+\frac{z_j}{\sqrt{D}}\right]+\frac{\partial^2}{\partial x^2}b\left[\frac{x}{n}+\frac{z_j}{\sqrt{D}}\right]\right]. \tag{16}$$

We have written the Fokker-Planck equation in the form of Eq. (13) in order to use the elimination technique discussed in paper I, for the powers of D written in (13) give the leading order of each term as $D\to\infty$. That is, we define the projection operator P,

$$Pf=P_s\int\prod_{q\,(\neq 0)}dy_q f(x,y),$$

where P_s is the solution to the stationary equation

$$L_1P_s=0$$

and f is an arbitrary function.

From Eqs. (14) and (15) we see

$$PL_1=L_1P=0$$

and (17)

$$PL_2P=PL_2=0.$$

Conditions (17) are all that is required to carry out the elimination procedure as described in paper I.

Following such methods, we find in the limit of rapid diffusion, $D\to\infty$, Eq. (13) becomes

$$\frac{\partial P(x)}{\partial t}=\lim_{D\to\infty}\{[\langle L_3(D)\rangle-PL_2(D)L_1^{-1}L_2(D)]P(x)\}. \tag{18a}$$

Here $\langle\cdots\rangle$ is an average over the stationary distribution of L_1. It is readily checked that, since L_2 contains only terms in which $\partial/\partial y(q)$ stands to the left, $PL_2(D)$ vanishes, so that the resultant equation is

$$\frac{\partial P(x)}{\partial t}=\lim_{D\to\infty}[\langle L_3(D)\rangle P(x)]$$
$$=n\left[\frac{\partial}{\partial x}a\left[\frac{x}{n}\right]+\frac{\partial^2}{\partial x^2}b\left[\frac{x}{n}\right]\right]P. \tag{18b}$$

The factor n results from summing over all n cells in the system. Elimination of the inhomogeneous modes results in the global description of Eq. (18b).

At this point, we have the following comments.

(i) Our method of proof has relied on the fact that the coefficient D was much larger than all other system parameters. This requires

$$D\lambda(1)\gg\kappa, \tag{19}$$

where κ is the reaction rate for the chemical reaction we are considering. As $\lambda(1)\sim(\pi/nl)^2$, Eq. (19) becomes

$$\left[\frac{D}{\kappa}\right]^{1/2}\gg\frac{L'}{\pi}, \tag{20}$$

where $L'=nl$ is the total length of the system.

The left-hand side of Eq. (20) represents the root-mean-square distance traveled in the time scale of the reaction. Thus, Eq. (20) shows that the elimination of q modes is valid only if this distance is very much greater than the length of the system—which is equivalent to requiring that diffusion homogenize the system.

(ii) If the Eq. (1) is viewed as a Poisson-representation[16] Fokker-Planck equation, corresponding to a multivariate master equation, then Eq. (18b) is the Poisson-representation Fokker-Planck equation corresponding to a global master equation. We thus recover Arnold's result: That in the the large diffusion limit the local master equation goes over into the global master equation.

III. COARSE-GRAINING OF SPACE IN THE REACTION-DIFFUSION MODEL

We consider now the same model as in Sec. II, but in this case we adiabatically eliminate so as to retain a band of small-q modes. The elimination of the larger-q modes results in a loss of spatial resolution and this is equivalent to a coarse-graining in space.

We start with almost the same Fokker-Planck equation as before, Eq. (1), and assume the simplest form for the diffusion operator D_{jk}, Eq. (2). This time however, we choose eigenfunctions appropriate to periodic boundary conditions, i.e.,

$$f_k(q)=\frac{1}{\sqrt{2n+1}}\exp(ikql), \tag{21}$$

where

$$q=\frac{2r\pi}{(2n+1)l},\quad r=-n,-n+1,\ldots,n \tag{22}$$

and this now corresponds to a system with $(2n+1)$ cells of length l. Thus where we wrote n in Sec. II, we must now write $(2n+1)$.

In order to define the system functions $a(x)$ and $b(x)$ in a way that is invariant under blocking, we define *band-limited polynomials* as linear combinations of quantities which we shall write as $[x_j^r]$, and which are defined by the following procedure. First, we define

$$x(q)=\sum_j f_j^*(q)x_j \tag{23}$$

and then we define

$$[x_j^r]=\sum_{Q,q,q',q'',\ldots}(2n+1)^{-r/2}\exp(ijQl)\times\delta_{q+q'+q''+\cdots,Q}\times x(q)x(q')x(q'')\cdots . \tag{24}$$

At first glance there appears to be no difference between $[x_j^r]$ and x_j^r. There is a difference, however, which occurs in the Kronecker delta, which is zero except when

$$Q=q+q'+q''+\cdots , \tag{25}$$

whereas a similar expression for x_j^r would contain a Kronecker delta which was zero unless

$$Q=q+q'+q''+\cdots+2N\pi/l \tag{26}$$

for any integral $N<r$. We then assume that the drift and diffusion constants are of the form

$$a(x_i)=[x_i^r] ,\qquad b(x_j)=[x_i^s] , \tag{27}$$

and the method for any linear combination of these forms follows by linearity.

