Intermittent diffusion: A chaotic scenario in unbounded systems

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In unbounded systems with discrete translational symmetry the Pomeau-Manneville scenario turns into a scenario involving intermittent diffusion. The velocity autocorrelation function, its power spectrum $S(\omega)$, and the mean-square displacements $\sigma^2(t)$ are calculated. We find excess noise ($S \sim \omega^{-2}$) at low frequencies and anomalous diffusion ($\sigma^2 \sim t^2$) of transient duration. We explain that the phenomenon can easily be observed in driven Josephson junctions.

In studies of chaotic systems, particular efforts are made to classify the transitions by which the chaotic state is approached when an external parameter is varied. The more likely roads to chaos have been classified as scenarios.[1] A prominent example is the Pomeau-Mannerville scenario, which has been studied extensively for iterative maps of a finite interval.[2–4] When in this scenario stable periodic orbits arise through a saddle-node (or tangent) bifurcation they are preceded by intermittent chaos. The latter is characterized by seemingly periodic episoides which are interrupted by short chaotic bursts. As the control parameter is varied this phenomenon emerges infinitely often, governed by bifurcation rates which tend to a universal constant.[5] This scenario has now been identified in chemical reactions,[6] Rayleigh-Bénard convection,[7] and other distinct physical systems.[8]

In the above cases the chaotic motions are restricted to a finite interval or to some other bounded region of phase space. In the present paper we study unbounded systems with discrete translational symmetry, in which the dynamics may extend to an entire axis [e.g., a particle in a periodic potential $V(x)$, $-\infty \leq x \leq \infty$]. Here the Pomeau-Manneville scenario turns into a scenario involving intermittent diffusion: Periodic motions can, in general, include a steady drift. When *drifting* periodic orbits arise through a saddle-node bifuracation they typically can be preceded by a *diffusive* chaotic motion. This diffusion is a deterministic "random" walk like the one we found in a previous work.[9,10] However, associated with intermittent characteristics it exhibits strongly correlated jumps.[11] We calculate the velocity autocorrelation function $C(t)$, its power spectrum $S(\omega)$, and the mean-square displacements $\sigma^2(t)$. We find that the scenario manifests itself by excess noise at low frequencies $[S(\omega) \sim \omega^{-2}]$ and by anomalous diffusion $[\sigma^2(t) \sim t^2]$ of transient duration. Both phenomena do not arise in the conventional Pomeau-Manneville scenario. We argue that this situation typically occurs abundantly in nonlinear systems with discrete translational symmetry and discuss how it can be detected in driven Josephson junctions, a prominent example.

The dynamics of strongly dissipative systems can in most cases be reduced to iterative one-dimensional (1D) maps via Poincaré section

$$X_{t+1} = F_\mu(X_t) \quad , \tag{1}$$

where X is a physical variable in a 1D phase space, t is a discrete time, and the map F_μ depends on an external control parameter μ.[1–15] We consider situations where the physical problem is invariant under translations $(X \to X+1)$ and reflections $(X \to -X)$, whence

$$F_\mu(X+n) = F_\mu(X)+n, \quad F_\mu(-X) = -F_\mu(X) \quad . \tag{2}$$

We may thus think in terms of unit cells of length 1 centered at the integers $X = 0, \pm 1, \pm 2, \ldots$. We assume F_μ to be analytic, with $F'_\mu(X) = 0$ only at one maximum and one minimum per cell. A paradigmatic example is

$$X_{t+1} = X_t - \mu \sin(2\pi X_t) \quad . \tag{3}$$

We have previously shown that these systems exhibit a deterministic diffusion with universal critical behavior at its onset.[9,10]

With increasing nonlinearity as the parameter μ is varied drifting periodic orbits arise through tangent bifurcations. We illustrate this in a case studied by Schell, Fraser, and Kapral.[11] Because of the existence of unit cells, it is convenient to identify all cells (mod 1) thereby constructing a reduced map.[12] Transforming to $x = X$ (mod 1) and restricting x to the zeroth cell $-\frac{1}{2} \leq x \leq \frac{1}{2}$ defines the reduced map

