One-soliton Korteweg–de Vries solutions with nonzero vacuum parameters obtainable from the generalized inverse scattering method

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Previously the inverse scattering method has been applied by various authors (Gardner, Greene, Kruskal and Miura, and Lax) to obtain solutions $u(x,t)$ of certain nonlinear equations [e.g., the Korteweg–de Vries (KdV) equation] under the boundary condition $u(x,t)\rightarrow 0$ as $x\rightarrow\pm\infty$. Recently via Bäcklund transformation, Au and Fung have obtained the KdV one-soliton solution which contains the vacuum parameter $b\neq 0$, and b has been shown to be of physical significance. In fact, b is the boundary value of $u(x,t)$: $u(x,t)\rightarrow b$ as $x\rightarrow\pm\infty$. In this investigation we provide the generalized inverse scattering theory under the more general boundary condition $u(x,t)\rightarrow b\neq 0$ as $x\rightarrow\pm\infty$. The one-soliton solution obtainable from this inverse scattering method is identical to the new solution just found by Au and Fung [Phys Rev. B 25, 6460 (1982)]. The solution containing nonzero b is outside the square-integrable class. This extension of the class of functions has a crucial feature in attempting to understand physical observables.

I. INTRODUCTION

Recently, via the Bäcklund transformation[1–3] we have found new solutions to the Korteweg–de Vries (KdV) equation $u_t+12uu_x+u_{xxx}=0$. In our previous work, we have emphasized that the new solutions contain one extra parameter b more than that occur in previous solutions. This parameter, which is nonzero in general, has been called the vacuum parameter, and represents the value of u as $x\rightarrow\pm\infty$. We have demonstrated,[1] in fact, that b is a physical observable parameter: A soliton can travel to the left or to the right, or remain stationary, depending on the values of b taken. Moreover, for various b values, a soliton with a smaller amplitude can travel *faster* than another soliton with a larger amplitude. Our investigation implies that we need to extend our study of the solutions of the KdV equation of infinite length.

Based on our discovery, we raise the following question: Since using the differential geometrical approach, via Bäcklund transformation, a nonzero parameter exists in our solutions, can we use another approach to find the same set of solutions to the KdV equation? Would the parameter b appear? It is therefore interesting and worthwhile to review other standard approaches, to see whether a more general solution should include the parameter b. If the answer to the stated question is positive, such an investigation would also substantiate our previous results. In this study, we shall reexamine the inverse scattering approach in such an aspect.

Previously, Gardner, Greene, Kruskal, and Miura[4] have developed the inverse scattering method to solve the KdV equation under the boundary condition $u\rightarrow 0$ as $x\rightarrow\pm\infty$. Lax[5] formulated the method in a more general form under the same boundary condition. In Sec. II we shall provide the essential mathematical theorems and expressions within the inverse scattering regime under the more general boundary condition $u\rightarrow b\neq 0$ as $x\rightarrow\pm\infty$. Using the results deduced in Sec. II, we obtain in Sec. III the more general one-soliton solution which does include one more parameter b. Section IV lists the crucial ideas and implications of our investigation.

II. RELEVANT THEOREMS

Let $u(x,t)$ be a solution of the KdV equation

$$u_t+12uu_x+u_{xxx}=0 \tag{1}$$

such that $u(x,t)$ is sufficiently smooth and satisfies the boundary condition

$$u(x,t)\rightarrow b \quad \text{as} \quad x\rightarrow\pm\infty\ , \tag{2}$$

and also, all its x derivatives tend to zero as $x\rightarrow\pm\infty$.

The inverse scattering method of solving the KdV equation is proposed by Gardner *et al.*[4] In their application of the method, $u(x,t)$ is required to vanished as $x\rightarrow\pm\infty$ and satisfies the condition

$$\int_{-\infty}^{\infty}(1+|x|)|-2u(x,t)|\,dx<\infty\ . \tag{3}$$

Clearly, under the nonvanishing boundary condition (2), $u(x,t)$ does not satisfy condition (3). Hence difficulties arise in solving $u(x,t)$ by using the inverse scattering method.

