Quasiperiodicity in lasers with saturable absorbers

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(Received 17 October 1983)

In this paper, we consider the mean-field equations for the laser with a saturable absorber (LSA) and concentrate on the low-intensity solutions. We show that the LSA equations may admit two successive bifurcations. The first bifurcation corresponds to the transition from the zero-intensity state to time-periodic intensities and is a Hopf bifurcation. The second bifurcation corresponds to the transition from these time-periodic intensities to quasiperiodic intensities which are characterized by two incommensurable frequencies. In order to describe these transitions, we investigate a particular limit of the parameters and propose a new perturbation method for solving the LSA equations. We give analytical conditions for the existence of both the primary and secondary bifurcations.

I. INTRODUCTION

The laser with saturable absorber (LSA) has recently been studied both experimentally and theoretically. Theoretical studies were mainly centered on the stationary and time-periodic solutions,[1–4] numerical analyses indicated the occurrence of (unstable) quasiperiodic solutions,[5–7] and experiments have demonstrated the existence of at least stationary and time-periodic behaviors.[8–13] In view of the possibility of realizing experimentally a LSA, it is worth investigating in somewhat more detail the properties of the semiclassical (i.e., deterministic) equations which describe the LSA. The purpose of this paper is to study analytically the properties of the low-intensity domain and to go beyond the usual description of the first bifurcation. In particular, we establish the existence of a domain in the parameter space where the LSA system undergoes two successive Hopf bifurcations leading first to a time-periodic solution, then to a quasiperiodic solution (i.e., two incommensurate frequencies). Furthermore, we prove analytically that this quasiperiodic solution has a finite domain of stability.

The LSA presents several advantages compared to the usual laser. First, it has two control parameters (called A and $\bar{A}$) corresponding to the pump parameters of the amplifying and absorbing atoms. From experimental and theoretical works in related fields,[14–16] it is known that two parameters allow a clearer approach of complex instabilities like those leading to chaotic regimes. Second, by contrast to the simple laser, it has been shown[3,7] that when $\bar{A}$ is fixed and A is progressively varied, the zero-intensity state may lose its stability through a Hopf bifurcation leading to time-periodic solutions.

With respect to the current trend in physics, the interest of our results stems from the fact that they arise in a system of ordinary differential equations. Indeed, quasiperiodicity in dissipative systems has recently been studied analytically by replacing the system of ordinary differential equations which is assumed to produce two successive Hopf bifurcations by a set of finite-difference equations modeling the Poincaré map just after the second Hopf bifurcation.[17,18] This transformation from the original equations to the discrete equations presents, however, many practical difficulties. On the contrary, we stick to our original system of differential equations and systematically construct the time-periodic and quasiperiodic solutions using a multitime perturbation procedure. A numerical investigation of these equations reveals no further bifurcation in the vicinity of the secondary Hopf bifurcation. As the bifurcation parameter is increased, the solution smoothly evolves from a nearly harmonic modulation to a pulsed solution.

Besides the description of quasiperiodicity in the LSA, our analysis has a second interest. Indeed, we show that complex time-dependent behavior may be observed in the low-intensity regime. This aspect has been frequently neglected in the recent literature of the LSA principally devoted to medium- or high-intensity regimes.[12]

Two aspects of the LSA will not be considered in this paper. First, we assume perfect tuning. This means that the state variables are real and we do not investigate the possible destabilizing effects of the phases. Second, since we concentrate on the low-intensity domain, we do not investigate the optically bistable character of the system and, in particular, the transition to the upper stable or unstable state.

The mean-field equations for the monomode, homogeneously broadened LSA corresponding to the class of solutions with perfect tuning are given by[2]

$$\begin{aligned}
\left[\frac{d}{dt}+\kappa\right]E &= Nv+\bar{N}\bar{v} ,\\
\left[\frac{d}{dt}+\gamma_\perp\right]v &= g^2 DE ,\\
\left[\frac{d}{dt}+\gamma_{||}\right]D &= \gamma_{||}\sigma-4vE ,\\
\left[\frac{d}{dt}+\bar{\gamma}_\perp\right]\bar{v} &= \bar{g}^2\bar{D}E ,\\
\left[\frac{d}{dt}+\bar{\gamma}_{||}\right]\bar{D} &= \bar{\gamma}_{||}\bar{\sigma}-4\bar{v}E .
\end{aligned} \tag{1.1}$$

Here E is the electric field amplitude and v ($\bar{v}$) and D ($\bar{D}$) describe the polarization and the inversion of population of the amplifying (absorbing) atoms. N ($\bar{N}$) is the total number of amplifying (absorbing) atoms; g^2 ($\bar{g}^2$) measures the strength of the interaction between the field and the amplifying (absorbing) atoms. By scaling the time t with the cavity decay rate κ and defining new dependent variables, the number of independent parameters can be reduced to seven.[6] They are (i) the two control parameters

$$A\equiv\frac{g^2N\sigma}{\kappa\gamma_\perp}>0 \tag{1.2}$$

and

$$\bar{A}\equiv\frac{\bar{g}^2\bar{N}\bar{\sigma}}{\kappa\bar{\gamma}_\perp}<0 \tag{1.3}$$

corresponding to the pump parameters for the amplifying and absorbing atoms, respectively; (ii) the atomic decay rates

$$d\equiv\gamma_\perp/\kappa , \tag{1.4}$$

$$\gamma_{||}/\kappa , \tag{1.5}$$

$$\bar{d}\equiv\bar{\gamma}_\perp/\kappa , \tag{1.6}$$

and

$$\bar{\gamma}_{||}/\kappa ; \tag{1.7}$$

and (iii) the ratio of the saturation intensity of the absorbing to the amplifying atoms

$$a\equiv\frac{\bar{g}^2\gamma_\perp\gamma_{||}}{g^2\bar{\gamma}_\perp\bar{\gamma}_{||}} . \tag{1.8}$$

Because of this large number of parameters, analytical indications are necessary before a sensible scan of the parameter space is undertaken numerically. The evolution equations (1.1) must be solved with the initial conditions for E, v, $\bar{v}$, D, and $\bar{D}$. They are suggested by the experimental situations,

$$\begin{aligned}
&E(0)=E_i\ll 1 ,\\
&v(0)=\bar{v}(0)=D(0)-\sigma=\bar{D}(0)-\bar{\sigma}=0 .
\end{aligned} \tag{1.9}$$

From numerical simulations of the LSA equations,[7] it is known that quite different behaviors can be observed when the control parameters are slightly changed. Moreover, even if the parameters are fixed, bistability and birythmicity (i.e., bistability between two time-periodic solutions) are possible responses of the LSA. In order to classify the possible solutions of the LSA equations and describe their properties, asymptotic approximations are particularly useful. Although these approximations are generally valid in sufficiently small neighborhoods of critical values of the parameters, they reveal the principal mechanisms leading to the existence of distinct solutions. In Ref. 3 we concentrated on the case where the zero-intensity state admits a double zero eigenvalue, i.e., when

$$A=A_c\equiv\frac{d(\bar{d}+1)}{d-\bar{d}} \tag{1.10}$$

and

$$\bar{A}=\bar{A}_c\equiv\frac{\bar{d}(d+1)}{\bar{d}-d} . \tag{1.11}$$

This analysis indicated that when $A\simeq A_c$ and $\bar{A}\simeq\bar{A}_c$, time-periodic solutions of large periods may exist in the low-intensity regime. In this paper, we examine a different case: the limit

$$\epsilon\equiv\gamma_{||}/\kappa\to 0 , \quad \bar{\gamma}_{||}/\kappa=O(\epsilon) \tag{1.12a}$$

with

$$d=O(1) \text{ and } \bar{d}=O(1) . \tag{1.12b}$$

This limit is in agreement with the experimental estimations of $\gamma_{||}$ and κ given in Ref. 12. Moreover, from the theoretical point of view, this case enables us to analyze the stability of time-periodic solutions in a simple way. In particular, as we shall demonstrate, the secondary bifurcation to quasiperiodic solutions can be described in detail.