$$x_{t+1} = F_\mu(x_t) \, (\mathrm{mod}\ 1) =: f_\mu(x_t) \quad . \tag{4}$$

As shown in Fig. 1 all parts of $F_\mu(X_t)$ lapping into neighboring cells are transferred back into the zeroth cell $[-\frac{1}{2}, \frac{1}{2}]$. Working with the reduced map one must therefore declare, in addition, that whenever X_t is in the transfer regions J_1 or J_2 a transfer occurs to the equivalent X_{t+1} in a neighboring cell. It is evident from Fig. 1 that the reduced map may become tangent to the bisector in the transfer regions. Fixed points are thus created which imply consecutive transfers into neighboring cells ("jumps") in every iteration. For Eq. (3) and $\mu = 1$ these drifting orbits are simply $X_t = \frac{1}{4} - t$ and $X_t = -\frac{1}{4} + t$. Before the tangent bifurcation one finds intemittency in the transfer regions as shown in Fig. 1. An orbit injected into J_1 sojourns in the transfer region thus performing consecutive jumps before being reejected from J_1. In this way intermittency implies correlated jumps which, in turn, imply diffusion.

Tangent bifurcations are abundant.[5] Since here they generally create *drifting* periodic orbits, we expect intermittent diffusion to be abundant as well. This in now analyzed in more detail. Depending on the external parameter μ the p-fold iterate $F^p_\mu(X_t) = X_{t+p}$ can map into a cell a distance q apart. A tangent bifurcation then creates drifting orbits $X_{t+p} = X_t \pm q$ with drift velocity $\langle V \rangle = \pm q/p$. For circle maps $\langle V \rangle$ is known as rotation number. The preceding in-

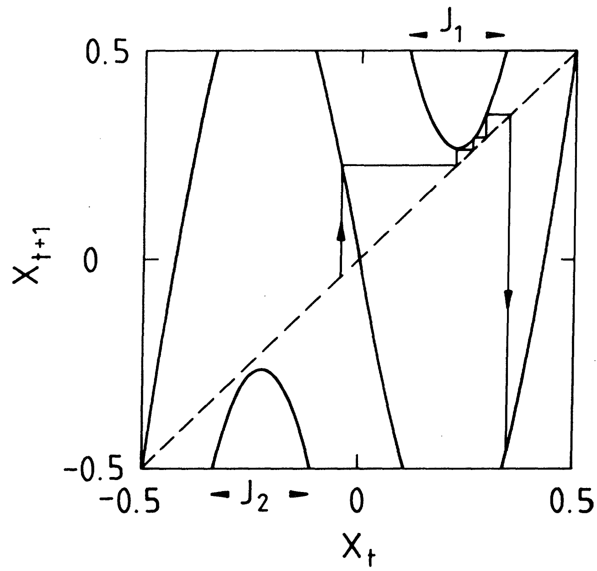

FIG. 1. Reduced map Eq. (4) for Eq. (3) with $\mu=0.97$. In the transfer regions J_1 and J_2 the overlap into neighboring cells was transferred back to the cell $[-\frac{1}{2},\frac{1}{2}]$. A visit of a transfer region thus implies a jump to a neighboring cell. Near tangency this happens successively as indicated.

termittent diffusion will be studied in detail for the case $p=1$: Here the reduced map Eq. (4) generally has two intermittent transfer regions analogous to J_1 and J_2 in Fig. 1. Whenever the orbit sojourns in a transfer region it has velocity $V\approx \pm q$ and $V\approx 0$ otherwise. The dynamics can thus be represented approximately by sequences with constant velocity $V=\pm q$ of different duration T shortly interrupted by $V=0$. T is simply the laminar time[2-4] arising in the reduced map Eq. (4) and can become arbitrarily long at the approach of tangency. A random switching between these sequences can arise. It is analogous to the random switching between laminar episodes and chaotic bursts in normal intermittency,[2-4] which is well understood. Here it implies a random walk. The following theory is carried out for $p=1$, but may also be considered a coarse-grained approximation for periods $p\neq 1$. In this case $V=\pm q$ must be replaced by the coarse-grained velocities $V=\pm q/p$ for the above sequences. Instead of $F(x)$ one must consider its p-fold iterate $F^p(x)$ in Eq. (4), which means a scaling tp of time. In general, the occurrence of intermittent diffusion requires that the orbits can switch between positive and negative velocities. Note that this is not always fulfilled. Another possibility consists in drifting intermittent orbits.