In our analysis, we do not attempt to solve $u(x,t)$ directly using the inverse scattering method. Instead, we replace $u(x,t)$ by $u(x,t)-b$, use

$$V(x,t)\equiv-2[u(x,t)-b] \tag{4}$$

as a scattering potential of the Schrödinger problem, and assume that

$$\int_{-\infty}^{\infty}(1+|x|)|V(x,t)|\,dx<\infty\ . \tag{5}$$

The reason for adopting such an approach will be clarified as we follow our analysis.

Define the operators $L(t)$, $B(t)$, and $U(t)$ by

$$L(t)\equiv\frac{-\partial^2}{\partial x^2}+V(x,t)=\frac{-\partial^2}{\partial x^2}-2[u(x,t)-b]\ , \quad (6)$$

$$B(t)\equiv-4i\frac{\partial^3}{\partial x^3}-6i\left[u(x,t)\frac{\partial}{\partial x}+\frac{\partial}{\partial x}u(x,t)\right]\ , \quad (7)$$

$$i\frac{\partial U(t)}{\partial t}\equiv B(t)U(t),\ \ U(0)=I\ . \quad (8)$$

By a straightforward manipulation, we obtain the following theorem specifying the properties $L(t)$, $B(t)$, and $U(t)$.

Theorem 1. Let $L(t)$, $B(t)$, and $U(t)$ be defined by (6), (7), and (8), then

$$i\frac{\partial L}{\partial t}(t)=[B(t),\ L(t)]\ , \quad (9)$$

$$L(t)=U(t)L(0)U^{-1}(t)\ , \quad (10)$$

and

$$L(t)\psi(t)=k^2\psi(t),\ i\frac{\partial\psi(t)}{\partial t}=B(t)\psi(t) \quad (11)$$

where k^2 is independent of the parameter t. The operators $-\partial^2/\partial x^2-2u(x,t)$ and $B(t)$ are the common Lax's pair. In our case of study, the Lax's pair is replaced by $L(t)=-\partial^2/\partial x^2-2[u(x,t)-b]$ and $B(t)$. In view of theorem 1 and the condition (5), we can employ the inverse scattering method to obtain $V(x,t)$ for each time t. The function $u(x,t)$ can then be determined by using the relation $u(x,t)=-\frac{1}{2}[V(x,t)-2b]$. Hence $u(x,t)$ can be deduced readily under the more general boundary condition $u\to b$ as $x\to\pm\infty$.

To carry out the inverse scattering method, we consider the following eigenvalue problem:

$$-\frac{\partial^2\psi}{\partial x^2}-2[u(x,t)-b]\psi=k^2\psi\ , \quad (12)$$

where k is a constant. Suppose $\psi(k,t)$ is a solution of (12) satisfying the following boundary conditions:

$$\psi(k,t)\to\begin{cases}e^{-ikx}+R(k,t)e^{ikx} & \text{as } x\to\infty \quad (13)\\ T(k,t)e^{-ikx} & \text{as } x\to-\infty \quad (14)\end{cases}$$

where $R(k,t)$ and $T(k,t)$ are, respectively, the reflection and transmission coefficients of the wave solution to (12). Using (8), (10), and (11) we obtain the following theorem (see Appendix A for proof).

Theorem 2. Under the boundary condition $u(x,t)\to b$ as $x\to\pm\infty$, the reflection and transmission coefficients $R(k,t)$ and $T(k,t)$ of the wave solution $\psi(k,t)$ are given by

$$R(k,t)=R(k,0)e^{i(8k^3-24kb)t}\ ,$$
$$T(k,t)=T(k,0)\ . \quad (15)$$

We see that as $b=0$, theorem 2 is reduced to the known result $R(k,t)=R(k,0)e^{8ik^3t}$ for the special case $u\to0$ as $x\to\pm\infty$.

