Atom in a damped cavity

Subir Sachdev*
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 9 December 1983)

The interaction of a damped cavity with a two-level atom is examined. The treatment is fully quantum mechanical. Expressions are obtained for the effect of damping and fluctuations in the walls of the cavity on the atom-cavity interaction, at zero temperature and in the limit of small damping. The physical situation can be realized with Rydberg atoms and a cavity.

I. INTRODUCTION

The spontaneous radiation of atoms in cavities is attracting increasing attention because of the emergence of new techniques, employing Rydberg atoms, which permit observation of spontaneous processes at microwave or millimeter wavelengths.[1-4] The fact that the spontaneous emission rate of an atom in a resonant cavity is approximately Q times greater than the free-space value was pointed out by Purcell.[5] The quantum-mechanical equation of motion of a two-level atom coupled to a single resonant mode of a cavity in the absence of dissipation was solved exactly by Jaynes and Cummings.[6] They found that this interaction led a to simple oscillation of energy between the atom and the cavity mode. Remarkably, this result is identical to the well-known semiclassical solution of a two-level system interacting with the electromagnetic field. More recently Kleppner[1] has pointed out the possibility of inhibiting spontaneous emission by placing an atom in a cavity that is small compared to the wavelength of the emission. The novel properties of Rydberg atoms—long transition wavelengths, long-lived high-l states, and large dipole moments—make the observation of such phenomena experimentally feasible. Experimental observation of inhibited blackbody absorption has been reported by Vaidyanathan *et al.*[2] Raimond *et al.*[3] have reported the observation of blackbody-induced radiation by an ensemble of atoms in a tuned resonator. The enhancement of the single-atom spontaneous emission in a cavity has been observed by Goy *et al.*[4]

In light of these advances it is of interest to examine theoretically the effect of the losses in the walls of the cavity on the interaction between the atom and the cavity. Damping in the interaction of atoms with the electromagnetic field is usually treated in a semiclassical manner by the introduction of a phenomenological damping constant. In the experimental situation of the spontaneous emission of an atom into a single mode of a cavity, however, it is no longer a valid approximation to treat the effect of the damping in a semiclassical way. The thermal fluctuations that must necessarily accompany the damping[7] introduce a fluctuation in the photon occupation number of the cavity and eventually bring the atom into thermal equilibrium with the radiation field. Since the system of a single atom in a cavity at zero temperature contains one photon, it is important to include the effect of the fluctuations quantum mechanically. In this paper we examine the quantum theory of the damped cavity and atom in the limit of small damping. The atom is modeled as a two-state quantum system. This can be an excellent approximation for the two highest angular momentum Rydberg states, for only a single electric dipole transition is possible and coupling to all other levels can be ignored. The cavity is modeled as a simple harmonic oscillator. We ignore the effects of other cavity modes which is valid for a high-Q cavity in one of its fundamental modes. The damping of the walls is treated by the reservoir approach of Lax.[8] This problem has also recently been solved by Haroche[9] using density operator methods in the dressed atom model.

The plan of the paper is as follows. In Sec. II the operators necessary to study the dynamics of the atom-cavity system are introduced. The coupling of the cavity to a thermal reservoir is carried out following the approach of Lax. In Sec. III the equations of motion are derived. These equations, which are expressed in terms of c-number variables, do not involve the dynamics of the reservoir explicitly. It is shown that these equations produce the correct limiting behavior of thermal equilibrium. In Secs. IV and V two exact solutions are obtained. In Sec. IV the exact solution is presented for arbitrary temperature but with zero damping. This solution corresponds to the solution of Jaynes and Cummings. Section V presents the solution for arbitrary damping at zero temperature. Two kinds of behavior are revealed. For small damping there are damped Rabi oscillations; for large damping the atom loses energy exponentially. In the latter case the decay rate is Q times faster than in free-space value. In Sec. VI the solutions found in Secs. IV and V are applied to obtain an approximate solution for small damping at finite temperatures. For small temperatures the result is damped Rabi oscillations. Thermal fluctuations wash out the oscillations at higher temperatures.