As outlined above we calculate some statistical quantities in continuous-time approximations[4] for the long-time limit. Details will be published elsewhere.[13] Expansion of the reduced map Eq. (4) around a contact point x_c leads to [4]

$$y_{t+1}=\epsilon+y_t+ay_t^2+\cdots \quad , \tag{5}$$

with $y_t=x_t-x_c$. The parameters ϵ and a are the only relevant parameters and depend on the details of the map $F_\mu(X)$. For Eq. (3), $x_c=\pm\frac{1}{4}$, $\epsilon=q-\mu$, and $a=2\pi^2q$. A continuum approximation yields the laminar time T and its distribution $\Psi(T)$ as in Ref. 4:

$$\Psi(T)=(\epsilon/2y_0)\cos^{-2}(\alpha-2\alpha T/T_m) \quad (T\leq T_m) \tag{6}$$

and $\Psi(T)=0$ for $T>T_m$. Here $T_m=2\alpha(\epsilon a)^{-1/2}$ is the maximum laminar time, $\alpha=\arctan[y_0(a/\epsilon)^{1/2}]$, and y_0 is the width of the laminar region.[4] Let $w_{0,t}$ denote the probability that at time 0 the system is in a laminar sequence with velocity $\pm q$ which persists at least to time t. Assuming random switching and applying renewal theory[11,14] this probability can be expressed as

$$w_{0,t}=\langle T\rangle^{-1}\int_0^\infty \Psi(T)(T-t)\theta(T-t)dT \quad . \tag{7}$$

From Eq. (7) we have computed the velocity autocorrelation function $C(t)=\langle V(0)V(t)\rangle$. For $t\leq T_m$

$$C(t)=q^2\{1-t/T_m+(ay_0T_m)^{-1} \times[\ln\cos\alpha-\ln\cos(\alpha-2\alpha t/T_m)]\} \quad , \tag{8}$$

whereas for $t>T_m$, $C(t)\equiv 0$. For small frequencies the velocity power spectrum $S(\omega)$ is obtained from $C(t)$ by a (continuous) Fourier transform

$$S(\omega)=\frac{4q^2}{T_m}\frac{\sin^2(\omega T_m/2)}{\omega^2} +\frac{2q^2\ln(2\cos\alpha)}{ay_0T_m}\frac{\sin(\omega T_m)}{\omega}+\Delta S \quad . \tag{9}$$

ΔS is a small remainder, which can be expressed as an infinite sum.[13] The mean-square displacements $\sigma^2(t)=\langle \Delta X^2(t)\rangle$ were obtained as usually from a convolution integral of $C(t)$:

$$\sigma^2(t)=q^2\begin{cases}(1+b/T_m)t^2-(3T_m)^{-1}t^3 & (t\leq T_m)\ ,\\ (T_m+2b)t-T_m^2/3-bT_m & (t>T_m)\ ,\end{cases} \tag{10}$$

where $b=(\ln 2\cos\alpha)/ay_0$, and a small remainder in terms of an infinite sum[13] was dropped.

We now discuss these results and compare with numerical data obtained for Eq. (3) with $\mu\lesssim 1$. The power spectrum (Fig. 2) exhibits oscillations due to the almost triangular shape of the correlation function $C(t)$. The spacing of these oscillations $2\pi/T_m$ depends on the maximum laminar time T_m and may become arbitrarily small. Similar spectra have been reported for driven Josephson junctions[15] although in a somewhat different context. The spectrum increases towards small frequencies. This so-called excess noise behaves like ω^{-2} as is seen from the maxima of Eq. (8). For still smaller frequencies $\omega<\pi/T_m$ it saturates at a finite value. The ω^{-2} regime may extend to an arbitrary number of decades as $\pi/T_m\to 0$ at the approach of tangency. The high-frequency regime is not expected to be reproduced accurately due to the continuous-time approximations used. The mean-square displacements in Fig. 3 exhibit transient anomalous diffusion. They grow like t^2 in a time domain depending on T_m. After the crossover time T_m, however, they follow the normal linear growth. *Asymptotically*, anomalous diffusion (without a crossover) was observed in a different intermittent situation and is reported elsewhere.[16] Transient t^2 growth was also observed in a piecewise linear map.[17] This behavior is accompanied by an enhancement of the diffusion coefficient.[11] More generally it also arises in Brownian motion in the Kramers regime (low friction limit).[18]