There is a physical difference introduced here. The simple monomial x_j^r contains terms in which $q+q'+q''+\cdots$ add up to a value outside the range of possible Q given in Eq. (22). In fact, the use of $[x_j^r]$ is appropriate where the lattice index j is regarded as a discretization of a continuous index—a formulation in terms of a continuous index would give a formula for x_j^r much like that for $[x_j^r]$, but the range of Q,q,q',q'' would be infinite. Discretization is imposed by band limiting of functions, and this then yields $[x_i^r]$ where x_i^r occured in the continuous formulation. We then find that Eq. (2) becomes

$$\sum_j D_{kj}f_j(q)=-D\lambda(q)f_k(q) ,$$

where

$$\lambda(q)=4\sin^2(ql/2)/l^2 . \tag{28}$$

The eigenfunctions have the orthogonality relations

$$\sum_j f_j^*(q)f_j(q')=\delta_{q,q'} ,\qquad \sum_q f_j^*(q)f_k(q)=\delta_{j,k} . \tag{29}$$

We now Fourier expand as

$$x(q)=\sum_j f_j^*(q)x_j ,\qquad x_j=\sum_q f_j(q)x(q) . \tag{30}$$

In this case however, instead of eliminating all modes but one [Eq. (10)] as in Sec. II, we now wish to retain the band of modes

$$x(Q),\quad Q\in R(N)\equiv\{-N,\ldots,N\}\pi/[(2n+1)l] \tag{31}$$

where

$$N\ll n .$$

We note that the diffusion term in Eq. (1) maps to q space as

$$\sum_{j,k}\frac{\partial}{\partial x_k}D_{kj}x_j\rightarrow -D\sum_q\lambda(q)\frac{\partial}{\partial x(q)}x(q) =-D\sum_{q\notin R(N)}\lambda\frac{\partial}{\partial x(q)}x(q) -D\sum_{Q\in R(N)}\lambda(Q)\frac{\partial}{\partial x(Q)}x(Q) . \tag{32}$$

As mentioned in Sec. II, the terms $a(x)$ and $b(x)$ are proportional to the rate constant κ. Thus the correct D dependence of L_3 requires N to be such that

$$\lambda(Q)D\sim\kappa . \tag{33}$$

Using Eq. (23) we find that Eq. (28) implies

$$\frac{N}{n}\sim\left[\frac{\kappa}{D}\right]^{1/2}\frac{l}{\pi} . \tag{34}$$

Again we anticipate the form of L_1, L_2, and L_3 required for elimination of high-q modes and change variables to $y(q)$ defined by

$$x(q)=\frac{y(q)}{\sqrt{D}},\quad q\notin R(N)$$

and $y(q)$ is to be eliminated.

Following the same procedure as in Sec. II we find, in the limit $D\rightarrow\infty$, the reduced Fokker-Planck equation

$$\frac{\partial P}{\partial t}=\left\{\sum_j\sum_Q f_j^*(Q)\frac{\partial}{\partial x(Q)}\left[\left[\sum_{Q'}f_j(Q')x(Q')\right]^r\right]-\sum_Q\lambda'(Q)\frac{\partial}{\partial x(Q)}x(Q) +\sum_j\sum_{Q,Q'}f_j^*(Q)f_j(Q')\frac{\partial^2}{\partial x(Q)\partial x(Q')}\left[\left[\sum_{Q''}f_j(Q'')x(Q'')\right]^s\right]\right\}P , \tag{35}$$

where we have used the explicit form of $a(x)$ and $b(x)$ as defined in Eq. (21), and $\lambda'(Q)\equiv\lambda(Q)D$, hence the term involving $\lambda'(Q)$ is the same order as the other terms.

We now wish to transform Eq. (30) back to a Fokker-Planck equation in coordinate space. The elimination of the large-q modes has effectively mapped the system to a new coarser-grained coordinate space. Consequently, the system is now characterized by a lattice of N, rather than n cells, where $N \ll n$ and $(N/n)\sim O(D^{-1/2})$.

We thus expect to be able to write our Fokker-Planck equation in the new coordinate space in terms of the system variables x_J, where J refers to one of the N (larger) cells. Hence we now consider a new basis, spanned by the eigenvectors

$$\tilde{f}_J(Q)=\frac{1}{\sqrt{2N+1}}\exp(iJQL)\ ,$$
$$[Q\in R(N)] \text{ and } L=\left[\frac{2n+1}{2N+1}\right]l \tag{36}$$

and thus write

$$x_J=\sum_Q \tilde{f}_J(Q)x(Q)\ ,$$
$$x(Q)=\sum_J \tilde{f}^*_J(Q)x_J\ . \tag{37}$$

Noting that the eigenfunctions $\tilde{f}_J(Q)$ obey orthogonality relations similar to those of the $f_j(q)$ [Eqs. (29)] we use Eqs. (37) to transform Eq. (35) to the coarser-grained coordinate space (J space).

Consider first the transport term of the drift; we transform this to J space via the eigenfunctions (37):

$$\sum_Q \frac{\partial}{\partial x(Q)}\lambda'(Q)x(Q)=\sum_Q\sum_J \frac{\partial}{\partial x_J}\tilde{f}_J(Q)\lambda'(Q)\sum_k \tilde{f}^*_k(Q)x_K$$
$$=\sum_{J,K}\frac{\partial}{\partial x_J}D_{J,K}x_K$$
$$\text{where } D_{J,K}=\sum_Q \tilde{f}_J(Q)\tilde{f}^*_K(Q)\lambda'(Q)\ . \tag{38}$$

The nonlinear part of the drift becomes

$$\sum_j\sum_Q f^*_j(Q)\frac{\partial}{\partial x(Q)}\left[\sum_{Q'} f_j(Q')x(Q')\right]^r$$
$$=\sum_{\{Q\}}\frac{\partial}{\partial x(Q)}[x(Q_1)\cdots x(Q_r)(2n+1)]\frac{\delta\left[Q,\sum_{\mu=1}^r Q_\mu\right]}{(2n+1)^{(r+1)/2}}. \tag{39}$$