II. DEFINITION OF THE OPERATORS

We start by defining operators suitable for describing the radiation field in a cavity, a two-level atom, and a thermal reservoir. We describe the single mode of the

cavity at frequency ω, by the creation and destruction operators $a^\dagger$ and a. These obey the commutation relation

$$[a, a^\dagger]=1 . \tag{2.1}$$

The Hamiltonian of the cavity field H_c is

$$H_c=\hbar\omega a^\dagger a . \tag{2.2}$$

The atom is assumed to behave as a two-level system. Such a system can be described by the Dicke[10] operators R_+, R_-, and R_3 which satisfy the commutation relations

$$[R_3, R_+]=R_+ , \tag{2.3}$$

$$[R_-, R_3]=R_- , \tag{2.4}$$

$$[R_+, R_-]=2R_3 , \tag{2.5}$$

$$R_3^2=\tfrac{1}{4},\ R_+^2=R_-^2=0 . \tag{2.6}$$

The atomic Hamiltonian H_a is

$$H_a=\hbar\omega R_3 . \tag{2.7}$$

The operator R_3 is therefore a measure of the energy of the cavity and its expectation value is directly measurable. The population of the upper atomic level is given by $\frac{1}{2}(1+\langle R_3\rangle)$. In the rotating wave approximation, the dipole coupling between the atom and the cavity is[6]

$$H_i=\frac{ih\lambda}{2}(R_+a-a^\dagger R_-) . \tag{2.8}$$

The coupling parameter λ is

$$\lambda=-\frac{2d}{\hbar}\left[\frac{2\pi\hbar\omega}{V}\right]^{1/2} . \tag{2.9}$$

d is the dipole matrix element between the two levels of the system. V is the effective volume of the cavity and is given by

$$\frac{1}{V}=\frac{\vec{u}^{\,2}(\bar{r})}{\int \vec{u}^{\,2}(r)dV} . \tag{2.10}$$

$\vec{u}(r)$ is the electric field of the cavity mode and the atom is at the position $\bar{r}$. The atom is assumed to be polarized in a direction parallel to $\vec{u}(\bar{r})$. The atom-cavity Hamiltonian H_t is therefore

$$H_t=H_c+H_a+H_i . \tag{2.11}$$

In addition the cavity is coupled to a thermal reservoir through the walls of the cavity. We follow the approach of Lax[8] in which all explicit reference to the reservoir is eliminated by introducing frequency shifts and dissipation coefficients in the Heisenberg equations of motion for the cavity operators. The exact mechanism of the coupling of the field to the atoms in the walls of the cavity is unimportant. The reservoir degrees of freedom can be averaged over provided that suitable noise sources with correct moments are added. Lax has shown that the interaction between the cavity mode and the reservoir can be represented by

$$\frac{da}{dt}=-(i\omega+\gamma)a+f(t) , \tag{2.12}$$

$$\frac{da^\dagger}{dt}=(i\omega-\gamma)a^\dagger+f^\dagger(t) . \tag{2.13}$$

γ is the dissipation coefficient and any frequency shifts have been absorbed into the definition of ω. $f(t)$ and $f^\dagger(t)$ are Langevin noise sources. Their values averaged over the thermal reservoir obey

$$\langle f(t)\rangle_R=\langle f^\dagger(t)\rangle_R=0 , \tag{2.14}$$

$$\langle f^\dagger(t)f(s)\rangle_R=2\gamma\bar{n}\delta(t-s) . \tag{2.15}$$

$\langle\ \rangle_R$ represents an expectation value over the reservoir degrees of freedom. $\bar{n}$ is the photon occupation number of the cavity in thermal equilibrium,

$$\bar{n}=\frac{1}{\exp(\hbar\omega/kT)-1} . \tag{2.16}$$

Using Eqs. (2.12)–(2.15), the equations of motion for arbitrary products of the creation and destruction operators can be shown to obey

$$\frac{d}{dt}\langle(a^\dagger)^r a^s\rangle_R=[i\omega(r-s)-\gamma(r+s)]\langle(a^\dagger)^r a^s\rangle_R + 2\gamma rs\bar{n}\langle(a^\dagger)^{(r-1)}a^{(s-1)}\rangle_R . \tag{2.17}$$

In the following discussion it will be found that the entire effect of the reservoir is described by this result.

III. EQUATIONS OF MOTION

The quantities of interest in the system are $\langle a^\dagger a\rangle$ and $\langle R_3\rangle$, the expectation values of the energy in the field and the atom. For a two-level atom $\langle R_3\rangle$ is related simply to the population of the two states, a convenient experimental observable. The rate of change of any system operator can be written as the sum of a contribution due to the total Hamiltonian H_t and a contribution arising from the interaction with the reservoir. Thus

$$\frac{d\langle\Theta\rangle}{dt}=\left[\frac{d\langle\Theta\rangle}{dt}\right]_t+\left[\frac{d\langle\Theta\rangle}{dt}\right]_R . \tag{3.1}$$

