# Diffusion on the Sierpiński gaskets: A random walker on a fractally structured object

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The motion of a random walker on a Sierpiński gasket is studied in $d$ dimensions using a renormalization-group approach. The relation $\tilde{\delta}=\bar{d}-2+w$ is established from the behavior of recursion relations and is verified in numerical studies (here $\bar{d}$ is the fractal dimension, $w$ describes coupling-constant scaling, and $\tilde{\delta}$ describes the distance-time relationship of a random walk). When this relation is combined with $\tilde{t}=d-2+\tilde{\zeta}$, an Einstein relation is derived.

## I. INTRODUCTION

Because of the mapping of a variety of physical problems onto percolation models, the structure of percolation clusters is a subject of considerable interest. Investigation of the dynamics of these systems involves the question of particle motion on a cluster, diffusive motion, motion driven by a field, etc. At or near the percolation threshold the clusters are believed to be structured like objects that are properly characterized as fractals;[1] e.g., the Sierpiński gasket. Thus the investigation of particle motion at threshold involves the assessment of the behavior of a random walker, of a current, etc. on a fractally structured object. An appreciation for these points has led to the development of a number of results for this and related problems. Gefen *et al.*[1] and Alexander and Orbach[2] have emphasized scaling relationships among the parameters that characterize the structure, the conductivity, etc. Ben-Avraham and Havlin[3] have explored these relationships with theoretical and numerical studies.

The purpose of this paper is to describe the motion of a random walker on the Sierpiński gaskets using a renormalization-group treatment of the appropriate diffusion equations.[4–6] In Sec. II the length-scale renormalization (LSR) procedure for handling the diffusion equation is reviewed and extended to describe the Sierpiński gaskets. From the form of the resulting recursion relations the relationship $\tilde{\delta}=\bar{d}-2+w$ is established and the Einstein relation is derived. In Sec. III the results of numerical implementation of the recursion relations appropriate to the Sierpiński gaskets in $d$ dimensions are discussed. The relationship $\tilde{\delta}=\bar{d}-2+w$ is confirmed and the physical content of the procedure is emphasized.

## II. LSR AND $\tilde{\delta}=\bar{d}-2+w$

In this section the behavior of the single-site probability is examined using the LSR procedure and the relationship $\tilde{\delta}=\bar{d}-2+w$ established. We begin by recalling some of the properties of the LSR procedure when applied to a simple one-dimensional ($1d$).[6]

Consider the motion of a particle on a uniform $1d$ lattice. This motion is described by

$$\frac{\partial}{\partial t}P_n(t)=-VP_n(t)+W[P_{n+1}(t)+P_{n-1}(t)] , \quad (1)$$

where $P_n(t)$ is the conditional probability that the particle is at site $n$ at time $t$ given that it was at site 0 at $t=0$, $P_n(0)=\delta_{n,0}$. $V$ and $W$ are the rates at which a particle leaves a site and comes onto a site (from the near-neighbor sites), respectively, usually $V=2W$. Upon Laplace transforming Eq. (1) we have

$$s\hat{P}_n=\delta_{n,0}-V\hat{P}_n+W(\hat{P}_{n+1}+\hat{P}_{n-1}) . \quad (2)$$

The LSR procedure employs the odd equations, those for $n=\ldots,-3,-1,+1,+3,\ldots$ in the even equations to find a new set of equations for $\hat{P}_0,\hat{P}_2,\ldots$ that have the same form as Eq. (2) with renormalized coupling constants given by

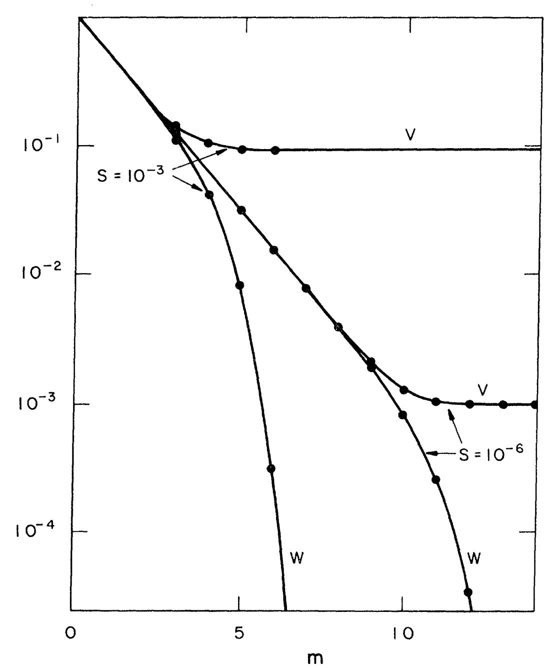


FIG. 1. Evolution of $V$ and $W$ for a homogeneous lattice $1d$. Values of $V$ and $W$ from the recursion relations, Eqs. (3), vs $m$, the number of iterations, for $s=10^{-3},10^{-6}$.