Inverting to J space [using (37)] gives

$$\sum_{\{Q\}}\sum_{\{J\}}\frac{\partial}{\partial x_J}[x_{J_1}\cdots x_{J_r}\tilde{f}_J(Q)\tilde{f}^*_{J_1}(Q_1)\cdots\tilde{f}^*_{J_r}(Q_r)]$$
$$\times\frac{\delta\left[Q,\sum_{\mu=1}^r Q_\mu\right]}{(2n+1)^{(r-1)/2}}\ . \tag{40}$$

Using the explicit form of the eigenfunctions we find

$$\sum_{\{J\}}\frac{\partial}{\partial x_J}\left\{x_{J_1}\cdots x_{J_r}\sum_{\{Q\}}\exp\left[iL\left[J\sum_\mu^r Q_\mu-\sum_\mu^r J_\mu Q_\mu\right]\right]\right\}\frac{1}{(2n+1)^{(r-1)/2}(2N+1)^{(r+1)/2}}$$
$$\Rightarrow\sum_{\{J\}}\frac{\partial}{\partial x_J}(x_{J_1}\cdots x_{J_r})\frac{1}{(2n+1)^{(r-1)/2}}\frac{1}{(2N+1)^{(r+1)/2}}$$
$$\times\sum_{Q_1}\exp[iLQ_1(J-J_1)]\sum_{Q_2}\exp[iLQ_2(J-J_2)]\cdots\sum_{Q_r}\exp[iLQ_r(J-J_r)]\ . \tag{41}$$

Summing over the Q's gives

$$\sum_J\frac{\partial}{\partial x_J}[x_J^r]\left[\frac{2N+1}{2n+1}\right]^{(r-1)/2}\ . \tag{42}$$

Similarly, we find the diffusion becomes

$$\sum_j\sum_{Q,Q'} f^*_j(Q)f_j(Q')\frac{\partial^2}{\partial x(Q)\partial x(Q')}\left[\sum_{Q''} f_j(Q'')x(Q'')\right]^S$$
$$=\sum_J\left[\frac{2N+1}{2n+1}\right]^{s/2}\frac{\partial^2}{\partial x_J^2}[x_J^s]\ . \tag{43}$$

We thus find the Fokker-Planck equation in J space,

$$\frac{\partial P}{\partial t}=\left[\sum_{J,K}\frac{\partial}{\partial x_K}D_{K,J}x_J+\sum_J\left[\frac{2N+1}{2n+1}\right]^{(r-1)/2}\frac{\partial}{\partial x_J}[x_J^r]\right.$$
$$\left.+\left[\frac{2N+1}{2n+1}\right]^{s/2}\frac{\partial^2}{\partial x_J^2}[x_J^s]\right]P\ . \tag{44}$$

[Note that the range of the Q_μ is restricted by the delta function in (40), to be such that their sum is in $R(N)$.] Using an order of magnitude estimate, we show (see Appendix A)

$$x_J=x\left[\frac{2n+1}{2N+1}\right]^{1/2},$$

where x is the concentration of the chemical substance. Thus, to obtain an equation in terms of the concentration, we scale the variables as

$$\bar{x}_J=x_J\left[\frac{2N+1}{2n+1}\right]^{1/2} \tag{45}$$

which is proportional to the concentration of the chemical substance, and Eq. (33) becomes

$$\frac{\partial P}{\partial t}=\left[\sum_{J,K}\frac{\partial}{\partial\bar{x}_K}D_{KJ}\bar{x}_J+\sum_J\frac{\partial}{\partial\bar{x}_J}[\bar{x}^r_J]\right.$$
$$\left.+\frac{2N+1}{2n+1}\sum_J\frac{\partial^2}{\partial\bar{x}_J^2}[\bar{x}^s_J]\right]P\ , \tag{46}$$

where

$$D_{KJ}=\sum_{Q}\tilde{f}_J^*(Q)\tilde{f}_k(Q)\lambda'(Q)\ .$$

Clearly,

$$\sum_{K}D_{J,K}\tilde{f}_K(Q)=-\lambda'(Q)\tilde{f}_J(Q)\ . \tag{47}$$

Thus, in the limit of rapid diffusion we have eliminated the large-q modes, and thus have transformed to a system defined on a coarser-grained lattice. The elimination procedure is thus equivalent to a blocking of the cells of the original system.

Hence the Fokker-Planck equation (46) describes the same system as does Eq. (1), but in a coarser-grained coordinate system. Note the following.

(a) This so-called blocking leads to an effective reduction in the noise. That is, the diffusion coefficient is multiplied by a factor of N/n which, from Eq. (34), implies the noise is now of order $D^{-1/2}$.

(b) The result of this section gives the basis for the applicability of the cell models of such continuum phenomena, for the reduced Fokker-Planck equation is valid when

$$D\gg\kappa\left[\frac{nl}{N\pi}\right]^2$$

(where κ is again a characteristic reaction rate), i.e., when diffusion homogenizes the distribution within one of the coarse-grained cells, during the characteristic reaction time. Thus although cells of arbitrarily small size may be chosen, the elimination of high-q modes enables a coarser description to be written in essentially the same form as originally written.

(c) There is, however, a difference. The definition of D_{KJ} by (47) gives the same form as D_{jk} only if $\lambda(q)=-q^2$. For any reasonable D_{kj} it turns out that $\lambda(q)$ has this form for small q: thus there is a degree of universality. The situations in which $\lambda(q)$ is not approximately $-q^2$ for small q give rise to the phenomena of Sec. IV.

(d) If the system is originally described in terms of x_j^r, we find that the blocking procedure still gives the result above, i.e., a description where $x_j^r\rightarrow[x_J^r]$. Thus, in blocking, a lattice system approaches a discretized continuum system, a result which is obviously reasonable.