The first term on the right obeys the Heisenberg equations of motion

$$\left[\frac{d\langle\Theta\rangle}{dt}\right]_{H_t}=\frac{1}{ih}[\Theta, H_t] . \tag{3.2}$$

$(d\langle\Theta\rangle/dt)_R$ vanishes if Θ is a function only of the atomic operators because the atom is not explicitly coupled to the reservoir. $(d\langle\theta\rangle/dt)_R$ can be obtained from Eq. (2.17) for arbitrary products of the creation and destruction operators. Using Eqs. (3.1), (3.2), and (2.17), the equations of motion for $\langle a^\dagger a\rangle$ and R_3 are

$$\frac{d\langle a^\dagger a\rangle}{dt}=-2\gamma\langle a^\dagger a\rangle-\frac{\lambda}{2}\langle a^\dagger R_-+R_+a\rangle+2\gamma\bar{n} , \tag{3.3}$$

$$\frac{dR_3}{dt}=\frac{\lambda}{2}\langle a^\dagger R_-+R_+a\rangle . \tag{3.4}$$

These equations involve the Hermitian operator $\langle a^\dagger R_-+R_+a\rangle$ which obeys

$$\frac{d}{dt}\langle a^\dagger R_- + R_+ a\rangle = -\lambda\langle R_3\rangle - 2\lambda\langle a^\dagger R_3 a\rangle - \frac{\lambda}{2} - \gamma\langle a^\dagger R_- + R_+ a\rangle \ . \tag{3.5}$$

This equation introduces another operator $\langle a^\dagger R_3 a\rangle$, whose equation of motion involves higher-order products of operators such as $\langle a^\dagger a^\dagger R_3 aa\rangle$, $\langle a^\dagger a^\dagger aa\rangle$, and $\langle (a^\dagger a^\dagger R_- + R_+ aa)\rangle$. To deal with such operators it is convenient to define the c-number variables

$$A_p = \langle (a^\dagger)^p a^p\rangle, \quad p \geq 0, \quad A_0 = 1 \ ; \tag{3.6}$$

$$B_p = \langle (a^\dagger)^p R_3 a^p\rangle, \quad p \geq 0, \quad B_0 = \langle R_3\rangle \ ; \tag{3.7}$$

$$C_p = \langle (a^\dagger)^p a^{p-1} R_- + R_+ (a^\dagger)^{p-1} a^p\rangle, \quad p > 0 \ . \tag{3.8}$$

The observables $\langle R_3\rangle$ and $\langle a^\dagger a\rangle$ are equal to B_0 and A_1, respectively. Hence solving the dynamical equations for A_p, B_p, and C_p is equivalent to finding the dynamics of the system. From Eqs. (2.15), (3.1), and (3.2) the following relations are eventually obtained:

$$\frac{dA_p}{dt} = -\frac{\lambda}{2} pC_p - 2\gamma pA_p + 2\gamma\bar{n}p^2 A_{p-1}, \quad p > 0 \ ; \tag{3.9a}$$

$$\frac{dB_p}{dt} = \frac{\lambda}{2} C_{p+1} + \frac{\lambda}{4} pC_p - 2\gamma pB_p + 2\gamma\bar{n}p^2 B_{p-1}, \quad p \geq 0 \ ; \tag{3.9b}$$

$$\frac{dC_p}{dt} = -\lambda pB_{p-1} - 2\lambda B_p - \frac{\lambda}{2} pA_{p-1} - \gamma(2p-1)C_p + 2\gamma\bar{n}p(p-1)C_{p-1}, \quad p > 0 \ . \tag{3.9c}$$

It is necessary to determine the initial and final values of the variables. If the field is initially in thermal equilibrium and the atom is in its excited state, then at $t = 0$,

$$A_p(t=0) = p!(\bar{n})^p \ , \quad B_p(t=0) = \tfrac{1}{2}p!(\bar{n})^p \ , \quad C_p(t=0) = 0 \ . \tag{3.10}$$

As $t \to \infty$ we expect both the atom and the field to approach thermal equilibrium. Assuming $dA_p/dt = dB_p/dt = dC_p/dt = 0$ as $t \to \infty$, we can solve Eqs. (3.9)–(3.11) to obtain the steady-state solutions

$$A_p(t\to\infty) = p!(\bar{n})^p \ , \quad B_p(t\to\infty) = -\tfrac{1}{2}p!(\bar{n})^p \frac{1}{1+2\bar{n}} \ , \quad C_p(t\to\infty) = 0 \ . \tag{3.11}$$

In particular we see from the expression for B_p that

$$\langle R_3(t\to\infty)\rangle = -\frac{1}{2}\left[\frac{\exp(\hbar\omega/kT)-1}{\exp(\hbar\omega/kT)+1}\right] . \tag{3.12}$$

This correctly describes the two-level atom in thermal equilibrium with the field.