 

$$W' = WDW\ ,$$
$$V' = V - 2W'\ , \tag{3}$$

where $D = (s+V)^{-1}$.

The new system of equations, when relabeled, is the same as the old so that further changes in length scale corresponds to iteration of Eq. (3). In Fig. 1 we show the behavior of $V$ and $W$ given by Eqs. (3) upon carrying through the iteration starting at $W=1$, $V=2W$, and $s=10^{-3}, 10^{-6}$. Note that initially $V' \simeq W' \simeq W/2$, $V'' \simeq W'' \simeq W/4 \ldots$ until at $n \geq m(s)$, $W^{(n)} \to 0$ and $V^{(n)} \to V^{(\infty)}$. At this point the system is being examined on time scale $t \simeq s^{-1}$ and on length scale $2^n b$ and is described by

$$s\hat{P}_n = \delta_{n,0} - V^{(\infty)}\hat{P}_n\ . \tag{4}$$

Thus $P_0(s) = (s + V^{(\infty)})^{-1}$. The value of $V^{(\infty)}$ is estimated by arguing that on time scale $t \simeq s^{-1}$ a particle can move a distance of order

$$\langle r^2 \rangle s \simeq W\ . \tag{5}$$

Thus for $n > m(s)$, where $2^{m(s)} b \simeq \sqrt{\langle r^2 \rangle} \simeq \sqrt{W/S}$ (here $b$ is the site-to-site separation), there should be no motion between sites $W \to 0$ and $V^{(n+1)} = V^{(n)} - 2W^{(n+1)} = V^{(n)}$. We have (use $b=1$, $W=1$)

$$V^{(\infty)} \simeq \frac{1}{2^{m(s)}} \simeq \sqrt{s} \tag{6}$$

and $P_0(s) \sim s^{-1/2}$ as $s \to 0$.

The two essential ingredients of this analysis are (1) $m(s)$ given by the distance-time relation $\langle r^2 \rangle s \simeq W$ and (2) the factor $\frac{1}{2}(V^{(n+1)} \simeq \frac{1}{2} V^{(n)})$ that measures the change in the effective rate of leaving a site that accompanies a change of length scale. This effective rate accounts for the reduction in the rate of leaving a site that occurs when a particle returns from neighboring sites.

Let us combine these ingredients to obtain an estimate of $P_0(s)$ on a fractal lattice of dimension $d$. We take the distance-time relation to be

$$\frac{\langle r^2 \rangle}{b^2} \simeq \left[ \frac{W}{s} \right]^{2/(2+\tilde{\delta})} \tag{7}$$

and the coupling-constant scaling to initially be

$$V^{(n+1)} = \lambda V^{(n)}\ .$$

(To make contact with other work note $\tilde{\delta} = \theta$, $2 + \tilde{\delta} = H^{-1}$). Then as $s \to 0$

$$P_0(s) \simeq \frac{1}{V^{(\infty)}} \simeq \frac{1}{\lambda^{m(s)}} = \frac{1}{(\frac{1}{2})^{m(s)w}}\ , \tag{8}$$

where $w = \ln\lambda / \ln(0.5)$ and $\sqrt{r^2} = s^{-1/(2+\tilde{\delta})} \equiv 2^{m(s)}$ from Eq. (7) (again $b=1$, $W=1$). Thus

$$P_0(s) \simeq s^{-w/(2+\tilde{\delta})}\ . \tag{9}$$

We compare this estimate of $P_0(s)$ to one made by taking

$$P_0(t) \simeq \frac{1}{\Omega(t)} \simeq \frac{1}{\langle r^2(t) \rangle^{\bar{d}/2}} \tag{10}$$

