Effects of rotation on radial heat flow in a gas

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A controversy which has appeared recently in the literature surrounding the topic of this paper has so far left open the question of whether the Boltzmann equation is invariant under rotation of the reference frame, and speculations regarding exotic consequences of the Coriolis acceleration have yielded physically paradoxical results. In this paper it is shown that by consistently using generalized canonical dynamical variables appropriate to a rotating reference frame, the exotic paradoxes dissolve, and a proof emerges that the physical consequences of the Boltzmann equation are invariant with respect to the rotation of the frame of reference.

I. INTRODUCTION

The question, "In a disc having radial temperature gradient does the flow of heat remain strictly radial when the disc is rotating?" has produced some interesting controversy. The traditional structure of continuum mechanics would demand frame indifference; the heat flow should, intuitively, remain purely radial. But the kinetic theory of heat flow is based on the Boltzmann ansatz of molecular chaos, which is in fact a violation of classical mechanics and, as pointed out by C. C. Wang,[1] the Boltzmann equation could very well fail to be frame indifferent. Wang concluded that the frame indifference of the Boltzmann equation was still an open question.

Müller[2] had studied the heat flow in a "rigidly rotating gas" and found that the Coriolis force should cause a net transport of energy across radial lines proportional to the temperature gradient. Hoover *et al.*[3] have recently claimed that computer simulations show that a rotating disc observed from a corotating reference frame does indeed show a transverse component of heat flow under the impact of a radial temperature gradient. Their motivation for this study was the observation that the Coriolis acceleration appears coupled with the temperature gradient in a second-order expansion term of the Boltzmann equation.

The argument implied in Ref. 3 is essentially as follows. The immediate effect of the Coriolis acceleration occurs during the free-path motion of the particle between collisions as visualized in the Boltzmann ansatz. This ansatz is the basis of the familiar Boltzmann equation which may be written, in its simplest form: If the unperturbed distribution function is f_0, the perturbed distribution function f can be expressed as a Taylor series either by

$$f=f_0-\tau DF-\tfrac{1}{2}\tau^2D^2f-\cdots$$

or, equivalently, by

$$f=f_0-\tau Df_0+\tfrac{1}{2}\tau^2D^2f_0-\cdots \quad . \tag{1}$$

We may write $D=D_0+D_1$, where D_0 is the derivative due to acceleration between collisions and D_1 that due to temperature gradient. It is claimed that the Coriolis acceleration gives rise to a term in $D_0D_1f_0$ which couples the angular velocity of the reference frame with the temperature gradient

$$2\Omega p_t\frac{\partial}{\partial p_t}\left[\frac{p_r}{m}\frac{dT}{dr}\frac{df_0}{dT}\right], \tag{2}$$

where p_t and p_r are the transverse and radial components of the linear momentum.

Because the Coriolis acceleration vanishes in any nonrotating reference frame, the term displayed in Eq. (2) must vanish even for a rotating gas if the gas is observed from a nonrotating frame. This raises serious doubt as to the "physical reality" of the effect. However, in this paper we shall derive a result equivalent to Eq. (2) from the canonical equations of motion of an ideal gas in rotation, without introducing the Coriolis acceleration, and prove the result to be independent of the rotation of the reference frame.

II. CANONICAL EQUATIONS OF MOTION FOR AN IDEAL GAS IN ROTATION

We consider an ideal gas contained in a perfectly smooth and rigid cylinder of radius R and unit length in the z direction. Collisions with the walls can change neither the kinetic energy nor the angular momentum. The question of how the gas is given its net angular momentum in the first place is not addressed. Cylindrical polar coordinates r,ϕ,z are employed.

A. Inertial reference system

The Lagrangian for a free particle[4] is

$$L_0=\tfrac{1}{2}m(\dot r^2+r^2\dot\phi^2+\dot z^2)\ . \tag{3}$$

The generalized momenta are then defined as

$$\begin{aligned}p_r&=\partial L_0/\partial\dot r=m\dot r\ ,\\ p_\phi&=\partial L_0/\partial\dot\phi=mr^2\dot\phi\ ,\\ p_z&=\partial L_0/\partial\dot z=m\dot z\ .\end{aligned} \tag{4}$$

Lagrange's equations of motion are

$$\dot{p}_r = \partial L_0/\partial r = mr\dot{\phi}^2 \, ,$$
$$\dot{p}_\phi = \partial L_0/\partial \phi = 0 \, , \quad (5)$$
$$\dot{p}_z = \partial L_0/\partial z = 0 \, .$$

The Hamiltonian is defined by

$$H_0 \equiv p_r\dot{r} + p_\phi\dot{\phi} - L_0 = (1/2m)(p_r^2 + p_\phi^2/r^2 + p_z^2) \, . \quad (6)$$