IV. SOLUTION FOR THE UNDAMPED SYSTEM

To help motivate the general solution of Eqs. (3.9) we consider the case $\gamma = 0$. Equations (3.9) become

$$\frac{dA_p}{dt} = -\frac{\lambda}{2} pC_p \ , \tag{4.1}$$

$$\frac{dB_p}{dt} = \frac{\lambda}{2} C_{p+1} + \frac{\lambda}{4} pC_p \ , \tag{4.2}$$

$$\frac{dC_p}{dt} = -\lambda pB_{p-1} - 2\lambda B_p - \frac{\lambda}{2} pA_{p-1} \ . \tag{4.3}$$

We can find the exact solution to these equations if initially there are exactly n photons in the field and the atom is in its excited state. Then at $t = 0$,

$$A_p(t=0) = \begin{cases} \dfrac{n!}{(n-p)!}, & p \leq n \\ 0, & p > n \end{cases} \tag{4.4}$$

$$B_p(t=0) = \begin{cases} \dfrac{1}{2}\dfrac{n!}{(n-p)!}, & p \leq n \\ 0, & p > n \end{cases} \tag{4.5}$$

$$C_p(t=0) = 0 \ . \tag{4.6}$$

These equations are difficult to solve directly, but by using the results of Jaynes and Cummings[6] it can be shown that for $p \leq n+1$ the solution is

$$A_p^{(n)} = \frac{n!}{(n-p+1)!}\left[(n - \tfrac{1}{2}p + 1) - \tfrac{1}{2}p\cos(\lambda\sqrt{n+1}\,t)\right] , \tag{4.7a}$$

$$B_p^{(n)} = \frac{n!}{2(n-p+1)!}\left[\tfrac{1}{2}p + (n - \tfrac{1}{2}p + 1)\cos(\lambda\sqrt{n+1}\,t)\right] , \tag{4.7b}$$

$$C_p^{(n)} = -\frac{n!}{(n-p+1)!}\sqrt{n+1}\sin(\lambda\sqrt{n+1}\,t) \ , \tag{4.7c}$$

while for $p > n+1$,

$$A_p = B_p = C_p = 0 \ .$$

Another useful situation to consider is the case where the field is initially in thermal equilibrium. The probability that the field has n photons is

$$P_n = r^n(1-r) \ , \tag{4.8}$$

where $r = \exp(-\hbar\omega/kT)$. The system is initally described by a weighted sum of the constants defined in (4.4)–(4.6) for n photons. The dynamical equations are linear so that the solution to Eqs. (4.1)–(4.3) is a weighted sum of the solutions in Eq. (4.7). The result is

$$A_p=(1-r)\sum_{n=0}^{\infty} r^n A_p^{(n)}$$
$$=\frac{p!r^{p-1}(1+r)}{2(1-r)^p}$$
$$-\frac{p(1-r)}{2}\sum_{n=p-1}^{\infty} r^n\frac{n!}{(n-p+1)!}\cos(\lambda\sqrt{n+1}t)\ , \tag{4.9a}$$

$$B_p=(1-r)\sum_{n=0}^{\infty} r^n B_p^{(n)}$$
$$=-\frac{p!r^{p-1}}{4(1-r)^{p-1}}$$
$$+\frac{1-r}{2}\sum_{n=p-1}^{\infty} r^n\frac{n!}{(n-p+1)!}(n-\tfrac{1}{2}p+1)$$
$$\times\cos(\lambda\sqrt{n+1}t) \tag{4.9b}$$

$$C_p=(1-r)\sum_{n=0}^{\infty} r^n C_p^{(n)}$$
$$=-(1-r)\sum_{n=p-1}^{\infty} r^n\frac{n!\sqrt{n+1}}{(n-p+1)!}\sin(\lambda\sqrt{n+1}t)$$
$$\text{for } p>0\ . \tag{4.9c}$$

In addition

$$B_0(t)=\frac{1-r}{2}\sum_{n=0}^{\infty} r^n\cos(\lambda\sqrt{n+1}t)\ . \tag{4.9d}$$

It can be verified that these are indeed solutions to Eqs. (4.1)–(4.3) with the initial conditions (3.10). Physically these solutions reveal the oscillation of energy between the atom and the cavity in a superposition of Rabi frequencies.