[here $\Omega(t)$ is the volume a particle can visit on time scale $t$] and using $\langle r^2(t) \rangle \simeq t^{2/(2+\tilde{\delta})}$ to find

$$P_0(t) \sim t^{-\bar{d}/(2+\tilde{\delta})}$$

or

$$P_0(s) \sim s^{1-\bar{d}/(2+\tilde{\delta})}\ . \tag{11}$$

Thus from Eqs. (9) and (11)

$$\tilde{\delta} = \bar{d} - 2 + w\ . \tag{12}$$

This equation involves $\tilde{\delta}$, characterizing the distance-time relation, $\bar{d}$, the fractal dimension, and $w$ a measure of coupling-constant scaling. It is similar to the relations $\tilde{t} = d - 2 + \tilde{\zeta}$ of Gefen *et al.* which involves $\tilde{\zeta}$ (the resistance scales as $L^{\tilde{\zeta}}$) and $\tilde{t}$ (the conductivity scales as $L^{-\tilde{t}}$). Below we make application of the LSR procedure to the Sierpiński gaskets, find $w$ from the structure of the recursion relations, and $\tilde{\delta}$ from their numerical implementation, and establish $\bar{d} - 2 + w = \tilde{\delta}$, $w = \tilde{\zeta}$. The relations $\bar{d} - 2 + w = \tilde{\delta}$ and $d - 2 + \tilde{\zeta} = \tilde{t}$ can be combined to yield the Einstein relation $\tilde{t} - \tilde{\delta} = d - \bar{d}$.[7]

## III. APPLICATION TO THE SIERPIŃSKI GASKETS

In this section we study the behavior of $P_0(s)$ on a Sierpiński gasket in $d$ dimensions. We use the phrase Sierpiński gasket in $d$ dimensions to denote the generalization to $d$ dimensions of the $2d$ gasket shown in Fig. 2. To construct the $d$-dimensional gasket we employ the following procedure: (1) Near the initial site 0 construct $d$ sites so that there are $d+1$ sites in total. (2) Form bonds between the $d+1$ sites [$d(d+1)/2$ bonds are required]. At the midpoint of each bond construct a new site [there are $d+1$ original sites and $d(d+1)/2$ new sites]. (4) Form bonds among the $d(d+1)/2$ new sites ($[d(d+1)/2][d(d+1)+1/2]/2$ new bonds are required). These bonds divide the space into $d+1$ units each of which contains $d+1$ bonds and is connected to each other unit at only one site. (5) Within each unit carry out steps (3)–(5) repeatedly. See Fig. 2 and the book by Mandelbrot.[8] The gaskets so formed have fractal dimension $\bar{d} = \ln(d+1)/\ln 2$ (see Table I).

To carry out the LSR procedure the equations for $\hat{P}_n(s)$ are written down (the connectivity is easily assessed from Fig. 2 or as in the Appendix for $d$ dimensions). Then, certain of the sites are removed from the equations of

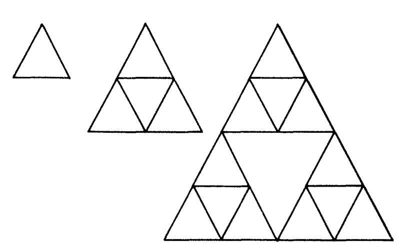

FIG. 2. Sierpiński gasket: $d=2$, $\bar{d}=1.585, \ldots$. The gasket is formed by cutting out material, constructing the $n+1$ unit from the $n$th unit, by the rules in the Appendix, etc.

TABLE I. Scaling parameters as a function of $d$; (a) spatial dimension, (b) fractal dimension $\bar{d}=\ln(d+1)/\ln 2$, (c) $w=\ln[(d+3)/(d+1)]/\ln 2$, (d) $k$ from analysis of $P_0(s)$ vs $s$ as in Fig. 5, (e) $\tilde{\delta}=(w/k)-2$ using $w$ from (c) and $k$ from (d), (f) $\tilde{\delta}=\bar{d}-2+w$ using $\bar{d}$ from (b) and $w$ from (c).