The paper is organized as follows. In Sec. II we describe the bifurcation possibilities of the zero-intensity state (the basic state). In Sec. III, we analyze the bifurcation from the basic state to a branch of time-periodic solutions and show that it becomes singular when ϵ approaches zero. We then propose in Sec. IV a new expansion of the time-periodic solutions for ϵ near zero. This analysis allows us to predict a secondary bifurcation point to quasiperiodic solutions. These solutions are characterized by two incommensurable frequencies and are studied in detail in Sec. V. Finally, Sec. VI gives a summary of the results and discusses future problems in the bifurcation analysis of the LSA equations.

II. STEADY STATES

Equations (1.1) admit steady-state solutions given by the following.

(1) The zero-intensity state:

$$E = v = \bar{v} = 0\ ,\quad D = \sigma\ ,\quad \bar{D} = \bar{\sigma}\ . \tag{2.1}$$

(2) The nonzero intensity states. Defining the intensity by $I \equiv SE^2$, where $S \equiv 4g^2/\gamma_\perp\gamma_{||}$, the nonzero intensity states correspond to the real and positive roots I_+ and I_- of the following quadratic equation:

$$aI^2 + I(a + 1 - aA - \bar{A}) + 1 - A - \bar{A} = 0\ , \tag{2.2}$$

with

$$D = \frac{\sigma}{1+I}\ ,\quad \bar{D} = \frac{\bar{\sigma}}{1+aI}\ ,\qquad v = \frac{g^2 DE}{\gamma_\perp}\ ,\quad \bar{v} = \frac{\bar{g}^2 \bar{D} E}{\bar{\gamma}_\perp}\ . \tag{2.3}$$

We consider A as our bifurcation parameter. Then, from the linear stability analysis,[13] we find that the basic state (2.1) becomes unstable under the following conditions: assuming that

$$d > \bar{d}\ , \tag{2.4}$$

there exist two distinct cases.

(i) If $\bar{A} < \bar{A}_c$, (2.1) is unstable when

$$A > A^* \equiv A_c \left[1 + \frac{(\bar{A} - \bar{A}_c)}{\bar{A}_c} \frac{\bar{d}^2}{d^2} \right]\ . \tag{2.5}$$

$A = A^*$ is a Hopf bifurcation to a branch of time-periodic solutions [see Fig. 1(a)]. As A tends to A^* from above, the oscillations tend to zero in amplitude and the frequency $\omega(A)$ tends to the critical frequency ω^* of the linear theory, i.e.,

$$\omega^* \equiv \bar{\gamma}_\perp \left| \frac{\bar{A} - \bar{A}_c}{\bar{A}_c} \right|^{1/2}\ . \tag{2.6}$$

The periodic solutions were constructed in Ref. 3 when A is near A^*. In Sec. III, we give the solution of the complete time-dependent problem when $|A - A^*| \to 0$ and show how the perturbation expansion becomes singular when $\epsilon \to 0$. Then, in Sec. IV, we propose a new expansion of the time-periodic solutions valid when $\epsilon \to 0$.

(ii) If $\bar{A} > \bar{A}_c$, (2.1) is unstable when

$$A > A^0 \equiv 1 - \bar{A}\ . \tag{2.7}$$

$A = A^0$ corresponds to the bifurcation from the basic state (2.1) to either I_+ or I_-. From (2.2), we find that the bifurcating solution is given by

$$I = \frac{A - A^0}{1 + \bar{A}(a-1)} + O((A - A^0)^2)\ . \tag{2.8}$$

From the linear stability analysis, we find that (2.8) describes a stable steady state when

$$a < 1 \quad \text{or} \quad a > 1 \quad \text{and} \quad \bar{A} > -(a-1)^{-1} \tag{2.9}$$

[see Fig. 1(b)]. Otherwise, when A surpasses A^0, the system presents a jump transition to a large-intensity regime [see Fig. 1(c)]. This final regime can be a steady state or a time-dependent state. We do not analyze these solutions here.

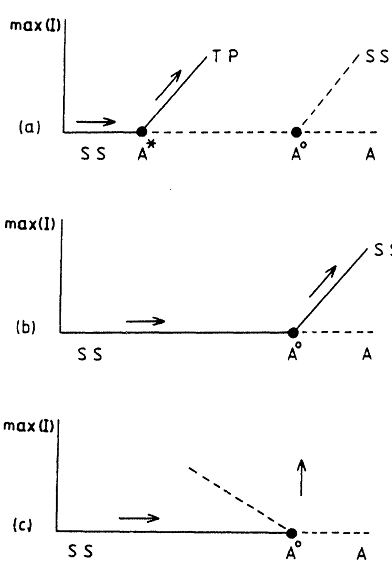

FIG. 1. The primary bifurcations. The bifurcation possibilities of the zero-intensity state are represented in (a)–(c). SS and TP in these figures correspond to steady and time-periodic states, respectively. Solid and dashed lines represent stable and unstable solutions. Arrows indicate the possible evolution of the LSA system when A is slowly increasing in time.

III. TIME-PERIODIC SOLUTIONS

From our previous analysis, we know that when $\bar{A} < \bar{A}_c$, $A = A^*$ is a bifurcation point to a branch of time-periodic solutions. In this section we give the time-periodic solutions of Eqs. (1.1) when A is near A^*. Then we show how our expansion of these solutions becomes singular when $\epsilon \to 0$. The bifurcation analysis directly follows the analysis given in Ref. 3. We shall use the same notation and briefly describe the main results.