V. SOLUTION AT ZERO TEMPERATURE

Exact solutions to Eqs. (3.9) can be obtained at zero temperature. The results should also apply in the limit $\bar{n}\ll 1$. Using the initial conditions in Eq. (3.10) for $\bar{n}=0$, the following solutions are easily found:

$$A_p=B_p=C_p=0 \quad \text{for } p\geq 2\ , \tag{5.1}$$

for all t. The vanishing of these variables has a simple interpretation: A cavity at zero temperature cannot emit photons. Consequently there is never more than one photon in the atom-cavity system. For $p\leq 2$, Eqs. (3.9) reduce to

$$\frac{dA_1}{dt}=-\frac{\lambda}{2}C_1-2\gamma A_1\ , \tag{5.1a}$$

$$\frac{dB_0}{dt}=\frac{\lambda}{2}C_1\ , \tag{5.1b}$$

$$\frac{dB_1}{dt}=\frac{\lambda}{4}C_1-2\gamma B_1\ , \tag{5.1c}$$

$$0=-2\lambda B_1-\lambda A_1\ , \tag{5.1d}$$

$$\frac{dC_1}{dt}=-\lambda B_0-2\lambda B_1-\frac{\lambda}{2}-\gamma C_1\ . \tag{5.1e}$$

These five equations are overdetermined because there are only four variables: A_1, B_0, B_1, and C_1. The situation is straightforward:

$$B_0(t)=\langle R_3(t)\rangle=-\frac{1}{2}+\frac{\exp(-\gamma t)}{2(\gamma^2-\lambda^2)}\left[-\lambda^2+\left[\gamma^2-\frac{\lambda^2}{2}+\gamma(\gamma^2-\lambda^2)^{1/2}\right]\exp[(\gamma^2-\lambda^2)^{1/2}t]\right.$$
$$\left.+\left[\gamma^2-\frac{\lambda^2}{2}-\gamma(\gamma^2-\lambda^2)^{1/2}\right]\exp[-(\gamma^2-\lambda^2)^{1/2}t]\right]\ , \tag{5.2a}$$

$$A_1(t)=\langle a^\dagger a(t)\rangle=-\frac{\lambda^2\exp(-\gamma t)}{2(\gamma^2-\lambda^2)}+\frac{\lambda^2\exp(-\gamma t)}{2(\gamma^2-\lambda^2)}\cosh[(\gamma^2-\lambda^2)^{1/2}t]\ . \tag{5.2b}$$

Two limiting cases of this solution are of interest.

A. Underdamped solution, $\gamma<\lambda$

In this case the arguments of the exponentials are imaginary and the solutions are oscillatory. As a result there is an exchange of energy between the atom and cavity. If $\gamma\ll\lambda$, the solution takes the simple and suggestive form

$$\langle R_3(t)\rangle=-\frac{1}{2}+\frac{\exp(-\gamma t)}{2}[1+\cos(\lambda t)]\ . \tag{5.3}$$

These are damped Rabi oscillations at the frequency λ.

B. Overdamped solution, $\gamma>\lambda$

In this case the energy of the atom decays as the sum of three exponentials. For $\gamma\gg\lambda$, a single exponential dominates and the solution is

$$\langle R_3(t)\rangle=\exp\left[-\frac{\lambda^2}{2\gamma}t\right]\ . \tag{5.4}$$

Using Eq. (2.9), the damping constant Γ becomes

$$\Gamma=\frac{\lambda^2}{2\gamma}\ ,$$
$$=\frac{4e^2\omega^3 d^2}{3\hbar c^3}\frac{\omega}{2\gamma}\frac{6\pi c^3}{V\omega^3}\ . \tag{5.5}$$

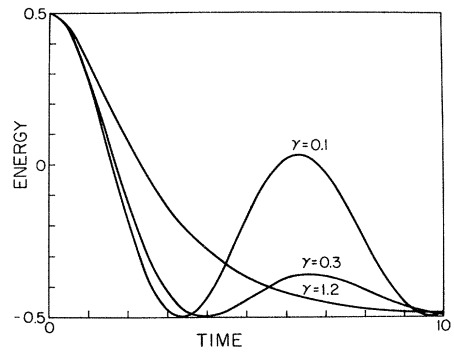

FIG. 1. Decay of the energy of an atom in a tuned cavity at zero temperature for different values of γ. Energy is measured in units of $\hbar\omega$. Time and γ are measured in units of $1/\lambda$ and λ, respectively.

The first factor is the decay rate of the atom in free space; the second is the quality factor Q of the cavity. The third factor depends upon the geometry of the cavity and is of order 1. (For the lowest mode of a simple cubic cavity it is $6/\pi^2$.) Thus, up to a geometric factor of order unity, the decay of the atom in an overdamped cavity is Q times faster than it is in free space. This is the result Purcell[5] obtained from phenomenological arguments.