| (a) $d$ | (b) $\bar{d}$ | (c) $w$ | (d) $k$ | (e) $\tilde{\delta}=(w/k)-2$ | (f) $\tilde{\delta}=\bar{d}-2+w$ |
|---|---|---|---|---|---|
| 2 | 1.584 963 | 0.736 965 | 0.318 | 0.318 | 0.322 |
| 3 | $2.\bar{0}$ | 0.584 962 | 0.229 | 0.555 | 0.585 |
| 4 | 2.321 928 | 0.485 426 | 0.173 | 0.806 | 0.807 |
| 10 | 3.459 431 | 0.241 008 | 0.0655 | 1.679 | 1.700 |

motion (see Fig. 3 for illustration) and the remaining equations cast in the same form as the original equations with the new set of coupling constants given by

$$V'=V-2dW_1' \tag{13}$$

and

$$W'=[1+2(d-1)DW]W_1' , \tag{14}$$

where

$$W_1'=WQW ,$$

$$D^{-1}=s+V-2(d-2)W ,$$

and

$$Q^{-1}=s+V-(d-1)W(1+2DW) .$$

These equations are derived in the Appendix. Note that $W_1'$, $D$, and $Q$ are auxiliary functions; $V,W$ determine $V',W'$.

The results of iteration of Eqs. (13) and (14) are shown in Figs. 4 and 5. Let us begin the discussion of these results by examining the behavior of $V$ and $W$ upon iteration. This behavior is qualitatively like that shown above for a $1d$ homogeneous lattice; $V$ and $W$ decrease by a constant factor as the renormalization proceeds until $W\rightarrow 0$ and $V$ saturates. Upon putting $s=0$, the recursion relations reduce to $V^{(n+1)}=2nW^{(n+1)}$ and $W^{(n+1)}=[(d+1)/(d+3)]W^{(n)}$. Thus in Eqs. (8) and (9) the coupling-constant scaling is $\lambda=(d+1)/(d+3)$ and

$$w=\frac{\ln[(d+1)/(d+3)]}{\ln(0.5)} , \tag{15}$$

in agreement with Gefen *et al.* for $\tilde{\zeta}$. The rate at which $V$ decreases upon recursion becomes less and less as $d\rightarrow+\infty$. See Table I. This rate, a measure of the probability that a particle that initially leaves a site will not return, goes to 1 as $d$ becomes large and the particle, moving in a system of more and more complex connectivity, has lower probability of returning. From Fig. 4 we see that as $s$ decreases, $V$ saturates at $m(s)$ related to $s$ as suggested by the distance-time relation, Eq. (6) or (7); i.e., saturation occurs where the system is viewed on a length scale beyond that accessible on time scale $s^{-1}$. Thus the qualitative behavior of the recursion relations contain

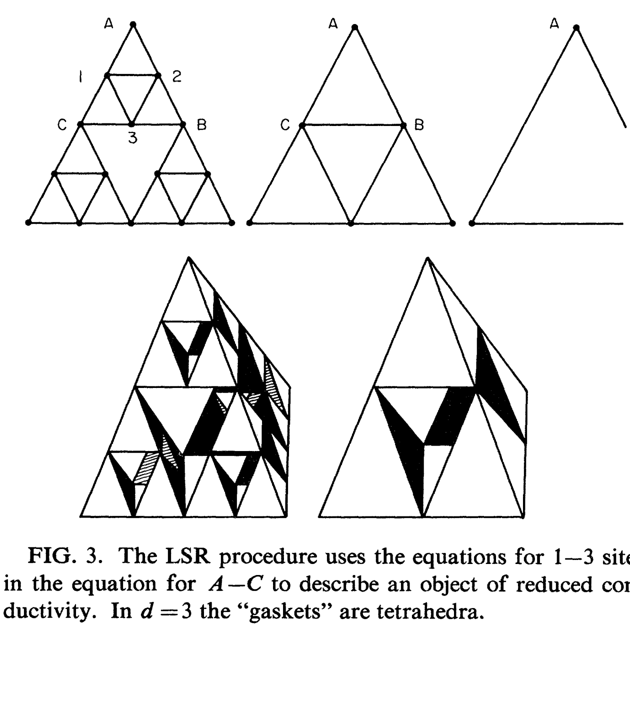


FIG. 3. The LSR procedure uses the equations for 1—3 sites in the equation for $A-C$ to describe an object of reduced conductivity. In $d=3$ the "gaskets" are tetrahedra.