IV. APPLICATION TO HYDRODYNAMIC-LIKE SYSTEMS

We now extend the methods of the previous sections to the study of the kind of fluctuations which occur about the critical point of the convective instability. Concerning the deterministic motion of a fluid heated from below, it is known that near the convective instability, description of all relevant hydrodynamical variables can be made in terms of a slow mode.[4,9] This mode is characterized by a slowly varying amplitude, the behavior of which yields insight into the critical nature of the system.

An approach commonly adopted in such a determination is the method of multiple scales:[4,6,9,10] That is, a perturbative solution to the hydrodynamic equations is gained by assuming an asymptotic expansion in of the system variables U,

$$U=\epsilon^{1/2}U_0(r_1,r_2)+\epsilon U_1(r_1,r_2)+\epsilon^{3/2}U_2(r_1,r_2)+\cdots\ , \tag{48}$$

where r_1 and r_2 are two space scales, and

$$r_2=\epsilon^{1/2}r_1\ . \tag{49}$$

The perturbation parameter ϵ represents the deviation of the Rayleigh number R from its critical value R_c.

A derivation of the slowly varying amplitude equation using this method in the Bénard problem has been presented by Newell and Whitehead[4] and Segel.[6] These approaches have been confined to a deterministic theory.

Graham[10] derived an amplitude equation corresponding to that of Newell and Whitehead, but included as well the effects of fluctuations in his theory. However, in order to formulate such a stochastic theory, Graham found it necessary to introduce noise terms in an essentially *ad hoc* fashion. That is, to reproduce the same deterministic amplitude equations of Newell and Whitehead, Graham includes noise terms only at a certain order in the perturbation expansion. We discuss this point later with regard to a simple model.

In the following subsections we will show the equivalence between Graham's application of the method of multiple scales and the adiabatic elimination procedure discussed in the previous sections. That is, we derive the stochastic amplitude equations via each method, using a simple hydrodynamic-like model.

The model, first introduced by Pomeau and Manneville,[15] represents a simplistic first approximation to a hydrodynamical system. As such it does not describe a realistic fluid, but displays the basic structure apparent in most hydrodynamic models.

We note that an equivalence between procedures is expected, as Cross[5] has derived a deterministic amplitude equation for the Bénard problem by projecting out the slow modes in a manner similar to the adiabatic elimination method.

A. Method of multiple scales

1. *Deterministic amplitude equation*

Consider the following one-dimensional deterministic hydrodynamic model:[15]

$$\partial_t U=[\epsilon-(\partial_r^2+q_0^2)^2]U-U^3\ , \tag{50}$$

where $U(r,t)$ is a real hydrodynamical variable satisfying

$$U(r,t)=\partial_r U(r,t)=0$$

at boundaries. The parameter ϵ is proportional to the deviation $(R-R_c)$, where R_c is the critical Rayleigh number.

Using this deterministic model, Cross *et al.*[9] derive an amplitude equation for the slowly varying mode, using the method of multiple scales. They consider only stationary solutions [i.e., set $\partial_t U=0$ in Eq. (50)] and assume solutions of the form

$$U=A(r_2)e^{iq_0r_1}\,, \tag{51}$$

where $A(r_2)$ is the slowly varying amplitude and $e^{iq_0r_1}$ is the rapidly varying part of the variable. They separate space scales by setting

$$r_2=\epsilon^{1/2}r_1$$

and thus

$$\partial_r\rightarrow\partial_{r_1}+\epsilon^{1/2}\partial_{r_2}\,. \tag{52}$$

The derivative ∂_{r_1} acts only on the rapidly varying functions $e^{iq_0r_1}$. Expanding U as

$$U=\epsilon^{1/2}U_0+\epsilon U_1+\epsilon^{3/2}U_2+\cdots \tag{53}$$

they solve Eq. (50) perturbatively and find the zeroth-order amplitude equation

$$4q_0^2A_0''+A_0-|A_0|^2A_0=0\,, \tag{54}$$

where $U_0=A_0(r_2)e^{iq_0r_1}$. We now wish to derive a similar amplitude equation, but using a stochastic approach—thus it is necessary to introduce fluctuations into the system.

2. *Stochastic amplitude equation*

We introduce an arbitrary noise term into the system as follows: Eq. (36) becomes

$$\partial_tU=[\epsilon-(\partial_r^2+q_0^2)^2]U-U^3+\epsilon^aT(r,t)\,, \tag{55}$$

where $T(r,t)$ is a Gaussian stochastic term satisfying

$$\begin{aligned}&\langle T(r,t)\rangle=0\,,\\&\langle T(r,t)T(r',t)\rangle=\delta(r-r')\delta(t-t')\,.\end{aligned} \tag{56}$$

As yet we have not specified the value of a in Eq. (55), which indicates the size of the noise.

There is no physical reason to attribute any ϵ dependence at all to the noise, for its magnitude is determined solely by the temperature through the statistical mechanical considerations outlined by Landau and Lifschitz.[16] Nevertheless, Graham[10] has introduced such a dependence, with $a=\frac{3}{4}$, since this allows a consistent expansion to be carried out in his study of the Bénard problem. He has been criticized for this[17] but has defended this choice on the grounds that, physically, the noise and ϵ are both small, and such an equivalence can make sense. The logical thing to do in such a situation is to expand in two small parameters: this would yield a perturbative expansion about the deterministic expansion like that of Cross *et al.*[9]

In this paper we wish to compare an adiabatic elimination method with Graham's method, and shall therefore preserve the ϵ^a dependence. When such an ϵ dependence is *not* assumed, we are lead to the study of nonclassical phenomena, which will be dealt with in another paper.