Equation (5.2a) is plotted in Fig. 1 for different values of γ. The transition from the underdamped case to the overdamped case can clearly be seen. To a factor of order unity this transition takes place when the Q of the cavity is

$$Q\simeq(\omega/2A)^{1/2}\,. \tag{5.6}$$

A is the decay rate of the atom in free space.

VI. DAMPED SYSTEM AT FINITE TEMPERATURE

We have found exact solutions to Eqs. (3.9) in two regimes: for zero damping at arbitrary temperature and for arbitrary damping at zero temperature. In this section we will match these two solutions to produce an approximate solution for small damping at finite temperatures. For times which are shorter than $1/\gamma$ the system cannot feel the effect of the damping. For such times the solution should be identical to Eqs. (4.9) up to a change in the amplitude and a shift in the baseline. We anticipate that the amplitude of the oscillations and the baseline will vary only over times which are larger than $1/\gamma$ From Eqs. (3.11) and (5.3) we also know that the amplitude of the oscillations will eventually go to zero and the system will approach thermal equilibrium. With these facts in mind we make the following ansatz:

$$A_p(t)=\frac{p!r^p}{(1-r)^p}+\frac{p!r^{p-1}}{2(1-r)^{p-1}}a_p(t)-\alpha(t)\frac{p(1-r)}{2}\sum_{n=p-1}^{\infty}r^n\frac{n!}{(n-p+1)!}\times\cos(\lambda\sqrt{n+1}t)\,. \tag{6.1}$$

$$B_p(t)=-\frac{p!r^p}{2(1-r)^{p-1}(1+r)}-\frac{p!r^{p-1}}{4(1-r)^{p-2}(1+r)}b_p(t)+\alpha(t)\frac{(1-r)}{2}\sum_{n=p-1}^{\infty}r^n\frac{n!}{(n-p+1)!}(n-\tfrac{1}{2}p+1)\times\cos(\lambda\sqrt{n+1}t)\,, \tag{6.2}$$

$$C_p(t)=c_p(t)-\alpha(t)(1-r)\sum_{n=p-1}^{\infty}r^n\frac{n!\sqrt{n+1}}{(n-p+1)!}\times\sin(\lambda\sqrt{n+1}t)\,, \tag{6.3}$$

$$B_0(t)=-\frac{1}{2}\left[\frac{1-r}{1+r}\right]+\frac{1-r}{2(1+r)}b_0(t)+\alpha(t)\frac{(1-r)}{2}\sum_{n=0}^{\infty}r^n\cos(\lambda\sqrt{n+1}t)\,. \tag{6.4}$$

We have introduced the new variables $a_p(t)$, $b_p(t)$, $c_p(t)$, and $\alpha(t)$ which are assumed to be constant over times that are short compared to $1/\gamma$. The forms of the equations have been chosen so that these variables satisfy the following conditions:

$$a_p(0)=b_p(0)=\alpha(0)=1\,,\qquad c_p(0)=c_p(\infty)=a_p(\infty)=b_p(\infty)=\alpha(\infty)=0\,. \tag{6.5}$$

Since $\alpha(t)$ multiplies a rapidly varying term, its behavior is difficult to determine. We simply use the solution obtained at zero temperature in Eq. (5.3) as a lowest-order approximation for $\alpha(t)$. Therefore,

$$\alpha(t)=\exp(-\gamma t)\,. \tag{6.6}$$

This solution holds for small T. A weak temperature dependence is expected at higher temperatures.

In solving for a_p, b_p, and c_p, the following dimensionless parameters are useful:

$$\tau=\gamma t,\quad x=\frac{\gamma}{\lambda}\ll 1\,. \tag{6.7}$$

If we now substitute Eqs. (6.1)–(6.4) into Eqs. (3.9) and perform a time average over the rapidly varying part, we obtain the following:

$$x\left[\frac{da_1}{d\tau}+2a_1\right]+c_1=0\,, \tag{6.8}$$

$$x\frac{db_0}{d\tau}=\frac{1+r}{1-r}c_1\,, \tag{6.9}$$

$$x\left[\frac{dc_1}{d\tau}+c_1\right]=\frac{1}{2}\left[\frac{1-r}{1+r}\right](b_1-b_0)\,, \tag{6.10}$$

$$x\left[\frac{db_1}{d\tau}+2b_1+\frac{4r}{1-r}b_0\right]=-\frac{1-r}{1+r}(2c_2+c_1)\,; \tag{6.11}$$