Since the potential $V(x,t)=-2[u(x,t)-b]$ satisfies condition (5), we can employ the inverse scattering method to determine $V(x,t)$ for each time t. Within the regime of this method, the potential $V(x,t)$ is given by

$$V(x,t)=-2\frac{dg(x,x)}{dx}\ , \quad (16)$$

where $g(x,y)$ for $x\le y$ is the solution of the Gel'fand-Levitan equation[6,7]

$$g(x,y)+K(x+y)+\int_x^\infty K(y+y')g(x,y')dy'=0 \quad (17)$$

with

$$K(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(k,t)e^{iky}dk+\sum_{n=1}^{N}M_ne^{-K_ny}\ , \quad (18)$$

where $-K_n^2$, $n=1,2,\ldots,N$ are the bound-state energies of the operator $L(t)=-\partial^2/\partial x^2+V(x,t)$, and M_n are the normalization constants for the bound states.

Given the reflection coefficient $R(k,t)$, the bound-state energies $-K_n^2$, and the normalization constants M_n, we can solve the Gel'fand-Levitan equation for $g(x,y)$, and thus $V(x,t)$ can be determined by using relation (16). By employing equations (8), (10), and (11), we obtain the following theorem.

Theorem 3. Under the boundary condition $u(x,t)\to b$ as $x\to\pm\infty$, the normalization constant $M_n(t)$ for the bound states of the operator $L(t)=-\partial^2/\partial x^2+V(x,t)$ is given by

$$M_n(t)=e^{(8K_n^3+24bK_n)t}M_n(0)\ . \quad (19)$$

Again, expression (19) becomes the conventional result $M_n(t)=e^{8K_n^3t}M_n(0)$ when $b=0$, as expected. The proof of the theorem is presented in Appendix B.

III. ONE-SOLITON SOLUTION FOR $b\neq0$

Let us consider the initial potential $-2[u(x,0)-b]$ which has only one bound state of energy $-K^2$, where K is a real number, and with no reflection, i.e., $R(k,0)=0$. From the Gel'fand-Levitan equation (17) and using relation (19), we yield

$$g(x,y,t)+M(0)e^{(8K^3+24bK)t}e^{-K(x+y)}+\int_x^\infty M(0)e^{(8K^3+24bK)t}e^{-K(y+y')}g(x,y',t)dy'=0\ . \quad (20)$$

We now express $g(x,y,t)$ in the form

$$g(x,y,t)=e^{-Ky}h(x,t) \quad (21)$$

and substituting (21) into (20), we arrive at

$$g(x,y,t)=-M(0)\frac{e^{(8K^3+24bK)t}e^{-K(x+y)}}{1+\dfrac{M(0)}{2K}e^{(8K^3+24bK)t}e^{-2Kx}}\ . \quad (22)$$

In view of relation (16), we obtain the solution $u(x,t)$:

$$u(x,t)=b+K^2\text{sech}^2\left[K\left[x-(4K^2+12b)t-\frac{\delta}{K}\right]\right]\ , \quad (23)$$

where the "phase" δ is

$$\delta=\tfrac{1}{2}\ln\frac{M(0)}{2k} . \tag{24}$$

Expressing a parameter λ such that

$$K=\sqrt{\lambda-2b} , \tag{25}$$

we can write (23) in the form

$$u(x,t)=b+(\lambda-2b)\operatorname{sech}^2\left[\sqrt{\lambda-2b}\left[x-4(b+\lambda)t-\frac{\delta}{\sqrt{\lambda-2b}}\right]\right] . \tag{26}$$

This one-soliton solution via the generalized inverse scattering method is identical to that obtained from the Bäcklund transformation approach obtained by Au and Fung[1] recently.

IV. CONCLUSIONS

From our investigation in this paper, we draw the following conclusions.