We first define the small parameter δ by the following relation:

$$A - A^* = \frac{Ng^2}{\kappa\gamma_\perp}\delta^2 c\ , \tag{3.1}$$

where $c = \pm 1$ when $A - A^* \gtrless 0$. Then, we seek a solution of the evolution equations (1.1) of the form

$$\begin{aligned} \underline{x} &= \mathrm{col}(E,\ v,\ \bar{v},\ D - \sigma,\ \bar{D} - \bar{\sigma}) \\ &= \underline{x}(T,\tau,\delta) = \underline{x}_1(T,\tau)\delta + \underline{x}_2(T,\tau)\delta^2 + \cdots\ , \end{aligned} \tag{3.2}$$

where T and τ are defined by

$$T = \omega(\delta)t = (\omega^* + \omega_2\delta^2 + \cdots)t\ , \tag{3.3}$$

$$\tau = t\delta^2\ . \tag{3.4}$$

We also assume a similar expansion of the initial conditions (1.9) with

$$E_i = e_1\delta + e_2\delta^2 + \cdots\ . \tag{3.5}$$

After introducing (3.1)–(3.5) into Eqs. (1.1) and equating to zero the coefficients of each power of δ, we obtain a sequence of linear problems for $\underline{x}_1, \underline{x}_2, \ldots$. Each problem is solved by using the solvability and initial conditions. We find that

$$\underline{x}=[\alpha(\tau)\underline{r}_1 e^{iT}+\text{c.c.}]\delta+\{[\alpha^2(\tau)\underline{r}_1 e^{iT}+\text{c.c.}]+(\alpha^2\underline{r}_2 e^{2iT}+\text{c.c.})+\alpha\alpha^*\underline{r}_0\}\delta^2+O(\delta^3)+\underline{y}(T,\tau)\,, \tag{3.6}$$

where $\underline{y}$ denotes a rapid, exponentially decaying function of T. The characteristic relaxation times of $\underline{y}$ are given by

$$T_1=\omega/\gamma_{\|}\,,\quad T_2=\omega/\bar{\gamma}_{\|}\,,\quad T_3=\omega/(\kappa+\gamma_\perp+\bar{\gamma}_\perp)\,. \tag{3.7}$$

The vectors $\underline{r}_1$, $\underline{r}_2$, and $\underline{r}_0$ are defined by

$$\begin{aligned}\underline{r}_1&=\text{col}(1,p,q,0,0)\,,\\ \underline{r}_2&=\text{col}(0,0,0,p_2,q_2)\,,\\ \underline{r}_0&=\text{col}(0,0,0,p_0,q_0)\,,\end{aligned} \tag{3.8}$$

where

$$p=\frac{\kappa\gamma_\perp}{N}A^*(\gamma_\perp+i\omega^*)^{-1}\,,\quad q=\frac{\kappa\bar{\gamma}_\perp}{\bar{N}}\bar{A}(\bar{\gamma}_\perp+i\omega^*)^{-1}\,, \tag{3.9}$$

$$p_2=-4p(\gamma_{\|}+2i\omega^*)^{-1}\,,\quad q_2=-4q(\bar{\gamma}_{\|}+2i\omega^*)^{-1}\,, \tag{3.10}$$

$$p_0=-4(p+p^*)\gamma_{\|}^{-1}\,,\quad q_0=-4(q+q^*)\bar{\gamma}_{\|}^{-1}\,. \tag{3.11}$$

In (3.6) and (3.11), α^*, p^*, and q^* denote the complex conjugates of α, p, and q. At the $O(\delta^3)$ stage of the perturbation analysis, α is obtained from the solvability conditions. They require that α satisfy the following ordinary differential equation:

$$\alpha_\tau=\alpha(R+\alpha\alpha^*S)\,,\quad \alpha(0)=\alpha_i(e_1)\,, \tag{3.12}$$

where

$$\begin{aligned}R&=-i\omega_2+g^2cp'^*/(1+pp'^*+qq'^*)\,,\\ S&=[g^2p'^*(p_0+p_2)+\bar{g}^2q'^*(q_0+q_2)]/(1+pp'^*+qq'^*)\,,\end{aligned} \tag{3.13}$$

$$p'=N(\gamma_\perp-i\omega^*)^{-1}\,,\quad q'=\bar{N}(\gamma_\perp-i\omega^*)^{-1}\,,$$

and p'^* and q'^* represent the complex conjugates of p' and q'. Equation (3.12) is the bifurcation equation and its steady-state solutions correspond to the time-periodic solutions of Eqs. (1.1).

We now concentrate on the limit $\epsilon\equiv\gamma_{\|}/\kappa\to 0$. The analysis is elementary but tedious, so that we only summarize the results and omit all details of the calculations. Defining ρ, θ, and k by

$$\alpha=\rho e^{i\theta}\,, \tag{3.14}$$

$$k(\epsilon)=\bar{\gamma}_{\|}/\gamma_{\|}=k_1+O(\epsilon)\,, \tag{3.15}$$

we find from (3.12) that the amplitude ρ and the phase θ satisfy the following equations:

$$\begin{aligned}\rho_\tau&=\rho(R'+\rho^2S')\,,\quad \rho(0)=\rho_i(e_1)\,,\\ \rho\theta_\tau&=\rho(R''+S''\rho^2)\,,\quad \theta(0)=\theta_i\,,\end{aligned} \tag{3.16}$$

where R' and S' are complicated functions of the parameters with the property that

$$R'=O(1)\ \text{and}\ S'=O(\epsilon^{-1})\,. \tag{3.17}$$

As $\epsilon\to 0$, the two steady-state solutions of (3.16) are in the first approximation given by

$$\rho=0\,, \tag{3.18}$$

$$\rho^2=\frac{g^2N}{d\bar{d}}\frac{(d-\bar{d})}{8(d+1)(\bar{d}+1)}\frac{\epsilon c}{(g^2/\bar{d}-\bar{g}^2/dk_1)}>0\,, \tag{3.19}$$

and the phase θ remains arbitrary. The ϵ in (3.19) appears because of (1.12a). From (3.6) and (3.14), we conclude that (3.18) and (3.19) correspond to the basic state and the time-periodic solutions of Eqs. (1.1), respectively. From (3.16), it is also possible to study their linear stability; we have found that (3.18) is stable if

$$c<0\,, \tag{3.20}$$

and (3.19) represents stable solutions if $c>0$ and

$$g^2/\bar{d}-\bar{g}^2/dk_1>0\,. \tag{3.21}$$

Under condition (3.21), the bifurcation to the stable time-periodic solutions is supercritical, as shown in Fig. 1(a). On the other hand, if (3.21) is not satisfied, the bifurcation is subcritical and the small-amplitude time-periodic solutions are unstable. When A surpasses A^*, the system jumps to a large-amplitude regime which cannot be described by our analysis (see Fig. 2).

When $\epsilon\to 0$, we observe from (3.19) that the amplitude of the time-periodic solutions is $\rho=O(\epsilon^{1/2})$. Moreover, from (1.12), (3.15), and (3.9)–(3.11), we note that p, q, p_2, and q_2 are $O(1)$ quantities but p_0 and q_0 are $O(\epsilon^{-1})$. Then, the analysis of the two first terms in the expansion (3.6) as $\epsilon\to 0$ indicates that the expansion becomes nonuniform when $\delta=O(\epsilon^{1/2})$ or equivalently when

$$A-A^*=O(\epsilon)\,. \tag{3.22}$$

We also note that in this critical regime, two of the characteristic relaxation times (3.7) become comparable to the slow time (3.4). Thus, we must reexamine the complete time-dependent problem when (3.22) is satisfied. In Sec. IV we present a new expansion of the solutions of (1.1) when $\epsilon\to 0$ and $|A-A^*|\to 0$ and show the existence of both time-periodic and quasiperiodic solutions.