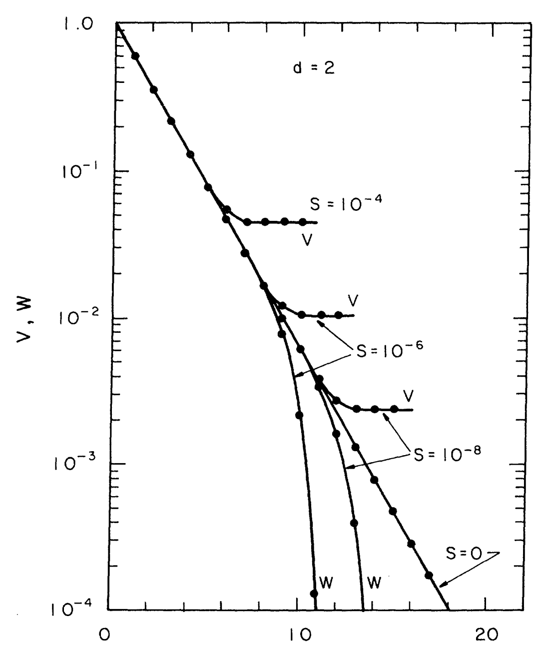


FIG. 4. Evolution of $V$ and $W$ for $d=2$, $\bar{d}=1.585, \cdots$. Values of $V$ and $W$ from the recursion relations, Eqs. (13) and (14), vs $m$ for $s=10^{-4}, 10^{-6}, 10^{-8}, 0$. Compare the qualitative behavior to that in Fig. 1.

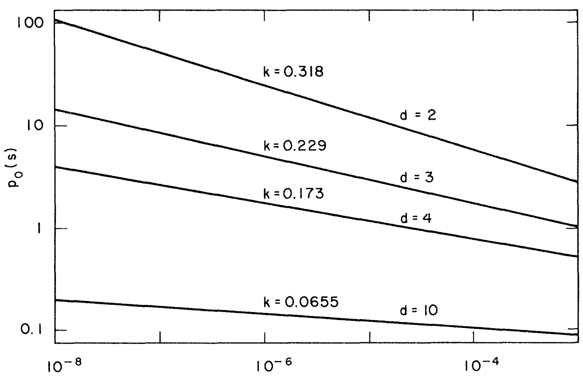


FIG. 5. $P_0(s)$ vs $s$. The single-site density $P_0(s)$ as a function of $s$ from $P_0(s)=(s+V^{(\infty)})^{-1}$; $V^{(\infty)}$ is found from numerical studies of the recursion relations. Values of $k$, $P_0(s)\sim s^{-k}$, are reported in Table I.

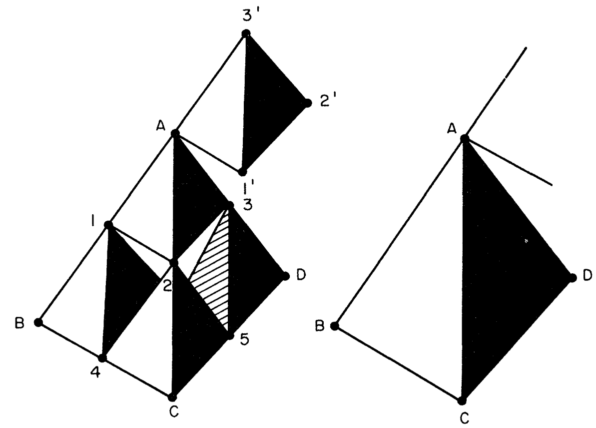


FIG. 6. Connectivity for $d=3$ and $\bar{d}=2$. Sites 1—3 are first-neighbor sites of site $A$; sites 4—6 are second-neighbor sites of site $A$. See Fig. 7.

physical information. In Fig. 5 we show $P_0(s)$ calculated from numerical analysis of the asymptotic form of the equations of motion (Eq. (4), $P_0(s)=[s+V^{(\infty)}(s)]^{-1}$) as a function of $s$. From the asymptotic behavior of $P_0(s)$, $P_0(s)\sim s^{-k}$ for $10^{-8}<s<10^{-3}$, we find $k$ and $\tilde{\delta}$ reported in Table I. Also shown in Table I is $\tilde{\delta}$ from Eq. (12), $\tilde{\delta}=\bar{d}-2+w$, using $w$ from Eq. (15). The excellent agreement between $\tilde{\delta}$ from analysis of $P_0(s)$ via the recursion relations and $\tilde{\delta}$ from Eq. (12) constitutes a demonstration of the usefulness of the $\tilde{\delta},\bar{d},w$ relation in a new context; $\bar{d}$, the fractal dimension, $\tilde{\delta}$ from the distance-time relation, and $w$ from the LSR recursion relation obey Eq. (12). In addition the LSR procedure provides recursion relations having qualitative behavior that embodies clear evidence of important features of the physical process being studied.

