PHYSICAL REVIEW A VOLUME 29, NUMBER 5 MAY 1984

Hydrodynamic stability of rotational gravity waves

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(Received 7 October 1983)

We generalize the Arnold functional to discuss Lyapunov stability of rotational water waves.

Usually gravity waves are studied through the velocity potential $\phi(x,y,z,t)$, i.e., by considering just the case of irrotational flows.[1–4] Recently the introduction of the stream function $\Psi(x,z,t)$ was shown to be rather useful in the study of two-dimensional motion of rotational flows.[5–9] In this paper we study gravity-wave motion, i.e., the two-dimensional divergence-free motion of an incompressible fluid bounded by a free surface $\eta(x,t)$ using the stream function $\Psi(x,z,t)$. We verify the hydrodynamic stability in the sense of Lyapunov for rotational gravity waves by using a generalization of a theorem of Arnold.[10,11]

Let us consider the vertical plane with coordinates x,z, where we define the velocity

$$\underline{v}\equiv(u,w)\equiv(-\partial_z\Psi(x,z,t);\partial_x\Psi(x,z,t)) .$$

The Euler equation can be written

$$\partial_t\nabla^2\Psi+\partial_x\Psi\partial_z\nabla^2\Psi-\partial_z\Psi\partial_x\nabla^2\Psi\equiv\partial_t q+J(\psi,q)=0 , \quad (1)$$

where $q\equiv\nabla^2\Psi$ represents the vorticity of the fluid. On the air-sea surface $z=\eta(x,t)$ the fluid must satisfy[12]

$$\tfrac{1}{2}\,|\,\underline{v}\,|^2+p_{\text{atm}}+gz\,|_{z=\eta}=\tfrac{1}{2}(\nabla\psi)^2+p_{\text{atm}}+gz\,|_{z=\eta} = \text{const} . \quad (2)$$

We will discuss the case of periodic traveling waves

$$\Psi(x,z,t)=\tilde{\Psi}(x+ct,z);\quad \eta(x,t)=\tilde{\eta}(x+ct) ,$$

where c is the constant phase velocity. Using the Galilean transformation

$$z\to z' ,$$
$$t\to t' ,$$
$$x\to x'+ct' ,$$

we may obtain the stationary case. Disregarding the primes in the above formulas, we obtain

$$\tilde{\Psi}\to\overline{\Psi}(x,z) ,$$
$$\tilde{\eta}\to\overline{\eta}(x) ,$$

which must obey

$$\overline{q}\equiv\nabla^2\overline{\Psi}(x,z)=F(\overline{\Psi}(x,z)) , \quad (3a)$$

$F(\cdots)$ being some regular function, and

$$\tfrac{1}{2}[\nabla\overline{\Psi}(x,z)]^2+g\overline{\eta}(x)=f(\overline{\Psi})= \text{const} , \quad (3b)$$

$$\overline{\Psi}=\text{const}' \quad (3c)$$

on the streamline $z=\overline{\eta}$ which corresponds to the air-sea surface and with

$$\overline{\Psi}= \text{const}'' \quad (3d)$$

on the bottom which is located at $z\to-\infty$.

We will show that for solutions of (3) in which $F(\cdots)$ is an increasing function, the configuration $\{\Psi,\eta\}$ is Lyapunov stable. We introduce the Lyapunov functional H (Ref. 13):

$$H(\Psi,\eta)=\int\int_{-\infty}^{\eta}[\tfrac{1}{2}(\nabla\Psi)^2+\Phi(\nabla^2\Psi)+gz]dx\,dz , \quad (4)$$

where $\Phi(\cdots)$ is a function to be determined. Let us check the following three properties of the functional.

(a) $\frac{d}{dt}H(\Psi,\eta)=0$,

(b) $\Delta H\equiv H(\overline{\Psi}+\delta,\overline{\eta}+\epsilon)-H(\overline{\Psi},\overline{\eta})=0$

to first order in $\{\delta(x,z,t),\epsilon(x,t)\}$:

(c) $\Delta^2H(\overline{\Psi},\overline{\eta})>0$.

Later we will prove that if (a), (b), and (c) hold, then we have the Lyapunov stability of the stationary configuration $(\overline{\Psi},\overline{\eta})$.

It is easy to prove that (a) holds if we write (4) as follows:

$$H=\int\int_{-\infty}^{\eta}[\tfrac{1}{2}(\nabla\Psi)^2+gz]dx\,dz+\int\int_{-\infty}^{\eta}\Phi(q)dx\,dz . \quad (5)$$

The first integral is the total energy of the system; the second integral is also a conserved quantity.