We assume the same expansion and scaling as Cross *et al.*, i.e., Eqs. (38) and (39). However, as we are now interested in time-dependent solutions, we must also incorporate the time scaling

$$\tau=\epsilon t\,, \tag{57}$$

a similar time scaling was adopted by Graham.[10]

With the use of Eqs. (53) and (57), the left-hand side of Eq. (55) becomes

$$\epsilon\frac{\partial}{\partial\tau}(\epsilon^{1/2}U_0+\epsilon U_1+\epsilon^{3/2}U_2+\cdots)\,. \tag{58}$$

Equation (58) shows that we must include a noise term into the system, at least to order $\epsilon^{3/2}$, in order to produce the correct time dependence in the problem. Noting also that the space and time scaling, Eqs. (52) and (57), will scale the magnitude of the noise (see Appendix B), we find the appropriate value of a is $\frac{3}{4}$. Thus Eq. (41) becomes

$$\partial_\tau U=[\epsilon-(\partial_r^2+q_0^2)^2]U-U^3+\epsilon^{3/4}T(r,t)\,. \tag{59}$$

Using Eqs. (52), (53), and (57) in Eq. (58) we find the following to order $\epsilon^{3/2}$. For $O(\epsilon^{1/2})$ we get

$$(\partial_{r_1}^2+q_0^2)^2U_0=0\,, \tag{60}$$

for $O(\epsilon)$

$$(\partial_{r_1}^2+q_0^2)^2U_1+4\partial_{r_1}\partial_{r_2}(\partial_{r_1}^2+q_0^2)U_0=0\,, \tag{61}$$

and for $O(\epsilon^{3/2})$

$$\begin{aligned}\partial_\tau U_0=&-(\partial_{r_1}^2+q_0^2)^2U_2-4\partial_{r_1}\partial_{r_2}(\partial_{r_1}^2+q_0^2)U_1\\&-(6\partial_{r_1}^2\partial_{r_2}^2+2q_0^2\partial_{r_2}^2+U_0^2-1)U_0\\&+T(r_2,\tau)\,.\end{aligned} \tag{62}$$

Equation (60) is solved by setting

$$U_1=\frac{1}{\sqrt{3}}A_0(r_2,\tau)e^{iq_0r_1}+\text{c.c.} \tag{63}$$

Substituting Eq. (63) in Eq. (61), we find Eq. (61) is solved by setting

$$U_1=\frac{1}{\sqrt{3}}A_1(r_2,\tau)e^{iq_0r_1}+\text{c.c.} \tag{64}$$

Equation (62) is solved by setting

$$U_2=\frac{1}{\sqrt{3}}[A_2(r_2,\tau)e^{iq_0r_1}+B_2(r_2,\tau)e^{3iq_0r_1}+\text{c.c.}]\,. \tag{65}$$

Then, requiring that terms proportional to $e^{3iq_0r_1}$ sum to zero, Eq. (62) becomes

$$\begin{aligned}&\partial_\tau(A_0e^{iq_0r_1}+\text{c.c.})\\&\quad=e^{iq_0r_1}[4q_0^2A_0''-|A_0|^2A_0+A_0+\tilde{T}(r_2,\tau)]\\&\qquad+\text{c.c.}\,,\end{aligned} \tag{66}$$

where we have defined

$$T(r_2,\tau)=\frac{1}{\sqrt{3}}[\tilde{T}(r_2,\tau)e^{iq_0r_1}+\tilde{T}^*(r_2,\tau)e^{-iq_0r_1}]\,. \tag{67}$$

We see that in the deterministic limit (where the left-hand side equals 0 and $\tilde{\tau}=0$) Eq. (66) is identical to the amplitude equation of Cross *et al.*, Eq. (54). In Sec. IV B we show how adiabatic elimination of large-q modes corresponds to the method of multiple scales, as described in this subsection.

B. Adiabatic elimination in hydrodynamical systems

We use the same model as in the previous method, i.e., Eq. (59), and thus include the same noise term as required for the perturbative analysis. We define a new variable v by

$$U=\epsilon^{1/2}v$$

which corresponds to the first-order term in the perturbation expansion of Sec. IV A. Thus Eq. (44) becomes

$$\partial_t v=\epsilon(v-v^3)-(\partial_{r_1}^2+q_0^2)^2 v+\epsilon^{1/4}T(r_1,t)\ . \tag{68}$$

Dividing the system into n cells and setting $t=\tau/\epsilon$ as before, Eq. (51) corresponds to the cell-model Fokker-Planck equation

$$\frac{\partial P}{\partial\tau}=\left[\frac{1}{\epsilon}\sum_{k,j}\frac{\partial}{\partial v_k}D_{kj}v_j-\sum_k\frac{\partial}{\partial v_k}a(v_k)+\epsilon^{-1/2}\sum_k\frac{\partial^2}{\partial v_k^2}\right]P\ , \tag{69}$$

where v_k describes the system variable in the kth cell, D_{kj} is the difference operator describing the term $(\partial_{r_1}^2+q_0^2)^2$ in Eq. (68), and again we use the band-limited polynomial form

$$a(v_k)=v_k-[v_k^3]\ . \tag{70}$$

Previously, we used the method of multiple scales to project out the fast hydrodynamic modes, and thus obtained an equation of motion for the amplitude of the slowly varying mode. We can achieve the same end by the projector-operator technique to eliminate the fast (large-q) modes and thus map the system to J (r_2) space.