IV. QUASIPERIODIC SOLUTIONS

The analysis presented in Sec. III indicates that expansion (3.6) of the time-periodic solutions becomes singular

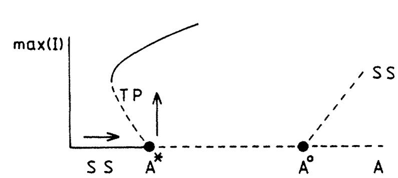

FIG. 2. Subcritical bifurcation of time-periodic solutions. In this figure we assume that the branch of time-periodic solutions admits a limit point and large-amplitude solutions. Then, when A is slowly increasing in time and surpasses A^*, the LSA system will jump to these large-amplitude solutions.

when (3.22) is satisfied. It also suggests that the appropriate expansions of $\underline{x}$ and $A - A^*$ in this critical regime are given by

$$\underline{x} = \underline{x}(T',\tau',\epsilon) = \epsilon \underline{x}_1(T',\tau') + \epsilon^2 \underline{x}_2(T',\tau') + \cdots , \quad (4.1)$$

$$A - A^* = \frac{Ng^2}{\kappa\gamma_\perp}(\epsilon\sigma_1 + \epsilon^2\sigma_2 + \cdots) , \quad (4.2)$$

where T' and τ' are defined by

$$T' = (\omega^* + \epsilon\omega_1 + \cdots)t , \quad \tau' = \epsilon t . \quad (4.3)$$

The leading-order term in (4.1) must be $O(\epsilon)$ because $\delta\alpha = O(\epsilon)$ in (3.6) when $A - A^* = O(\epsilon)$. Similarly, the definitions of T' and τ' are motivated by (3.3) and (3.4) since $\delta^2 = O(\epsilon)$ when $A - A^* = O(\epsilon)$. As in Sec. III, we find the different unknown vectors $\underline{x}_1, \underline{x}_2, \ldots$ by inserting (4.1)–(4.3), (3.15), and (1.12) into Eqs. (1.1) and applying the solvability conditions. We then obtain the following results:

$$\begin{bmatrix} E \\ v \\ \bar{v} \end{bmatrix} = \epsilon \left[\alpha(\tau') \begin{bmatrix} 1 \\ p \\ q \end{bmatrix} e^{iT'} + \text{c.c.} \right] + O(\epsilon^2) , \quad (4.4)$$

$$\begin{bmatrix} D - \sigma \\ \bar{D} - \bar{\sigma} \end{bmatrix} = \epsilon \begin{bmatrix} \beta(\tau') \\ \bar{\beta}(\tau') \end{bmatrix} + O(\epsilon^2) , \quad (4.5)$$

where p and q are defined by (3.9). The amplitudes α, β, and $\bar{\beta}$ must satisfy a system of three ordinary differential equations given by

$$\begin{aligned} &\alpha_{\tau'} = \alpha(-i\omega_1 + P\sigma_1 + P\beta + Q\bar{\beta}) , \\ &\beta_{\tau'} = -\kappa\beta + B\alpha\alpha^* , \\ &\bar{\beta}_{\tau'} = -\kappa k_1\bar{\beta} + C\alpha\alpha^* , \\ &\alpha(0) = \alpha_i , \quad \beta(0) = \bar{\beta}(0) = 0 , \end{aligned} \quad (4.6)$$

where the initial conditions are obtained from (1.9) and an expansion of E_i in a power series of ϵ. The coefficients P, Q, B, and C are defined by

$$\begin{aligned} &P = \frac{g^2 p'^*}{1 + pp'^* + qq'^*} , \quad Q = \frac{\bar{g}^2 q'^*}{1 + pp'^* + qq'^*} , \\ &B = -4(p^* + p) , \quad C = -4(q^* + q) , \end{aligned} \quad (4.7)$$

where p' and q' are defined by (3.13). With the use of (3.14), Eqs. (4.6) can be rewritten as

$$\rho_{\tau'} = \rho(P_1\sigma_1 + P_1\beta + Q_1\bar{\beta}) , \quad (4.8a)$$

$$\beta_{\tau'} = -\kappa\beta + B\rho^2 , \quad (4.8b)$$

$$\bar{\beta}_{\tau'} = -\kappa k_1\bar{\beta} + C\rho^2 , \quad (4.8c)$$

$$\rho(\theta_{\tau'} + \omega_1 - P_2\sigma_1 - P_2\beta - Q_2\bar{\beta}) = 0 , \quad (4.8d)$$

$$\rho(0) = \rho_i , \quad \theta(0) = \theta_i , \quad \beta(0) = \bar{\beta}(0) = 0 , \quad (4.8e)$$

where $P = P_1 + iP_2$ and $Q = Q_1 + iQ_2$. Equations (4.8) are the new bifurcation equations: corresponding to each solution of (4.8), we obtain an asymptotic approximation to a solution of the LSA equations (1.1) for small ϵ. The expressions of P_1, Q_1, B, and C which appear in the first three equations of (4.8) are given by

$$\begin{aligned} &P_1 = \frac{g^2 N}{\gamma_\perp}\frac{1}{\bar{d}^2}\frac{(d+1)}{d}G > 0 , \\ &Q_1 = \frac{\bar{g}^2 \bar{N}}{\bar{\gamma}_\perp}\frac{1}{d^2}\frac{(\bar{d}+1)}{\bar{d}}G > 0 , \end{aligned} \quad (4.9)$$

$$B = -8\frac{\kappa}{N}A_c < 0 , \quad C = -8\frac{\kappa}{\bar{N}}\bar{A}_c > 0 , \quad (4.10)$$

where G is a complicated, positive function of the parameters. It is not necessary to know its explicit form for finding the possible solutions of Eqs. (4.8). Moreover, P_2 and Q_2 only appear in (4.8d) and the determination of ρ, β, and $\bar{\beta}$ by Eqs. (4.8a)–(4.8c) does not depend on them.

We now examine the steady-state solutions of Eqs. (4.8). They are given by (i)

$$\rho = \beta = \bar{\beta} = 0 , \quad (4.11)$$

which is the basic state solution of (1.1). From (4.8), we find that it is stable if

$$\sigma_1 < 0 , \quad (4.12)$$

which implies from (4.2) that $A < A^*$; and (ii)

$$\beta = B\rho^2/\kappa , \quad \bar{\beta} = C\rho^2/k_1\kappa ,$$

$$\rho^2 = -[\sigma_1 P_1 \kappa k_1/(P_1 B k_1 + Q_1 C)] > 0 , \quad (4.13)$$

$$\omega_1 = P_2\sigma_1 + P_2\beta + Q_2\bar{\beta} , \quad (4.14)$$

which corresponds to the time-periodic solutions of (1.1). For example, E is obtained from (4.4) and is given by

$$E = \epsilon(\rho e^{i[\omega^* + \epsilon\omega_1 + O(\epsilon^2)]t} + \text{c.c.}) + O(\epsilon^2) , \quad (4.15)$$