Let us prove (b) by considering the variation of H for $\overline{\Psi}(x,z)\to\overline{\Psi}(x,z)+\delta(x,z,t)$, we have

$$\begin{aligned}\Delta_\Psi H&\equiv H(\overline{\Psi}+\delta,\overline{\eta})-H(\overline{\Psi},\overline{\eta})\\&=\int\int_{-\infty}^{\overline{\eta}}\left[(\vec{\nabla}\overline{\Psi}\cdot\vec{\nabla}\delta)+\frac{d}{dq}\Phi(q)\bigg|_{q=\overline{q}}\delta q\right]dx\,dz\\&=-\int\int_{-\infty}^{\overline{\eta}}\left[\overline{\Psi}-\frac{d}{dq}\Phi(q)\bigg|_{q=\overline{q}}\right]\delta q\,dx\,dz\\&\quad+\int_{\text{boundary}}\overline{\Psi}\vec{\nabla}\delta\cdot d\vec{s} ,\end{aligned}$$

where $q=\nabla^2\Psi$, $\delta q=\nabla^2\delta$. Since $\delta=0$ on the boundary (see Whitham, Ref. 3, p. 435), it follows that

$$\Delta_\Psi H=\int\int_{-\infty}^{\bar\eta}\left[(\vec\nabla\Psi\cdot\vec\nabla\delta)+\frac{d}{dq}\Phi(q)\Big|_{q=\bar q}\delta q\right]dx\,dz$$
$$=-\int\int_{-\infty}^{\bar\eta}\delta q\left[\bar\Psi-\frac{d}{dq}\Phi(q)\Big|_{q=\bar q}\right]dx\,dz\ . \qquad (6)$$

If we choose Φ such that

$$\frac{d}{dq}\Phi=F^{-1}(q)\ ,$$

[where $F(\cdots)$ is defined by (3)], then we have the second integral zero. Clearly $F(\cdots)$ must be an invertible function, hence irrotational flows are excluded from this analysis.

For variations of the boundary $\bar\eta(x)\to\bar\eta(x)+\epsilon(x,t)$ it is easy to prove that

$$\Delta_\eta H\equiv H(\bar\Psi,\bar\eta+\epsilon)-H(\bar\Psi,\bar\eta)$$
$$=\int\epsilon\{[\tfrac{1}{2}(\vec\nabla\bar\Psi)^2+gz]+\Phi(\bar q)\}_{z=\eta}dx\ . \qquad (7)$$

The term inside the curly brackets is constant along the streamline $z=\eta$; hence it is possible to write from the Bernoulli equation and from the conservation of vorticity

$$\Delta_\eta H=\{[\tfrac{1}{2}(\nabla\bar\Psi)^2+gz]+\Phi(q)\}_{z=\eta}\int\epsilon\,dx\ .$$

If we consider only perturbations that conserve the mass, we have $\int\epsilon\,dx=0$, hence $\Delta_\eta H=0$.

Finally we discuss property (c) by evaluating the second variation of $H(\bar\Psi,\eta)$. For $\bar\Psi\to\bar\Psi+\delta$ from Eq. (6) we have

$$\text{(i)}\ \ \Delta_\Psi(\Delta_\Psi H)=\int\int_{-\infty}^{\bar\eta}\left[(\vec\nabla\delta)^2+\frac{d^2}{dq^2}\Phi(q)\Big|_{q=\bar q}(\delta q)^2\right]dx\,dz\ .$$

For $\bar\eta\to\bar\eta+\epsilon$, from Eq. (6) we have

$$\text{(ii)}\ \ \Delta_\eta(\Delta_\Psi H)=\int\left[\vec\nabla\bar\Psi\cdot\vec\nabla\delta+\frac{d}{dq}\Phi(q)\Big|_{q=\bar q}\delta_q\right]\epsilon\,dx=\int\left[\bar\Psi-\frac{d}{dq}\Phi(q)\Big|_{q=\bar q}\right]\delta q\,\epsilon\,dx\ .$$

For $\bar\Psi\to\bar\Psi+\delta$, from Eq. (7) we have

$$\text{(iii)}\ \ \Delta_\Psi(\Delta_\eta H)=\int\left[\Delta\bar\Psi\cdot\nabla\delta+\frac{d}{dq}\Phi(q)\Big|_{q=\bar q}\delta q\right]\epsilon\,dx\ .$$

For $\bar\eta\to\bar\eta+\epsilon$, from Eq. (7) we have

$$\text{(iv)}\ \ \Delta_\eta(\Delta_\eta H)=\tfrac{1}{2}g\int\epsilon^2dx\ .$$

From the Euler equation we see that (ii) and (iii) are zero. The integral (iv) is positive, hence by inspection of the expression (i) we see that property (c) holds if

$$\frac{d^2}{dq^2}\Phi(q)\equiv\frac{d}{dq}F^{-1}(q)\equiv\left[\frac{d}{d\Psi}F(\Psi)\right]^{-1}>0\ .$$

This is the same as requiring that $F(\cdots)$ is an increasing function. In the simplest case when $F(\Psi)$ is linear, Eq. (3) shows that this condition implies the vertical length scale is less than the horizontal one.

As a consequence of the above three properties of H, we will now show the Lyapunov stability of the system defined by $\bar\Psi$ and $\bar\eta$. Let us consider the perturbed initial system defined by $(\bar\Psi+\delta;\bar\eta+\epsilon)$. Because of property (a) we have that H will remain constant regardless of the time evolution of the initial system. Now if we assume that the perturbation increased with the time, then it would follow that H must increase, since the configuration defined by $(\bar\Psi,\bar\eta)$ represents a minimum of H [properties (b) and (c)]. From this contradiction we can deduce that the perturbation cannot increase with time and hence the system is stable. Finally, if H has a relative minimum, the initial perturbation has to be sufficiently small; if H has an absolute minimum, larger classes of initial perturbations are stable. Internal waves, nonperiodical surface waves, and irrotational waves will be discussed in a subsequent paper.

We would like to thank Dr. Gibbons, Dr. Benzi, Dr. Purini, and Professor Zirilli for reading the manuscript.

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