(1) Motivated by our previous work[1] on obtaining new solutions to the KdV equation, we have presented the generalized inverse scattering method under the more general boundary condition $u\to b\neq 0$ as $x\to\pm\infty$. As an example, we have deduced a set of one-soliton solutions which is identical to that obtained recently by Au and Fung[1] via the Bäcklund transformation method. In treating our problem, in the first step, instead of solving for $u(x,t)$ in the KdV equation, we solve for the potential $V(x,t)=-2[u(x,t)-b]$ using the inverse scattering method under the generalized boundary condition $u(x,t)$ tends to different constant values as $x\to\pm\infty$. In the second (obvious) step, $u(x,t)$ falls out readily once $V(x,t)$ is known.

(2) Perhaps one of the most important aspects of this investigation is that we have learned from our mathematical deduction (as summarized in theorem 1) to extend our class of $u(x,t)$ (solution) function to a non-square-integrable class. That is, the length $\int_{-\infty}^{\infty} u^* u\, dx$ is infinite. In the past, it was generally believed that most of the physical observables are embedded in the restricted square-integrable class. Our discovery indicates that in the nonlinear world, we may obtain physical information in the non-square-integrable class. For future research it is meaningful, therefore, to study certain special properties of functions in the non-square-integrable class, and see what new information about the physical world can arise.

(3) Under the general boundary condition, theorem 2 implies that the phase difference between the reflection and transmission coefficients (R and T) of the wave solution to the Schrödinger equation is, in general, nonzero and depends on the vacuum parameters b. For a one-soliton solution, the fact that there is a new phase difference in R and T implies that the velocity, and amplitude (and pulse width, in general) depend on the parameters b and λ. It is clear from expression (19) that the normalization constant M_n depends on b also. This result implies that the vacuum parameter b influences the energy spectrum and the properties of the bound states also. In fact, it is not difficult to see that when the potential $V(x,t)$ has changed from $-2u(x,t)$ (old case) to $-2[u(x,t)-b]$, $M_n(t)$ is necessarily changed.

(4) Some time ago, it was stated in Ref. 8 that "Interesting conditions that we have not treated include the case where $u(x,t)$ approaches different constants as $x\to\pm\infty$, and the case where $u(x,t)$ is periodic in x; the former would seem to yield to essentially the same approach here." This article has provided a concrete answer to the first part of the query raised. It appears that, with the inclusion of the more general boundary condition, more physical features can be deduced from solutions to nonlinear equations.[9–11] This investigation strengthens our belief, which Au and Fung have stated earlier:[1] that different vacuum states, as specified by the vacuum parameter b, of the nonlinear processes represented by the KdV and other nonlinear equations have different effects on the observable physical state.

APPENDIX A

Let $\psi(k,t)$ be a solution of the Schrödinger problem

$$L(t)\psi(k,t)=k^2\psi(k,t) \tag{12'}$$

satisfying the following boundary conditions:

$$\psi(k,t)\simeq\begin{cases} e^{-ikx}+R(k,t)e^{ikx} & \text{as } x\to\infty \quad (13)\\ T(k,t)e^{-ikx} & \text{as } x\to-\infty . \quad (14)\end{cases}$$

Define $\phi(k,t)$ by

$$\phi(k,t)\equiv U(t)\psi(k,0) , \tag{A1}$$

where $U(t)$ is given by (8). By using the time-evolution equation $i\partial U(t)/\partial t=B(t)U(t)$ and (A1) we obtain

$$i\frac{\partial\phi(k,t)}{\partial t}=B(t)\phi(k,t) . \tag{A2}$$

Next, using the relation $L(t)=U(t)L(0)U^{-1}(t)$ [Eq. (10)] and (12), we have

$$L(t)\phi(k,t)=k^2\phi(k,t) . \tag{A3}$$

In view of (12) and (A3), $\phi(k,t)$ satisfies the same equation as $\psi(k,t)$.