At present, the most prominent physical examples show-

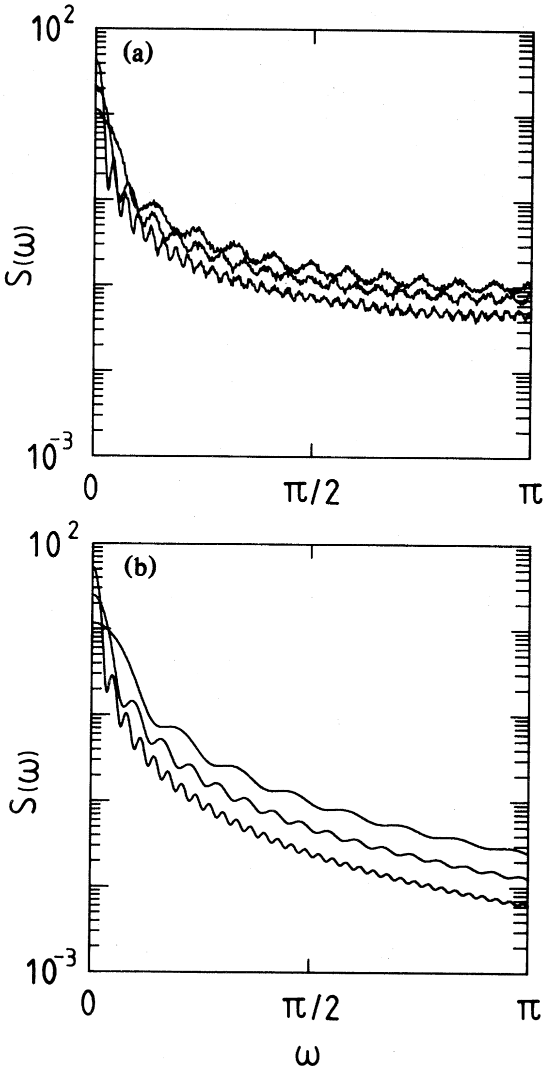

FIG. 2. Velocity power spectrum $S(\omega)$ for three parameters μ approaching tangency at $\mu=1$ in Eq. (3). Top: numerical results; bottom: theory [Eq. (9)]; parameters: $1-\mu=10^{-3}$ (slowest oscillations), 3×10^{-4}, 10^{-4}. There is excess noise at small frequencies falling off like ω^{-2}.

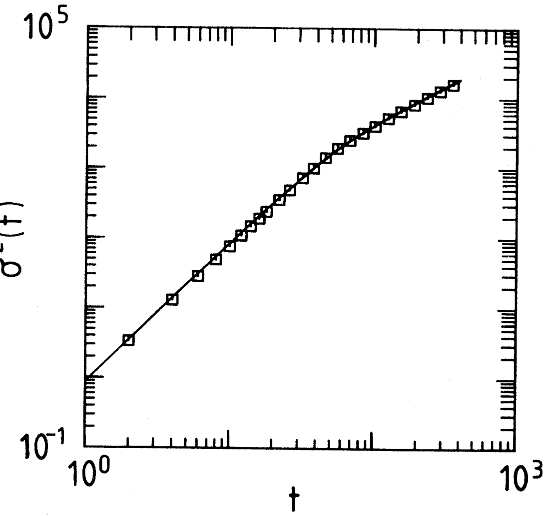

FIG. 3. Mean-square displacements $\langle \Delta X^2(t)\rangle = \sigma^2(t)$ for Eq. (3) with $1-\mu=10^{-4}$. Squares: numerical result; line: theory [Eq. (10)]. The anomalous t^2 growth crosses over to a linear growth at $t=T_m$. The anomalous regime becomes arbitrarily large for $\mu \to 1$.

ing deterministic diffusion are rf-driven Josephson junctions. Besides diffusion, drifting periodic orbits have been observed abundantly which set in through saddle-node bifurcations.[15,19] From general grounds[20] the dynamics of the junctions is expected to reduce to a 1D map (at least for strong dissipation), i.e., to the model Eqs. (1) and (2) considered here. Indeed, a 1D map has recently been extracted from numerical simulations.[21] In the particular example the simulation exhibited intermittent diffusion previous to a saddle-node bifurcation. An infinity of similar situations must exist. In the present paper we have given analytical expressions for the velocity autocorrelation function, power spectrum, and the mean-square displacements. The power spectrum is of particular importance, as spectral analysis is a typical and very convenient experimental method applied to Josephson junctions. It is also ideal for detecting and studying a diffusive motion as we now discuss: Following from the definition $D=\lim\langle \Delta X^2(t)\rangle/2t$, the diffusion coefficient can generally be expressed as the integral over the velocity correlation function, i.e., $D=(\frac{1}{2})S(\omega=0)$. For the junction, the diffusing variable is the phase difference ϕ, and according to Josephson its velocity $\dot{\phi}=(2e/\hbar)U$ is related to the voltage U across the junction. Thus $D=(2e^2/\hbar^2)\times S_{UU}(\omega=0)$, where $S_{UU}(\omega=0)$ is the low-frequency limit of the spectrum of voltage fluctuations, which can be measured easily. By the same argument the excess noise reported here for intermittent diffusion will directly show up in the low-frequency regime of the voltage spectrum.

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