A random walker on a percolation cluster near threshold sees events on three time scales. At short times, large $s$, he sees the local structure of the cluster [the contents of the basic unit from which the cluster (gasket) is built]. An intermediate time scales he sees the factal structure of the object on which he moves. At long time scales he sees connected, fractally structured objects. In this paper we have emphasized the intermediate process, motion on a fractal. Suppose the size of the fractal is such that on the time scale of interest the renormalization has not saturated. The LSR procedure transverses the fractal; the final set of coupling constants can be usefully employed to describe the bulk properties of the material.

## ACKNOWLEDGMENT

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## APPENDIX: DERIVATION OF THE RECURRENCE RELATIONS

We do this for $d=3$ and argue for the general form for these relations from the results shown here. See Fig. 6. The equation of motion for $\hat{P}_A(s)$ reads [$P_n(t=0)=\delta_{n,A}$; sites $A$—$D$ remain after the first step of the LSR procedure in which sites 1—6 are removed]

$$sP_A=1-VP_A+W(P_1+P_2+P_3+P_1'+P_2'+P_3')\ , \qquad (A1)$$

sites 1—3 are the near neighbors of $A$ in the tetrahedron shown, and sites 1′—3′ are the near neighbors of $A$ in the tetrahedron not completely shown. The equations for $P_1,P_2,P_3\dots$ are, like the equation for $P_1$,

$$sP_1=-UP_1+W(P_A+P_B+P_1+P_2+P_4+P_6)\ , \qquad (A2)$$

coupled to $A$—$D$, to themselves, and to sites that are second neighbors of site $A$. Sites 1—3 act in concert as a first-neighbor site and sites 4—6 act in concert as a second-neighbor site. We have

$$(s+U)R_1=W(3P_A+P_B+P_C+P_D+2R_1+2R_2)\ , \qquad (A3)$$

$$(s+U)R_2=W[2(P_B+P_C+P_D)+2R_1+2R_2]\ , \qquad (A4)$$

where

$$R_1=\sum_{i=1}^{3}P_i$$

and

$$R_2=\sum_{i=4}^{6}P_i\ .$$

Thus

$$sP_A=1-(V-6WQW)P_A + WQW(1+4DW)(P_B+P_C+P_D+P_{B'}+P_{C'}+P_{D'}) \qquad (A5)$$

from which the recursion relations follow. In $d$ dimensions site $A$ is coupled to $d$ near-neighbor sites that act in concert. The $d$ near-neighbor sites are coupled by 1 bond to $B\cdots Z$, among themselves with $d-1$ bonds each and to $d(d-1)/2$ second-neighbor sites with $d-1$ bonds each. The second-neighbor sites in turn couple to $B\cdots Z$

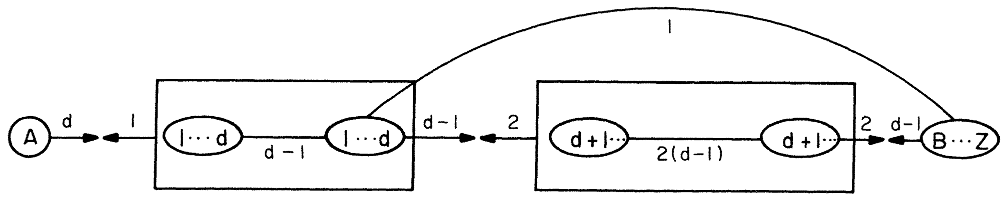


FIG. 7. Connectivity for $d$.

with 2 bonds each among themselves with $2(d-2)$ bonds each and to the first-neighbor sites with 2 bonds each. See Fig. 7. Then in place of (A3) and (A4) we have

$$(s+U)R_1=(d-1)WR_1+2WR_1 + W(dP_A+P_B+\cdots+P_Z)\ , \quad \text{(A6)}$$

$$(s+U)R_2=(d-1)WR_1+2(d-2)WR_2 +(d-1)(P_B+\cdots+P_Z)\ , \quad \text{(A7)}$$

and for $P_A$

$$sP_A=1-VP_A+W(R_1+R_1')\ , \quad \text{(A8)}$$

where $R_1'$ refer to the set of near-neighbor sites in the "tetrahedron" not completely shown, Fig. 6. Equations (A6), (A7), and (A8) lead to the recursion relations in the text.

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