Let us now find out the asymptotic form of $\phi(k,t)$. Since $V(x,t)=-2[u(x,t)-b]\to 0$ as $x\to\pm\infty$, we have

$$\phi(k,t)\simeq\begin{cases} a_+(k,t)e^{ikx}+a_-(k,t)e^{-ikx} & \text{as } x\to\infty\\ b_+(k,t)e^{ikx}+b_-(k,t)e^{-ikx} & \text{as } x\to-\infty .\end{cases} \tag{A4}$$

Now, as $u(x,t)\to b$ when $x\to\pm\infty$, we have

$$B(t)=-4i\frac{\partial^3}{\partial x^3}-6i\left[u(x,t)\frac{\partial}{\partial x}+\frac{\partial}{\partial x}u(x,t)\right]$$

$$\simeq -4i\frac{\partial^3}{\partial x^3}-12ib\frac{\partial}{\partial x} \quad \text{as } x\to\pm\infty . \tag{A5}$$

Using (A2) and (A5), we arrive at

$$\begin{aligned} a_+(k,t)&=a_+(k,0)e^{i(4k^3-12kb)t} , \\ a_-(k,t)&=a_-(k,0)e^{-i(4k^3-12kb)t} , \\ b_+(k,t)&=b_+(k,0)e^{i(4k^3-12kb)t} , \\ b_-(k,t)&=b_-(k,0)e^{-i(4k^3-12kb)t} . \end{aligned} \tag{A6}$$

From (A1) and (13) and (14),

$$\begin{aligned} \phi(k,0)&=U(0)\psi(k,0) \\ &=\psi(k,0)\simeq \begin{cases} e^{-ikx}+R(k,0)e^{ikx} & \text{as } x\to\infty \\ T(k,0)e^{-ikx} & \text{as } x\to-\infty \end{cases} \end{aligned} \tag{A7}$$

hence we have

$$\begin{aligned} &a_+(k,0)=R(k,0),\quad a_-(k,0)=1 , \\ &b_+(k,0)=0,\quad b_-(k,0)=T(k,0) . \end{aligned} \tag{A8}$$

Substituting (A6) and (A8) into (A4), we obtain

$$\phi(k,t)\simeq \begin{cases} e^{-i(4k^3-12kb)t}e^{-ikx}+R(k,0)e^{i(4k^3-12kb)t}e^{ikx} & \text{as } x\to\infty \\ T(k,0)e^{-i(4k^3-12kb)t}e^{-ikx} & \text{as } x\to-\infty . \end{cases} \tag{A9}$$

From (A9), we have

$$\phi(k,t)e^{i(4k^3-12kb)t}\simeq \begin{cases} e^{-ikx}+R(k,0)e^{i(8k^3-24kb)t}e^{ikx} & \text{as } x\to\infty \\ T(k,0)e^{-ikx} & \text{as } x\to-\infty . \end{cases} \tag{A10}$$

Since $\psi(k,t)$ and $\phi(k,t)e^{i(4k^3-12kb)t}$ satisfy the same equation, (12), and both of them satisfy the same boundary condition, as can be seen from (13), (14), and (A10), we conclude that

$$\psi(k,t)=\phi(k,t)e^{i(4k^3-12kb)t} , \tag{A11}$$

hence we arrive at

$$R(k,t)=R(k,0)e^{i(8k^3-24kb)t}$$

and (15)

$$T(k,t)=T(k,0) ,$$

which are the required expressions.