Once again we transform the Fokker-Planck equation to q space by defining the eigenfunctions $f_k(q)$ of D_{kj} as in Eq. (21). However, the eigenvalues now have the form

$$\lambda(q)=\left[\frac{4\sin^2\left[\frac{ql}{2}\right]}{l^2}-q_0^2\right]^2\ . \tag{71}$$

We transform Eq. (69) to q space via the expansion

$$v_j=\sum_q f_k(q)v(q)\ ,\qquad v(q)=\sum_j f_j^*(q)v_j\ . \tag{72}$$

As in Sec. III, we wish to eliminate all q modes except the $v(Q)$ where the Q satisfy

$$\epsilon^{-1}\lambda(Q)\sim 1 \tag{73}$$

and so produce the correct form for L_3.

Equations (71) and (73) thus imply Q has the range

$$R=\left[\pm\tilde{q}_0+\frac{2\pi}{(2n+1)l}(-N,-N+1,\ldots,N)\right], \tag{74}$$

where

$$\left[\frac{2N+1}{2n+1}\right]\sim\sqrt{\epsilon}$$

and

$$\tilde{q}_0=\frac{2}{l}\sin^{-1}\left[\frac{q_0 l}{2}\right]\simeq q_0\ \text{ for } l \text{ small}\ . \tag{75}$$

To enable the elimination of all other modes we set

$$v(q)=y(q)\epsilon^{1/4},\quad q\notin R\ . \tag{76}$$

Hence writing

$$v_j=\sum_Q f_j(Q)v(Q)+\epsilon^{1/4}w_j\ , \tag{77}$$

where

$$w_j=\sum_{q\notin R}f_j(q)y(q)\ , \tag{78}$$

the Fokker-Planck equation (69) becomes

$$\frac{\partial P}{\partial\tau}=(\epsilon^{-1}L_1+\epsilon^{-3/4}L_2+\epsilon^{-1/4}L_2'+\epsilon^{-1/4}L_2''+\epsilon^{-1/2}L_3)P\ , \tag{79}$$

where

$$L_1=-\sum_{q\notin R}\lambda(q)\frac{\partial}{\partial y(q)}y(q)+\sum_k\sum_{q,q'\notin R}f_k^*(q)f_k^*(q')\frac{\partial^2}{\partial y(q)\partial y(q')}\ , \tag{80}$$

$$L_2=2\sum_k\sum_{q\notin R}\sum_{Q'}f_k^*(q)\frac{\partial}{\partial y(q)}f_k(Q')\frac{\partial}{\partial v(Q')}\ , \tag{81}$$

$$L_2'=\sum_k\sum_{q\notin R}f_k^*(q)\frac{\partial}{\partial y(q)}\times a\left[\sum_Q f_k(Q)v(Q)+\epsilon^{1/4}w_k\right], \tag{82}$$

$$L_2''=\sum_k\sum_Q f_k^*(Q)\frac{\partial}{\partial v(Q)}\times a_2\left[\sum_Q f_k(Q)v(Q),\epsilon^{1/4}w_k\right], \tag{83}$$

and

$$L_3=-\sum_Q\lambda'(Q)\frac{\partial}{\partial v(Q)}v(Q)+\sum_k\sum_Q f_k^*(Q)\frac{\partial}{\partial v(Q)}a\left[\sum_{Q'}f_k(Q')v(Q')\right]+\epsilon^{-1/2}\sum_k\sum_{Q',Q}f_k^*(Q)f_k^*(Q')\frac{\partial^2}{\partial v(Q)\partial v(Q')}\ , \tag{84}$$

where

$$a\left[\sum_Q f_k(Q)v(Q)+\epsilon^{1/4}w_k\right]$$

$$=a\left[\sum_Q f_k(Q)v(Q)\right]+\epsilon^{1/4}a_2\left[\sum_Q f_k(Q)v(Q),\epsilon^{1/4}w_k\right] \tag{85}$$

and

$$\lambda'(Q)\equiv\frac{\lambda(Q)}{\epsilon}\simeq 4q_0^2Q^2/\epsilon\ . \tag{86}$$

We define the projection operator

$$P'f=P_s\int\prod_{q\notin R}dy(q)f(y)\ , \tag{87}$$

where P_s is the solution of $L_1P_s=0$.

Thus

(i) $L_1P'=0$;

(ii) as L_2 and L_2' begin with derivatives of $y(q)$, $P'L_2=P'L_2'=0$;

(iii) $[P',L_3]=0$;

and Eqs. (83) and (84) show that L_2'' is $O(\epsilon^{-1/4})$ to highest order.

Conditions (i)—(iii) are all that is required to carry out the elimination procedure, and in the limit $\epsilon\to 0$ we find the Fokker-Planck equation becomes

$$\frac{\partial P}{\partial\tau}=\left[-\sum_Q\lambda'(Q)\frac{\partial}{\partial v(Q)}v(Q)+\sum_k\sum_Q f_k^*(Q)\frac{\partial}{\partial v(Q)}a\left[\sum f_k(Q)v(Q)\right]+\epsilon^{-1/2}\sum_k\sum_{Q,Q'}f_k^*(Q)f_k^*(Q')\frac{\partial^2}{\partial v(Q)\partial v(Q')}\right]P\ . \tag{88}$$

We now wish to invert this equation and rewrite it as a Fokker-Planck equation in J space.

First, we note that Q has the range

$$\left[\pm\tilde{q}_0+\frac{2\pi}{(2n+1)l}(-N,-N+1,\ldots,N)\right]$$

and thus we must consider both the positive and negative regions of the range. Hence we define,

$$f_k^{\pm}(Q)=\frac{1}{\sqrt{2n+1}}\exp[ikl(\pm\tilde{q}_0+\tilde{Q})]\ , \tag{89}$$

where $\tilde{Q}$ has the range

$$[2\pi/(2n+1)l](-N,-N+1,\ldots,N)\ .$$

We transform Eq. (62) to r_2 space (J space) in a manner similar to that followed in Sec. III. The transport part of the drift transforms as before [i.e., Eq. (33a)].