Hamilton's equations of motion are

$$\partial H_0/\partial p_r = \dot{r}, \;\; \partial H_0/\partial p_\phi = \dot{\phi}, \;\; \partial H_0/\partial p_z = \dot{z} \, , \quad (7)$$

$$\partial H_0/\partial r = -\dot{p}_r, \;\; \partial H_0/\partial \phi = -\dot{p}_\phi, \;\; \partial H_0/\partial z = -\dot{p}_z \, . \quad (8)$$

From Eq. (7) we then have

$$\dot{r} = p_r/m, \;\; \dot{\phi} = p_\phi/mr^2, \;\; \dot{z} = p_z/m \quad (9)$$

and

$$\dot{p}_r = p_\phi^2/mr^3, \;\; \dot{p}_\phi = 0, \;\; \dot{p}_z = 0 \, . \quad (10)$$

Note that p_ϕ is angular momentum, not the linear momentum p_t used in the discussion above of the Coriolis acceleration.

B. Rotating reference frame

Here[5] the Lagrangian for a free particle is

$$L = \tfrac{1}{2}m(r^2 + r^2\dot{\Phi}^2 + z^2) + m\Omega_f r^2\dot{\Phi} + \tfrac{1}{2}m\Omega_f^2 r^2 \, . \quad (11)$$

Here the coordinates r and z are presumably unchanged by the rotation, but $\Phi \neq \phi$, it being measured with respect to the rotating frame. Ω_f is the angular speed of rotation of the frame about the z axis. The new generalized momenta are now given by

$$P_r \equiv \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \;\; P_z \equiv \frac{\partial L}{\partial z} = m\dot{z} \, ,$$
$$P_\phi \equiv \frac{\partial L}{\partial \dot{\Phi}} = mr^2\dot{\Phi} + m\Omega_f r^2 \, . \quad (12)$$

Lagrange's equations of motion are

$$\dot{P}_r = \frac{\partial L}{\partial r} = mr\dot{\Phi}^2 + 2m\Omega_f r\dot{\Phi} + m\Omega_f^2 r \, ,$$
$$\dot{P}_\phi = \frac{\partial L}{\partial \Phi} = 0, \;\; \dot{P}_z = \frac{\partial L}{\partial z} = 0 \, . \quad (13)$$

Again the free-particle Hamiltonian is defined by

$$H_1 = P_r\dot{r} + P_\phi\dot{\Phi} - L \quad (14)$$
$$= m\dot{r}^2 + (mr^2\dot{\Phi} + m\Omega_f r^2)\dot{\Phi} - L + m\dot{z}^2$$
$$= \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\Phi}^2 + m\Omega_f r^2\dot{\Phi}$$
$$- m\Omega_f r^2\dot{\Phi} - \tfrac{1}{2}m\Omega_f^2 r^2 + \tfrac{1}{2}m\dot{z}^2$$
$$= \frac{P_r^2}{2m} + \frac{P_z^2}{2m} + \tfrac{1}{2}mr^2\frac{(P_\phi - m\Omega_f r^2)^2}{m^2 r^4} - \tfrac{1}{2}m\Omega_f^2 r^2 \, , \quad (15)$$

which may also be written in the form

$$H_1 = \frac{P_r^2}{2m} + \frac{P_z^2}{2m} + \frac{P_\phi^2}{2mr^2} - P_\phi\Omega_f \, . \quad (16)$$

Hamilton's equations of motion

$$\frac{\partial H_1}{\partial P_r} = \dot{r}, \;\; \frac{\partial H_1}{\partial P_\phi} = \dot{\Phi}, \;\; \frac{\partial H_1}{\partial P_z} = \dot{z} \, ,$$
$$\frac{\partial H_1}{\partial r} = -\dot{P}_r, \;\; \frac{\partial H_1}{\partial \Phi} = -\dot{P}_\phi, \;\; \frac{\partial H_1}{\partial z} = -\dot{P}_z \, , \quad (17)$$

yield the results

$$\dot{r} = \frac{P_r}{m}, \;\; \dot{\Phi} = \frac{P_\phi}{mr^2} - \Omega_f, \;\; \dot{z} = \frac{P_z}{m} \, ,$$
$$\dot{P}_r = \frac{P_\phi^2}{mr^3}, \;\; \dot{P}_\phi = 0, \;\; \dot{P}_z = 0 \, . \quad (18)$$

C. Maxwell-Boltzmann distribution, inertial frame

The most probable distribution having given total energy is the familiar Maxwell-Boltzmann distribution function. (This is a density-in-phase, with the phase element $dr\,d\phi\,dz\,dP_r dP_\phi dP_z$.)

$$f_0 = C_0 e^{-H_0/kT} \, , \quad (19)$$

where

$$C_0 = \rho(2\pi mkT)^{-3/2}, \;\; \rho = N/\pi(R^2 - r_0^2) \, . \quad (20)$$

We recall that $(1/kT)$ is the "undetermined multiplier" in the Lagrangian process of maximizing probability. The angular momentum given by this distribution is easily seen to be zero:

$$M = 2\pi C_0 \int_{r0}^{R} dr \int_{-\infty}^{\infty} \int \int p_\phi e^{-H_0/kT} d^3p \equiv 0 \, . \quad (21)$$