where ρ is given by (4.13) and the frequency correction is defined by (4.14). From (4.13), we note that the bifurcation is supercritical (subcritical), i.e., ρ is defined in the regions $\sigma_1 > 0$ and $\sigma_1 < 0$ if, respectively,

$$P_1 B k_1 + Q_1 C < 0 , \quad (4.16)$$

$$P_1 B k_1 + Q_1 C > 0 , \quad (4.17)$$

or, using (4.9) and (4.10), if

$$k_1 > \frac{\bar{g}^2\bar{d}}{g^2 d} , \quad (4.18)$$

$$k_1 < \frac{\bar{g}^2\bar{d}}{g^2 d} . \quad (4.19)$$

Thus, if the state is supercritical (subcritical) then it exists for $A > A^*$ ($A < A^*$). The linear stability of these solutions is found from Eqs. (4.8) and is determined by the Routh-Hurwitz conditions.[19] In addition to (4.16), they require that

$$-\kappa^2 k_1(1 + k_1) + 2\rho^2(P_1 B + Q_1 C k_1) < 0 . \quad (4.20)$$

By using (4.9) and (4.10), we note that the coefficient of ρ^2 in (4.20) is positive if

$$k_1 > \frac{g^2 d}{\bar{g}^2\bar{d}} . \quad (4.21)$$

If (4.18) and (4.21) are simultaneously satisfied, it can then be shown that the solutions given by (4.13) and (4.14) change stability at a critical amplitude defined by

$$\rho_c^2 \equiv \frac{\kappa^2 k_1(1+k_1)}{2(P_1 B + Q_1 C k_1)} . \tag{4.22}$$

Using (4.22) and (4.13), we find the corresponding value of σ_1:

$$\sigma_1 = \sigma_1^{**} \equiv -\rho_c^2 \frac{(P_1 B k_1 + Q_1 C)}{P_1 \kappa k_1} . \tag{4.23}$$

When $A^{**} \equiv A(\sigma_1^{**})$, where $A^{**} > A^*$ is related to σ_1^{**} by (4.2), i.e.,

$$A^{**} = A^* + \frac{N g^2}{\kappa \gamma_\perp}[\epsilon \sigma_1^{**} + O(\epsilon^2)] , \tag{4.24}$$

a Hopf bifurcation from the nontrivial steady states (4.13) to time-periodic solutions of (4.8) is then possible. Thus, it corresponds to a secondary bifurcation from the time-periodic solutions of (1.1) to quasiperiodic solutions. In Sec. V this bifurcation is shown to be supercritical and thus exists for $A > A^{**}$, as shown in Fig. 3. These quasiperiodic solutions are characterized by two distinct frequencies. As $|A - A^{**}|$ tends to zero, the first frequency approaches the frequency of the time-periodic solutions, evaluated at $A = A^{**}$, i.e.,

$$\omega(1) = \omega^* + \epsilon \omega_1(\sigma_1^{**}) + O(\epsilon^2) . \tag{4.25}$$

On the other hand, the second frequency tends to the critical frequency associated with (4.22) and obtained by the linear stability analysis of (4.13). It is given by

$$\omega(2) = \epsilon \left[-\frac{2\rho_c^2(P_1 B k_1 + Q_1 C)}{(1+k_1)} \right]^{1/2} \ll 1 . \tag{4.26}$$

In summary, our stability analysis of the steady-state solutions of Eqs. (4.8) reveals the possibility of a Hopf bifurcation for these equations. This bifurcation appears if (4.18) and (4.21) can simultaneously be satisfied. Since [from (4.4) and (3.14)] the possible time-periodic solutions of (4.8) correspond to quasiperiodic solutions of (1.1), we can describe these quasiperiodic solutions by just studying the time-periodic solutions of (4.8). This analysis is presented in Sec. V. Assuming that such solutions exist for the amplitudes ρ, β, and $\bar{\beta}$, we then find from (4.8d) the following expression for θ:

$$\theta(\tau') = -\omega_1 \tau' + \Gamma(\tau') ,$$
$$\Gamma(\tau') = P_2 \sigma_1 \tau' + \int_0^{\tau'} [P_2 \beta(s) + Q_2 \bar{\beta}(s)] ds . \tag{4.27}$$

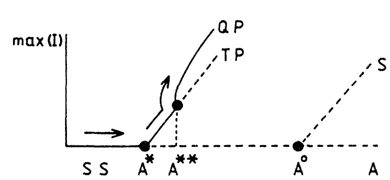

FIG. 3. Secondary bifurcation to quasiperiodic solutions. We represent the two successive Hopf bifurcations leading to stable quasiperiodic (QP) solutions.

For example, E is given by

$$E = \epsilon[\rho(\tau') e^{i\Gamma(\tau')} e^{i[\omega^* + O(\epsilon^2)]t} + \text{c.c.}] + O(\epsilon^2) , \tag{4.28}$$

where $\rho(\tau')$ and $\Gamma(\tau')$ are time-periodic functions of τ'. As A approaches A^{**}, they are characterized by a frequency close to $\omega(2)/\epsilon$.

V. NUMERICAL RESULTS

In this section, we analyze the time-periodic solutions of the bifurcation equations (4.8a)—(4.8c) and describe numerically the Hopf bifurcation. By defining new dependent and independent variables, Eqs. (4.8a)—(4.8c) simplify considerably:

$$x_t = x(p_1 - y + p_2 z) ,$$
$$y_t = -y + x^2 , \tag{5.1}$$
$$z_t = -p_3 z + x^2 ,$$

where x, y, z, and t are proportional to ρ, β, $\bar{\beta}$, and τ', respectively. The bifurcation equations now depend on three parameters: p_1 which is proportional to $-\sigma_1$, p_2, and p_3 which are given by

$$p_2 = \frac{\bar{d}\bar{g}^2}{d g^2} , \quad p_3 = k_1 . \tag{5.2}$$

Then, the nontrivial steady-state solutions of (5.1) are given by

$$y = x^2 , \quad z = x^2/p_3 , \quad x^2 = p_1/(1 - p_2/p_3) > 0 \tag{5.3}$$

and the Hopf bifurcation is possible under the following conditions:

$$p_3 > p_2 , \quad p_3 > p_2^{-1} . \tag{5.4}$$

The bifurcation equations (5.1) are analyzed numerically[20] by a computer program which finds steady and periodic solution branches using continuation methods, and determines their stability properties. Typical parameter values of $p_2 = 0.2$ and $p_3 = 6.0$ are used in the computations. They satisfy conditions (5.4). The numerical analysis finds the predicted Hopf bifurcation point at $p_1 = p_{1c} \simeq 101.5$. A supercritical branch of periodic solutions bifurcates from this point and is stable at least as far as $p_1 \simeq 5 p_{1c}$. The bifurcation diagram is shown in Fig. 4 (see also Fig. 3), where the maximum value of x is plotted against the bifurcation parameter p_1. In this figure, the lower and upper branches represent the steady and periodic solutions of (5.1), respectively. The periodic oscillations are harmonic near p_{1c} and tend to become pulsed as p_1 is increased. This may be noticed by looking at Figs. 5 and 6(a), where x is plotted against ωt (ω is defined here as the frequency of the periodic oscillations). In Fig. 6(b), we represent the corresponding x-z phase plane. We observe that for $p_1 = p_1(3)$, the limit cycle is elliptical and the time response is sinusoidal. By contrast, when $p_1 = p_1(5)$, the ellipse is largely deformed and the time response is sharply pulsed. From our numerical simulations, we also note that the period of the oscillations does not increase significantly along the periodic branch: from $p_1 = p_1(1)$ to $p_1 = p_1(5)$ the period increases from 0.4808 to 0.5793.