$$x\left[\frac{da_p}{d\tau}+2pa_p-2pa_{p-1}\right]=-\frac{(1-r)^{p-1}}{r^{p-1}(p-1)!}c_p\,,\qquad p>1\,; \tag{6.12}$$

$$x\left[\frac{db_p}{d\tau}+2pb_p-2pb_{p-1}\right]$$
$$=-\frac{(1-r)^{p-2}(1+r)}{r^{p-1}p!}(2c_{p+1}+pc_p)\ ,\quad p>1\ ; \tag{6.13}$$
$$x\left[\frac{dc_p}{d\tau}+(2p-1)c_p-\frac{2r}{1-r}p(p-1)c_{p-1}\right]$$
$$=-\frac{p!r^{p-2}}{4(1-r)^{p-2}(1+r)}[(1-r)b_{p-1}+2rb_p$$
$$-(1+r)a_{p-1}],\quad p>1\ . \tag{6.14}$$

To solve the equations for $a_p(t)$, $b_p(t)$, and $c_p(t)$ subject to the conditions (6.5), we expand the solution in powers of the parameters $x\ll 1$ as follows:

$$a_p=a_p^{(0)}+xa_p^{(1)}+x^2a_p^{(2)}+\cdots\ ,$$
$$b_p=b_p^{(0)}+xb_p^{(1)}+x^2b_p^{(2)}+\cdots\ , \tag{6.15}$$
$$c_p=c_p^{(0)}+xc_p^{(1)}+x^2c_p^{(2)}+\cdots\ .$$

We will only determine the leading terms $a_p^{(0)}(\tau)$, $b_p^{(0)}(\tau)$, and $c_p^{(0)}(\tau)$. We assume that x is small enough to make the following approximation for the boundary conditions:

$$a_p^{(0)}(0)=b_p^{(0)}(0)=1\ , \tag{6.16}$$
$$c_p^{(0)}(0)=a_p^{(0)}(\infty)=b_p^{(0)}(\infty)=c_p^{(0)}(\infty)=0\ . \tag{6.17}$$

Inserting the expansions (6.15) into Eqs. (6.8)—(6.14) and equating terms with the same order of x yields

$$c_p^{(0)}=0 \tag{6.18}$$

for all p. Inserting this into Eqs. (6.10)—(6.14) we obtain

$$b_1^{(0)}=b_0^{(0)}\ , \tag{6.19}$$

$$(1-r)b_{p-1}^{(0)}+2rb_p^{(0)}-(1+r)a_{p-1}^{(0)}=0\ . \tag{6.20}$$

These equations provide one set of constraints between the variables $a_p^{(0)}(\tau)$ and $b_p^{(0)}(\tau)$. Another set is obtained by eliminating the c_p's between Eqs. (6.12) and (6.13). The result is an equation involving the variables $a_{p+1}^{(0)}$, $a_p^{(0)}$, $a_{p-1}^{(0)}$, $b_p^{(0)}$, and $b_{p-1}^{(0)}$. These two sets of equations are recursive: given the values of $a_1^{(0)}(\tau)$ and $b_0^{(0)}(\tau)$ the remaining variables can be solved for successively. For example, eliminating c_1 between Eqs. (6.8) and (6.9), gives

$$\frac{da_1^{(0)}}{d\tau}+2a_1^{(0)}+\frac{1-r}{1+r}\frac{db_0^{(0)}}{d\tau}=0\ . \tag{6.21}$$

If we now write the trial solution

$$a_1^{(0)}(\tau)=\exp(-\Gamma\tau)\ , \tag{6.22}$$

which is consistent with the conditions (6.16) and (6.17), we obtain

$$b_0^{(1)}(\tau)=1+\frac{(\Gamma-2)(1+r)}{(1-r)\Gamma}[1-\exp(-\Gamma\tau)]\ . \tag{6.23}$$

This can satisfy Eqs. (6.16) and (6.17) only if

$$\Gamma=1+r\ , \tag{6.24}$$

in which case

$$a_1^{(0)}(\tau)=b_0^{(0)}(\tau)=\exp[-(1+r)\tau]\ . \tag{6.25}$$

The remaining $a_p^{(0)}(\tau)$ and $b_p^{(0)}(\tau)$ can now be constructed systematically from Eq. (6.20) and the equations obtained from eliminating the c_p's from Eqs. (6.12) and (6.13). These solutions are correct to zeroth order in x. It might seem that the ansatz in Eq. (6.22) was arbitrary, but in general it will not be possible to satisfy all the boundary conditions with an arbitrary ansatz. The fact that the ansatz of Eq. (6.22) satisfies the boundary conditions to zeroth to zeroth order is a check on its validity.