Consider now the remaining drift terms. With the use of Eq. (63) and defining

$$v^{\pm}(Q)=\sum_k f_k^{*\pm}(Q)v_k$$

the cubic drift term becomes

$$\sum_k\sum_{\{Q\}}f_k^*(Q)\frac{\partial}{\partial v(Q)}f_k(Q_1)f_k(Q_2)f_k(Q_3)v(Q_1)v(Q_2)v(Q_3)$$

$$\to\sum P\left[\sum_k\sum_{\{\underline{Q}\}}\frac{\partial}{\partial v^+(Q)}f_k^{*+}(Q)f_k^{\pm}(Q_1)f_k^{\pm}(Q_2)f_k^{\pm}(Q_3)v^{\pm}(Q_1)v^{\pm}(Q_2)v^{\pm}(Q_3)\right.$$

$$\left.+\sum_k\sum_{\{\underline{Q}\}}\frac{\partial}{\partial v^-(Q)}f_k^{*-}(Q)f_k^{\pm}(Q_1)f_k^{\pm}(Q_2)f_k^{\pm}(Q_3)v^{\pm}(Q_1)v^{\pm}(Q_2)v^{\pm}(Q_3)\right]\ , \tag{90}$$

where $\sum P$ denotes the sum of all the different arrangements of the $v^+(Q)$, $v^-(Q)$, etc.

Consider the first term of Eq. (64). The sum over k will generate all possible arrangements of the $(+,-)$ in the following:

$$\sum_k\exp\{ikl([-\tilde{q}_0-\tilde{Q}]+[\pm\tilde{q}_0+\tilde{Q}_1]+[\pm\tilde{q}_0+\tilde{Q}_2]+[\pm\tilde{q}_0+\tilde{Q}_3])\}\ .$$

Because of the values the k and Q can assume, this expression vanishes unless the argument of the exponential itself vanishes. The range of the Q is very much smaller than that of $\tilde{q}_0$. For sufficiently large q_0 the vanishing of the argument is only possible for those arrangements of the $(+,-)$ in which all the $\tilde{q}_0$ cancel out. This corresponds to the term

$$v^+(Q_1)v^+(Q_2)v^-(Q_3)\ .$$

Thus Eq. (90) becomes

$$\sum_{\{\underline{Q}\}}\left[\frac{\partial}{\partial v^+(Q)}v^+(Q_1)v^+(Q_2)v^-(Q_3)(2n+1)^{-1}\delta(\tilde{Q},\tilde{Q}_1+\tilde{Q}_2+\tilde{Q}_3)\right.$$

$$\left.+\frac{\partial}{\partial v^-(Q)}v^-(Q_1)v^-(Q_2)v^+(Q_3)(2n+1)^{-1}\delta(\tilde{Q},\tilde{Q}_1+\tilde{Q}_2+\tilde{Q}_3)\right]\ . \tag{91}$$

As in Sec. III, we introduce a coarse-grained J space in which

$$v_J^{\pm}=\sum_Q \tilde{f}_J^{\pm}(Q)v^{\pm}(Q)\ , \tag{92}$$

$$v^{\pm}(Q)=\sum_J \tilde{f}_J^{*\pm}(Q)v_J^{\pm}\ , \tag{93}$$

where

$$\tilde{f}_J^{\pm}(Q)=\frac{1}{\sqrt{2N+1}}\exp[iJL(\pm\tilde{q}_0+\tilde{Q})]\ . \tag{94}$$

We thus invert Eq. (91) to J space, following a similar procedure to that of Sec. III, to find that Eq. (62) becomes

$$\frac{\partial P}{\partial\tau}=\left[\sum_{J,K}\frac{\partial}{\partial\tilde{v}_J^{+}}D_{J,k}\tilde{v}_k^{+}+\sum_J\frac{\partial}{\partial\tilde{v}_J^{+}}(\tilde{v}_J^{+}+\tilde{v}_J^{+}v_J^{+}v_J^{-})+\frac{2N+1}{2n+1}\epsilon^{-1/2}\sum_J\frac{\partial^2}{\partial\tilde{v}_J^{+}\partial\tilde{v}_J^{-}}+(+\leftrightarrow-)\right]P\ , \tag{95}$$

where, as in Sec. III, we have scaled the variables as

$$\tilde{v}_J^{\pm}=v_J^{\pm}\left[\frac{2N+1}{2n+1}\right]^{1/2}\equiv c\ ,$$

where c is the concentration, and

$$D_{J,K}=\sum_Q \tilde{f}_J(Q)\tilde{f}_K^{*}(Q)\lambda'(Q)\ .$$

Equations (74) and (75) indicate that $N/n\sim\epsilon^{1/2}$ and so the noise term in Eq. (95) has no ϵ dependence.

Thus the stochastic differential equations corresponding to Eq. (95) are

$$\partial_\tau v^{\pm}(r_2,\tau)=-Lv^{\pm}+(v^{\pm}-v^{\pm}v^{\pm}v^{\mp})+T^{\pm}(r_2,\tau)\ , \tag{96}$$

where r_2 is the coordinate of J space and L is the operator corresponding to $D_{J,k}$ in Eq. (95). We note that in Q space

$$-Lv=\lambda'(Q)v(Q)\sim 4q_0^2\frac{\partial^2}{\partial r_2^2}v(Q)\ .$$

Thus Eq. (96) represents essentially the same amplitude equation as that predicted by the method of multiple scales, Eq. (66). We have therefore demonstrated the equivalence between the methods of Secs. IV A and IV B.