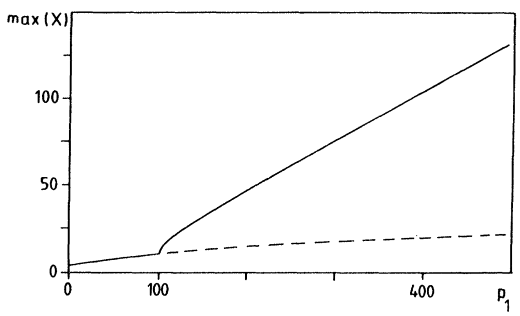

FIG. 4. The bifurcation diagram of Eqs. (5.1). The upper and lower branches correspond to time-periodic and steady solutions of (5.1), respectively. x is proportional to ρ, the amplitude of the electric field; p_1 is the bifurcation parameter and is related to the small deviation $A - A^*$. The values of the other parameters appearing in Eqs. (5.1) are given by $p_2 = 0.2$ and $p_3 = 6$.

The periodic solutions of (5.1) correspond to quasiperiodic solutions of the evolution equations (1.1). When $|p_1 - p_{1c}| \to 0$, i.e., when $|A - A^{**}| \to 0$, the quasiperiodic intensities are of the form:

$$I(t) = \epsilon^2[a + b(p_1 - p_{1c})^{1/2}\cos(\sigma_1 \epsilon t)] \times \cos^2(\sigma_2 t) + O(\epsilon^3) ,$$

where a, b, σ_1, and σ_2 are $O(1)$ quantities (see the Appendix) and t is now the real time as defined in the LSA equations (1.1). Thus, the quasiperiodic solutions essentially correspond to the primary time-periodic oscillations modulated periodically on a long $O(\epsilon^{-1})$ time scale. Since σ_1 and σ_2 are in general rationally independent frequencies, the general behavior is clearly not periodic: for example, a given maximum of $I(t)$ will never reappear at a later time. In Figs. 7 and 8, the response of the LSA is given for two different values of p_1, using the numerical solution of Eqs. (5.1).

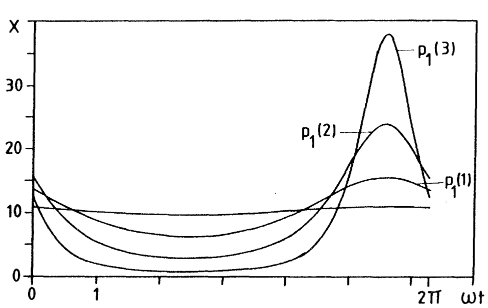

FIG. 5. The time-periodic oscillations. We represent x as a function of ωt, where $\omega = 2\pi/T$ is the frequency and T is the period of the oscillations. $p_1(1) = 106.4$ $(T = 0.4808)$, $p_1(2) = 125.7$ $(T = 0.4966)$, and $p_1(3) = 170.2$ $(T = 0.5243)$.

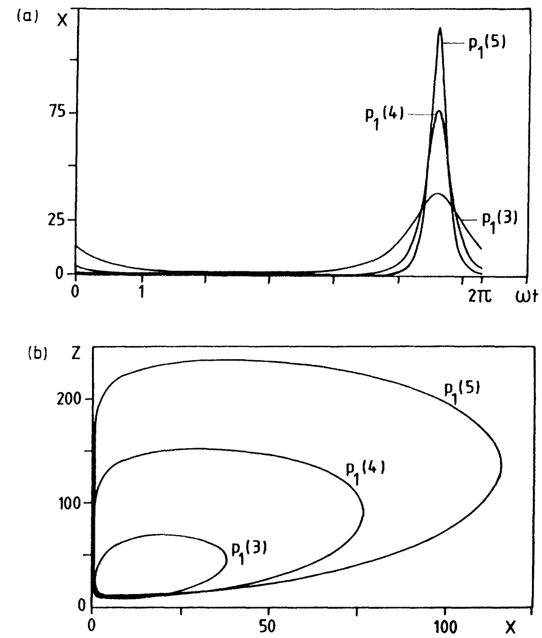

FIG. 6. The time-periodic pulses. In (a) we represent x as a function of t for increasing values of p_1. $p_1(3) = 170.2$ $(T = 0.5243)$, $p_1(4) = 305$ $(T = 0.5635)$, and $p_1(5) = 441$ $(T = 0.5793)$. In (b) we give the corresponding limit cycles in the x-z phase plane.

VI. SUMMARY AND DISCUSSION

The principal purpose of this paper was to describe the two successive bifurcations of the LSA equations which lead, in the low-intensity regime, to quasiperiodic solutions. To this end, we have developed a new perturbation method which is based on the fact that $\epsilon = \gamma_{\parallel}/\kappa < 1$ and $\bar{\gamma}_{\parallel}/\kappa = O(\epsilon)$. This limiting case of the LSA is suggested by recent experimental estimations of the LSA parameters.[12] When ϵ is too small, the usual (Hopf) bifurcation analysis of the transition from the zero-intensity state to time-periodic oscillations is singular, i.e., the perturbation expansion of the solution becomes nonuniform as ϵ approaches zero. We then propose a new perturbation analysis valid for small ϵ. This analysis reveals a secondary bifurcation from the time-periodic solutions to quasiperiodic solutions.

We are thus able to predict analytically the following sequence of transitions.

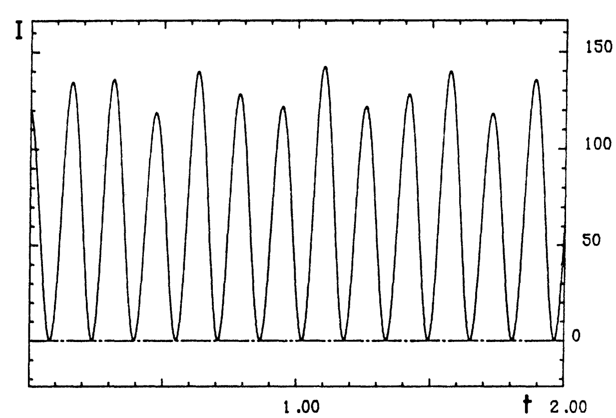

FIG. 7. Quasiperiodic intensities. Intensity is given as a function of time: $I = x^2(\epsilon t)\cos^2(t)$ where x is obtained from solving numerically Eqs. (5.1) with $p_2 = 0.2$, $p_3 = 6$, $p_1 = 125.7$ [i.e., $p_1 = p_1(2)$], and $\epsilon = 0.05$.