APPENDIX B

Let $f(k,t)$ be the solution to Eq. (12) satisfying the following boundary condition:

$$f(x,k,t)\to e^{ikx} \quad \text{as } x\to\infty . \tag{B1}$$

The bound states of the operator $L(t)=-\partial^2/\partial x^2-2[u(x,t)-b]$ are $f(iK_n,t)$, where $-K_n^2$, $n=1,2,\ldots,N$ are the bound-state energies of $L(t)$. The normalization constant for the bound states is

$$M_n(t)=\left[\int_{-\infty}^{\infty} f^2(x,iK_n,t)dx\right]^{-1} . \tag{B2}$$

Let us determine the time dependence of $M_n(t)$. Consider the solution ψ_n, $n=1,2,\ldots,N$ to the equation

$$-\frac{\partial^2\psi_n}{\partial x^2}-2[u(x,0)-b]=-K_n^2\psi_n \tag{B3}$$

satisfying the following boundary conditions:

$$\psi_n\simeq \begin{cases} R_n(0)e^{-K_nx} & \text{as } x\to\infty \\ T_n(0)e^{K_nx} & \text{as } x\to-\infty \end{cases} \tag{B4}$$

where $R_n(0)$ and $T_n(0)$ are constants. Define

$$\psi_n(t)\equiv U(t)\psi_n , \tag{B5}$$

where $U(t)$ is given by Eq. (8). Since $i\partial U/\partial t(t)=B(t)U(t)$, we have

$$i\frac{\partial\psi_n}{\partial t}(t)=B(t)\psi_n(t) . \tag{B6}$$

Also, by using the relation $L(t)=U(t)L(0)U^{-1}(t)$, we obtain

$$L(t)\psi_n(t)=-K_n^2\psi_n(t) . \tag{B7}$$

Let

$$\psi_n(t)=\begin{cases} R_n(t)e^{-K_nx} & \text{as } x\to\infty \\ T_n(t)e^{K_nx} & \text{as } x\to-\infty . \end{cases} \tag{B8}$$

In fact, $R_n(t)$ and $T_n(t)$ can be determined by using the $i\partial\psi_n(t)/\partial t=B(t)\psi_n(t)$ [Eq. (B6)]. The result is

$$R_n(t)=R_n(0)e^{(4K_n^3+12bK_n)t} \tag{B9}$$

and

$$T_n(t)=T_n(0)e^{-(4K_n^3+12bK_n)t} . \tag{B10}$$

It is clear from (B8) that

$$\psi_n(t)=R_n(t)f_1(iK_n,t) . \tag{B11}$$

Hence, by using Eq. (B9), we have [recalling that $\psi_n(t)$ is also a function of x]

$$\begin{aligned} \int_{-\infty}^{\infty}\psi_n^2(x,t)dx&=R_n^2(t)\int_{-\infty}^{\infty}f_1^2(x,iK_n,t)dx \\ &=R_n^2(0)e^{(8K_n^3+24bK_n)t} \\ &\quad\times\int_{-\infty}^{\infty}f_1^2(x,iK_n,t)dx . \end{aligned} \tag{B12}$$

As U is unitary, we have

$$\int_{-\infty}^{\infty}\psi_n^2(x,t)dx=\int_{-\infty}^{\infty}\psi_n^2(x,0)dx . \tag{B13}$$

Hence

$$R_n^2(0)e^{(8K_n^3+24bK_n)t}\int_{-\infty}^{\infty} f_1^2(x,iK_n,t)dx = R_n^2(0)\int_{-\infty}^{\infty} f_1^2(x,iK_n,0)dx \; ,$$

or

$$\int_{-\infty}^{\infty} f_1^2(x,iK_n,t)dx = e^{-(8K_n^3+24bK_n)t}\int_{-\infty}^{\infty} f_1^2(x,iK_n,0)dx \; . \quad \text{(B14)}$$

That is,

$$M_n(t) \equiv \left[\int_{-\infty}^{\infty} f_1^2(x,iK_n,t)dx\right]^{-1} = e^{(8K_n^3+24bK_n)t}\left[\int_{-\infty}^{\infty} f_1^2(x,iK_n,0)dx\right]^{-1} = e^{(8K_n^3+24bK_n)t}M_n(0) \; . \quad \text{(B15)}$$

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