V. DISCUSSION

In this paper we have demonstrated the procedure of eliminating high-q Fourier modes in spatially distributed systems in the limit of rapid diffusion. The adiabatic elimination techniques used were those described by Gardiner.[3,11]

This elimination in q space corresponds to an effective loss of spatial resolution in r space. Thus we have systematically formalized the method of coarse-graining or "cell blocking" in coordinate-space. Physically, coarse-graining follows as in the limit of rapid diffusion, the system tends to homogenize.

Complete homogenization was discussed in Sec. II, and in Sec. III we showed that partial homogenization resulted in mapping the system to a coarser-grained lattice, with an effective reduction in system noise. In fact, given that the coarse-graining procedure maps the system from a lattice of n cells to one consisting of a lattice of N, larger cells, the noise term in the Fokker-Planck equation is reduced by a factor of $N/n\sim O(D^{-1/2})$, where D is proportional to the strength of the diffusion process.

Analysis also showed that this coarse-graining procedure was equivalent to the method of multiple scales used in hydrodynamical systems to derive amplitude equations for slowly varying system variables. In this case however, blocking of cells did not follow from rapid diffusion but instead from the asymptotic behavior of the system around the point of convective instability.

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APPENDIX A: RELATION BETWEEN CONCENTRATION OF CHEMICAL SUBSTANCE AND SYSTEM VARIABLES

We note the relationship

$$x_J=\sum_Q \tilde{f}_J(Q)x(Q)\ . \tag{A1}$$

With the use of Eq. (25) this becomes

$$x_J=\sum_Q \tilde{f}_J(Q)\sum_j f_j^{*}(Q)x_j\ . \tag{A2}$$

We now assume x_j is a constant, that is x_j is equivalent to the concentration of chemical substance in the system. Thus we set

$$x_j=x=c\ ,$$

where c is the concentration. Hence (A2) gives

$$x_J=x\sum_Q\frac{1}{\sqrt{2n+1}}\exp(iJQL)\sum_j\frac{1}{\sqrt{2n+1}}\exp(-ijQl) \tag{A3}$$

[using Eqs. (22) and (31)] and becomes

$$x_J=\frac{x}{\sqrt{(2n+1)(2N+1)}}\sum_Q\exp(iJQl)2n\delta_{Q,0}=x\left[\frac{2n+1}{2N+1}\right]^{1/2}\ . \tag{A4}$$

Thus we scale the variables as

$$\bar{x}_J=x_J\left[\frac{2N+1}{2n+1}\right]^{1/2} \tag{A5}$$

and $\bar{x}_j$ is then proportional to the concentration of chemical substance in the system.

APPENDIX B: SCALING OF NOISE TERM

Consider the noise term from Eq. (41),

$$\epsilon^a T(r,t) ,$$

where, at present, the value of a is undetermined. We assume the scaling

$$r_2=\epsilon^{1/2}r ,$$

$$\tau=\epsilon t ,$$

thus from Eq. (42) we find

$$\langle \epsilon^a T(r,t)\epsilon^a T(r',t')\rangle = \left\langle \epsilon^a T\left[\frac{r_2}{\epsilon^{1/2}},\frac{\tau}{\epsilon}\right]\epsilon^a T\left[\frac{r_2'}{\epsilon^{1/2}},\frac{\tau'}{\epsilon}\right]\right\rangle = \epsilon^{2a}\delta\left[\frac{r_2}{\epsilon^{1/2}}-\frac{r_2'}{\epsilon^{1/2}}\right]\delta\left[\frac{\tau}{\epsilon}-\frac{\tau'}{\epsilon}\right]$$

$$= \epsilon^{2a}\epsilon^{3/2}\delta(r_2-r_2')\delta(\tau-\tau') = \langle \epsilon^{a+3/4}T(r_2,\tau)\epsilon^{a+3/4}T(r_2',\tau')\rangle$$

$$\Longrightarrow \epsilon^a T(r,t) \Longrightarrow \epsilon^{a+3/4}T(r_2,\tau) .$$

Hence, as we require noise at least to order $\epsilon^{3/2}$ in this problem, we must set $a=\frac{3}{4}$.

[1]Y. Kuramoto, Prog. Theor. Phys. 52, 711 (1974).
[2]L. Arnold, in *Dynamics of Synergetic Systems,* edited by H. Haken (Springer, Berlin and Heidelberg, 1980).
[3]C. W. Gardiner, preceding paper, Phys. Rev. A 29, 2814 (1984).
[4]A Newell and J. Whitehead, J. Fluid Mech. 38, 279 (1969).
[5]M. C. Cross, Phys. Fluids 23, 1727 (1980).
[6]L. A. Segel, J. Fluid Mech. 38, 203 (1969).
[7]J. Swift and P. C. Hohenberg, Phys. Rev. A 15, 319 (1977).
[8]A. Schluter, D. Lortz, and F. Busse, J. Fluid Mech. 23, 129 (1965).
[9]M. C. Cross, P. G. Daniels, P. C. Hohenberg, and E. D. Siggia (unpublished).
[10]R. Graham, Phys. Rev. A 10, 1762 (1974).
[11]C. W. Gardiner, *A Handbook of Stochastic Methods for Physics, Chemistry and Natural Sciences* (Springer, Berlin and Heidelberg, 1983).
[12]S. Grossmann, J. Chem. Phys. 65, 2007 (1976).
[13]C. W. Gardiner, J. Stat. Phys. 15, 451 (1976).
[14]C. W. Gardiner and S. Chaturvedi, J. Stat. Phys. 17, 429 (1977).
[15]Y. Pomeau and P. Manneville, Phys. Lett. 75A, 296 (1980).
[16]L. D. Landau and E. M. Lifschitz, *Fluid Mechanics* (Pergamon, London, 1959).
[17]A. Nitzan, Phys. Rev. A 17, 1513 (1978).