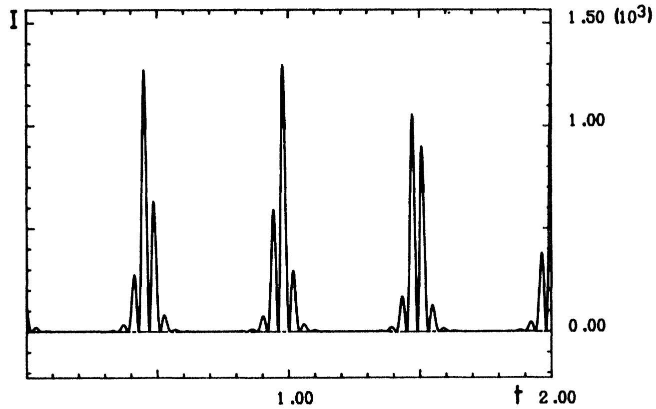

FIG. 8. Quasiperiodic pulsating intensities. As in Fig. 7, we represent the intensity $I=x^2(\epsilon t)\cos^2(t)$ as a function of t. x is obtained from Eqs. (5.1) with $p_2=0.2$, $p_3=6$, $p_1=170.2$ $[p_1=p_1(3)]$, and $\epsilon=0.013$. At this larger value of the bifurcation parameter p_1, we observe a quasiperiodic pulsed behavior.

(1) When $A=A^*$ [A^* refers to the primary Hopf bifurcation point and is defined by (2.5)], the LSA equations (1.1) admit a bifurcation from the zero-intensity state to a branch of time-periodic intensities given by

$$I(t)=\epsilon\frac{(A-A^*)}{\Gamma_1}\cos^2(\omega^* t)+O(\epsilon^2)\;, \tag{6.1}$$

where

$$A-A^*=O(\epsilon) \tag{6.2}$$

and Γ_1 is a constant coefficient independent of the bifurcation parameter A. The frequency ω^* of the time-periodic oscillations is given by (2.6). When $A>A^*$ and if

$$\Gamma_1>0\;, \tag{6.3}$$

i.e.,

$$\frac{g^2}{\bar{d}}\frac{\kappa}{\gamma_{\parallel}}-\frac{\bar{g}^2}{d}\frac{\kappa}{\bar{\gamma}_{\parallel}}>0\;,$$

then the zero-intensity state transfers its stability to time-periodic solutions. Otherwise, i.e., when $A>A^*$ but $\Gamma_1<0$, the system jumps to large intensities as it is suggested in Fig. 2.

(2) If, in addition to condition (6.3), the following condition is satisfied:

$$\frac{g^2}{\bar{d}}\frac{\kappa}{\bar{\gamma}_{\parallel}}-\frac{\bar{g}^2}{d}\frac{\kappa}{\gamma_{\parallel}}<0\;, \tag{6.4}$$

then when $A=A^{**}$ [A^{**} refers to the secondary bifurcation point and is defined by (4.24)], the LSA equations (1.1) admit a bifurcation from the time-periodic intensities to quasiperiodic intensities given by

$$I(t)=\epsilon^2\rho^2(\epsilon t)\cos^2[\omega^* t+\Gamma(\epsilon t)]+O(\epsilon^3)\;, \tag{6.5}$$

still with

$$A-A^*=O(\epsilon)\;. \tag{6.6}$$

The function $\Gamma(\epsilon t)$ is defined by (4.27) with $\tau'=\epsilon t$ and $\rho(\epsilon t)$ is obtained by solving Eqs. (4.8a)–(4.8c). As A approaches A^{**}, ρ and Γ have the following asymptotic behavior (see the Appendix):

$$\rho(\epsilon t)=\left[\frac{A^{**}-A^*}{\epsilon\Gamma_1}\right]^{1/2}+\left[\frac{A-A^{**}}{\epsilon\Gamma_2}\right]^{1/2}\cos(\omega^{**}\epsilon t)+O\left[\frac{A-A^{**}}{\epsilon}\right]\;, \tag{6.7}$$

$$\Gamma(\epsilon t)=\epsilon t\Gamma_3+O([(A-A^{**})/\epsilon]^{1/2})\;, \tag{6.8}$$

where $\omega^{**}=\omega(2)/\epsilon$ and $\omega(2)$ is defined by (4.26) and Γ_2 and Γ_3 are constant coefficients independent of A. The numerical simulations of Sec. V, as well as the analytical study presented in the Appendix, indicate that when $A>A^{**}$, the LSA system may present a transition from stable oscillations to stable quasiperiodic oscillations (see Fig. 3).

In this paper, we have analyzed the limit (1.12) of the LSA equations. The question remains open whether a different order of magnitude for d and $\bar{d}$ may qualitatively change our bifurcation results. To answer this question, we have examined the asymptotic behavior of the expansion (4.1)–(4.3) when

$$d=O(\bar{d})\to\infty \tag{6.9}$$

and when

$$d=O(\bar{d})\to 0\;. \tag{6.10}$$

In the first case, we observe that our perturbation analysis remains valid for all values of d and $\bar{d}$, i.e., the expansion (4.1) of the solutions remains uniform even if d and $\bar{d}$ become large. In the second case, however, we find that the expansion (4.1) of the solutions become nonuniform when $d=O(\bar{d})=O(\epsilon)$. In this critical regime, a new expansion of the solutions is needed in order to give a correct description of the solutions. This new regime will not be investigated in this paper. In conclusion, our perturbation analysis in the limit $\epsilon\to 0$ remains qualitatively valid for all d and $\bar{d}$ in the range $\epsilon<O(d)=O(\bar{d})<\infty$.

The results of our analysis are limited by two major points. First, we do not study the complete initial value problem of the LSA equations. Therefore, we cannot discuss the chances that these low-intensity regimes can be observed experimentally. In future work, we intend to explore in more detail this problem by numerical integration of the LSA equations. Second, from conditions (6.3) and (6.4), we observe that quasiperiodic solutions are possible when $\epsilon<1$ and $A-A^*=O(\epsilon)$ only if

$$\frac{\gamma_{\parallel}}{\kappa}\neq\frac{\bar{\gamma}_{\parallel}}{\kappa}\;. \tag{6.11}$$

Antoranz *et al.*[7] have concluded from their numerical analysis that quasiperiodic solutions may also exist if $\gamma_{\parallel}/\kappa=\bar{\gamma}_{\parallel}/\kappa$. In order to explain this different result, we started a new perturbation analysis which investigates the simultaneous limit $\epsilon\to 0$ and $\bar{A}\to\bar{A}_c$, where $\bar{A}_c$ is defined by (1.11). Moreover, we have directly studied the properties of the LSA equations by integrating numerically these equations when $\gamma_{\parallel}/\kappa=\bar{\gamma}_{\parallel}/\kappa$. The results will be presented elsewhere. They indicate the existence of three families of periodic solutions with quite distinct properties. Thus, the complexity of the time-dependent response, even in the low-intensity conditions, is rather surprising and emphasizes the need for asymptotic theories to reveal the principal mechanisms which are responsible.