The final result for the energy transfer between the atom and the field is

$$\langle a^\dagger a(t)\rangle=\tfrac{1}{2}\exp[-\gamma(1+r)t]+\frac{r}{1-r}$$
$$-\frac{1-r}{2}\sum_{n=0}^{\infty}r^n\exp(-\gamma t)\cos(\lambda\sqrt{n+1}t)\ , \tag{6.26}$$

$$\langle R_3(t)\rangle=\frac{1-r}{2(1+r)}\exp[-\gamma(1+r)t]-\frac{1-r}{2(1+r)}$$
$$+\frac{1-r}{2}\sum_{n=0}^{\infty}r^n\exp(-\gamma t)\cos(\lambda\sqrt{n+1}t)\ . \tag{6.27}$$

This result is valid over the regime $\gamma\ll\lambda$. The expression for $\langle R_3(t)\rangle$ is plotted in Fig. 2 for different temperatures. At low temperatures there is an oscillation of energy between the atom and the cavity. However, as the temperature rises, an increasing number of cavity states are excited—the oscillations eventually interfere and cancel out. The energy then decays from the atom to the cavity and the system attains thermal equilibrium.

VII. DISCUSSION

We have solved the problem of a two-level atom interacting with a single mode of a damped cavity in two

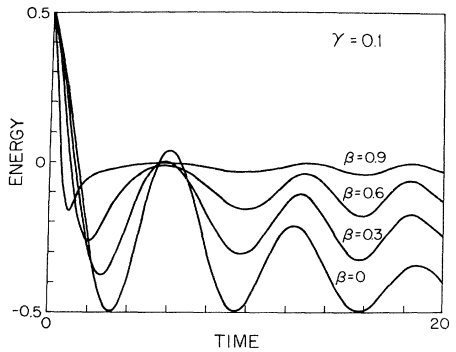

FIG. 2. Decay of the energy of an atom in a tuned cavity at different temperatures. γ is taken to be 0.1. Energy is measured in units of $\hbar\omega$. Time and γ are measured in units of $1/\lambda$ and λ, respectively.

limits. (a) Low temperatures: Two kinds of phenomena are observed. At high Q values there is an oscillation of energy between the atom and the cavity. At lower Q values the energy of the atom simply decays into the cavity. The transition occurs at a Q of the order of $(\omega/A)^{1/2}$. (b) High Q's: At low temperatures there is an oscillation of energy between the atom and the cavity. As the temperature is raised, the fluctuations in the photon occupation number of the cavity increase and the oscillations are eventually washed out.

This simple model of an interacting quantum system is an excellent approximation for the interaction of a Rydberg atom with a tuned cavity. Experiments to observe the oscillation of energy between the atom and the cavity have already been suggested and are currently in progress.[2,4]

ACKNOWLEDGMENTS

This problem was suggested to me by D. Kleppner. I acknowledge numerous enlightening discussions with him throughout the course of the work. I also acknowledge a useful discussion with S. Haroche. This work is based in part upon a thesis submitted at Massachusetts Institute of Technology (M.I.T.) in partial fulfillment of the requirements for the degree of Bachelor of Science. It was supported in part by the Undergraduate Research Opportunities Program, M.I.T., and the National Science Foundation.

---

*Present address: Department of Physics, Harvard University, Cambridge, MA 02138.

[1]D. Kleppner, Phys. Rev. Lett. 47, 233 (1981).

[2]A. G. Vaidyanathan, W. P. Spencer, and D. Kleppner, Phys. Rev. Lett. 47, 1592 (1981).

[3]J. M. Raimond, P. Goy, M. Gross, C. Fabre, and S. Haroche, Phys. Rev. Lett. 49, 117 (1982).

[4]P. Goy, J. M. Raimond, M. Gross, and S. Haroche, Phys. Rev. Lett. 50, 1903 (1983) and references therein.

[5]E. M. Purcell, Phys. Rev. 69, 681 (1946).

[6]E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963).

[7]H. B. Callen and T. A. Welton, Phys. Rev. 83, 34 (1951).

[8]M. Lax, Phys. Rev. 145, 110 (1965) and references therein.

[9]S. Haroche, *Proceedings of the Les Houches Summer School, New Trends in Atomic Physics,* edited by A. S. Greenberg (North-Holland, Amsterdam, 1982) and private communication.

[10]R. H. Dicke, Phys. Rev. 93, 99 (1954).