Our study of the LSA equations in the low-intensity regime has another interest: the LSA solutions exhibit multiple similarities with the solutions described for double-diffusive convection.[21–25] The latter problem also depends on two control parameters and the conduction state admits a bifurcation to time-periodic solutions as well as a bifurcation to steady solutions. We hope that our analysis of the mechanism leading to quasiperiodic solutions in the LSA equations will suggest a new approach to obtain the same type of solutions for the more complicated double-diffusive convection equations.

ACKNOWLEDGMENTS

This research was supported by the U. S. Air Force Office of Scientific Research under Grant No. AFOSR-80-0016A, the U. S. Navy Office of Naval Research under Contract No. 00014-80-C-0196, the North Atlantic Treaty Organization (NATO) Research Grant No. 0348/83, and the Fonds National de la Recherche Scientifique (Belgium).

APPENDIX: HOPF BIFURCATION IN EQS. (5.1)

In order to study Eqs. (5.1), it is mathematically convenient to analyze an equation for $R \equiv x^2$, only. This equation can be obtained from (5.1) by decoupling the variable y and z. We find that R must satisfy the following equation:

$$\left[\frac{R'}{2R}\right]'' + \left[\frac{R'}{2R}\right]'(p_3+1) + \left[\frac{R'}{2R}\right][p_3+2R(1-p_2)] + R(p_3-p_2) - p_1p_3 = 0 , \quad \text{(A1)}$$

where the prime denotes d/dt and y and z are related to R by

$$y=(1-p_3)^{-1}\left[p_3\left[\frac{R'}{2R}\right]-p_1p_3+\left[\frac{R'}{2R}\right]'+R(1-p_2)\right] , \quad \text{(A2)}$$

$$z=[p_2(1-p_3)]^{-1}\left[\left[\frac{R'}{2R}\right]-p_1+\left[\frac{R'}{2R}\right]'+R(1-p_2)\right] . \quad \text{(A3)}$$

Equation (A1) admits a nontrivial steady-state solution given by

$$R=R_0=\frac{p_1p_3}{p_3-p_2}>0 \quad \text{(A4)}$$

and from the linear stability analysis of (A4) we observe that a Hopf bifurcation is possible if and only if

$$p_3>p_2 \text{ and } p_3>p_2^{-1} . \quad \text{(A5)}$$

Under the conditions (A5), we expect a transition from $R=R_0$ to time-periodic solutions $R=R(t)$ at a critical amplitude defined by

$$R_0=R_c=\frac{(p_3+1)p_3}{2(p_2p_3-1)} . \quad \text{(A6)}$$

Moreover, as R_0 tends to R_c, the deviation $R(t)-R_0$ approaches zero and the frequency σ of the oscillations tends to the critical frequency of the linear theory, given by

$$\sigma_c=\left[\frac{2R_c(p_3-p_2)}{1+p_3}\right]^{1/2} . \quad \text{(A7)}$$

In order to study the time-periodic solutions, we first simplify Eq. (A1) by defining the new variable

$$W=\ln R \quad \text{(A8)}$$

and by considering the following equation for W:

$$W'''+(1+p_3)W''+W'[p_3+2(1-p_2)e^W] + 2(p_3-p_2)e^W-2p_1p_3=0 . \quad \text{(A9)}$$

Then, we construct the time-periodic solutions by seeking solutions of the form

$$w(T,\eta)=W-\ln R_0=\eta w_1(T)+\eta^2 w_2(T)+\cdots , \quad \text{(A10)}$$

where $T=\sigma t$ (σ is the frequency of the oscillations) and η is the "amplitude" of w, defined according to the Poincaré-Lindstedt method by

$$\eta \equiv \frac{1}{2\pi}\int_0^{2\pi}[w(s,\eta)e^{-is}]ds . \quad \text{(A11)}$$

In addition to (A10), we also expand the unknown frequency σ and the bifurcation parameter R_0 as

$$\sigma(\eta)-\sigma_c=\eta^2\sigma_2+\cdots , \quad \text{(A12)}$$

$$R_0(\eta)-R_c=\eta^2 R_2+\cdots . \quad \text{(A13)}$$

When (A11)–(A13) are substituted in (A9), we obtain a sequence of reduced problems, of which the first three are

$$Lw_1\equiv\sigma_c^3(w_1'''+w_1')+\sigma_c^2(1+p_3)(w_1''+w_1)=0 , \quad \text{(A14)}$$

$$Lw_2=-2\sigma_c w_1'(1-p_2)R_c w_1-(p_3-p_2)R_c w_1^2 , \quad \text{(A15)}$$

$$\begin{aligned}Lw_3=&-2\sigma_c w_1'(1-p_2)R_c(w_2+w_1^2/2)-2\sigma_c w_2'(1-p_2)R_c w_1\\&-2(p_3-p_2)R_c(w_1w_2+w_1^3/6)-R_2[2\sigma_c w_1'(1-p_2)+2(p_3-p_2)w_1]\\&-\sigma_2\{3\sigma_c^2 w_1'''+2(1+p_3)\sigma_c w_1''+[p_3+2(1-p_2)R_c]w_1'\} .\end{aligned} \quad \text{(A16)}$$

The homogeneous problem (A14) has the solution

$$w_1 = e^{iT} + \text{c.c.} \, , \tag{A17}$$

where c.c. denotes complex conjugate. Problems (A15) and (A16) are inhomogeneous forms of (A14). In order to admit bounded solutions in T, the right-hand sides of (A15) and (A16) must satisfy a solvability or orthogonality condition. For Eq. (A15), this condition is always satisfied and the solution of (A15) is given by

$$w_2 = -1 + q_2 e^{2iT} + \text{c.c.} \tag{A18}$$

where q_2 is a complex coefficient defined by

$$\begin{aligned} q_2 &= \frac{R_c}{3\sigma_c^2} \frac{(\alpha + i\beta)}{(1+p_3)^2 + 4\sigma_c^2} \, , \\ \alpha &= (1+p_3)(p_3 - p_2) + 4\sigma_c^2(1-p_2) \, , \\ \beta &= 2\sigma_c(1 - p_2 p_3) \, . \end{aligned} \tag{A19}$$

We now substitute (A17) and (A18) into the right-hand side of (A16). Problem (A16) will have bounded solutions only if the forcing term is orthogonal to e^{iT}. The orthogonality condition requires that

$$\begin{aligned} -\sigma_c 2(1-p_2)R_c(-i + iq_2 + i/2) - 2(p_3 - p_2)R_c(-1 + q_2 + \tfrac{1}{2}) \\ - R_2[2(1-p_2)i\sigma_c + 2(p_3 - p_2)] - \sigma_2[-2i\sigma_c^2 - (1+p_3)2\sigma_c] = 0 \, . \end{aligned} \tag{A20}$$

Then, from the real and imaginary parts of (A20), we determine the unknown quantities R_2 and σ_2. The time-periodic solutions bifurcate supercritically if $R_2 > 0$ and as a consequence are linearly stable solutions. We have verified that this will be the case when $p_3 \gg 1$, $p_2 = O(1)$. Thus, we have shown analytically that Eqs. (5.1) admit stable time-periodic solutions as suggested by our numerical simulations for $p_2 = 0.2$ and $p_3 = 6$.

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