If we ask for the most probable distribution of a gas for which the total (conserved) angular momentum is not zero, we need another undetermined multiplier, conveniently written Ω/kT which is to be determined by the total angular momentum and the result is

$$f_1 = C_1 e^{-H_0/kT} e^{\Omega P_\phi/kT} \, , \quad (22)$$

where

$$C_1 = \rho(2\pi mkT)^{-3/2} \frac{m\Omega^2(R^2 - r_0^2)/2kT}{e^{m\Omega^2R^2/2kT} - e^{m\Omega^2r_0^2/2kT}} \quad (23)$$

and the total angular momentum is easily found to be

$$M = Nm\Omega\langle r^2\rangle \, ,$$

where $\langle r^2\rangle$ is the average of r^2 in the given distribution, and from this it is clear that Ω is to be interpreted physically as the mean velocity of rotation of the gas relative to the inertial frame.

D. Maxwell-Boltzmann distribution, rotating frame

Here the appropriate Hamiltonian is H_1 of Eq. (16). If we ask for the most probable distribution with only the restriction that this Hamiltonian be conserved, we find

$$f=C_1e^{-H_1/kT}\,, \tag{24}$$

and because the angular momentum in this frame is given by integrating the factor $mr\dot{\Phi}=P_\phi/r-mr\Omega_f$ [see Eq. (12)] over the distribution of Eq. (24), this turns out to be identically zero.

More generally we are interested in a gas with an average angular velocity, in the laboratory frame Ω, and observed from a rotating reference frame having an angular velocity Ω_f relative to the inertial laboratory frame. This requires that we add an undetermined multiplier $(\Omega-\Omega_f)/kT$ to conserve the angular momentum relative to the noninertial frame. The new distribution function is now

$$f_1=C_1e^{-H_1/kT}e^{P_\phi(\Omega-\Omega_f)/kT}\,. \tag{25}$$

When we define the angular momentum relative to the rotating frame as above, it integrates very easily to yield

$$M=Nm(\Omega-\Omega_f)\langle r^2\rangle\,. \tag{26}$$

To see this and other later developments it is convenient to write Eq. (25) in the form

$$f_1=C_1\exp[-(P_r^2+P_z^2)/2mkT-(P_\phi-\Omega mr^2)^2/2mr^2kT+mr^2\Omega^2/2kT]\,. \tag{27}$$

It is to be particularly emphasized that this distribution function contains no term depending explicitly on the rotation of the reference system itself, Ω_f. We thus have the theorem that the form of the distribution function in terms of the appropriate generalized canonical dynamical variables is invariant with respect to the rotation of the reference frame. Therefore if we can express Boltzmann's equation in these same dynamical variables, then that equation must be frame indifferent in the same sense.

Before proceeding with this, we present a couple of useful approximations valid to first order in the ratio $m\Omega^2R^2/kT$:

$$C_1=\rho(2\pi mkT)^{-3/2}[1-m\Omega^2(R^2+r_0^2)/4kT+\cdots]\,, \tag{28}$$

$$\langle r^2\rangle=\tfrac{1}{2}(R^2+r_0^2)+(m\Omega^2/24kT)(R^2-r_0^2)^2+\cdots\,, \tag{29}$$

when evaluated under the distribution of Eq. (24).

III. BOLTZMANN EQUATION IN CANONICAL DYNAMICAL VARIABLES: ROTATING REFERENCE SYSTEM

We now review the Boltzmann equation—Eq. (1) using generalized canonical dynamical variables, Eq. (12), and the canonical equations of motion, Eq. (17), appropriate to the rotating reference frame. The appropriate distribution function is f_1 of Eq. (27).

The Boltzmann equation now reads

$$f=f_1-\tau Df_1+\tfrac{1}{2}\tau^2D^2f_1-\cdots\,, \tag{30}$$

where $D=D_0+D_1$ with

$$D_0f_1=\dot{P}_r\frac{\partial f_1}{\partial P_r}+\dot{r}\frac{\partial f_1}{\partial r} \tag{31}$$

and

$$D_1f_1=\frac{P_r}{m}\left[\frac{dT}{dr}\frac{\partial f_1}{\partial T}+\frac{d\rho}{dr}\frac{\partial f_1}{\partial\rho}\right]. \tag{32}$$

[Here we have omitted those terms from the complete expansion that we know are individually zero, namely, $\dot{P}_\phi\partial f_1/\partial P_\phi$ and $\dot{P}_z\partial f_1/\partial P_z$; see Eq. (13), $\dot{z}\partial f_1/\partial z$ and $\dot{\Phi}\partial f_1/\partial\Phi$ which are all zero].

From Eq. (18)

$$\dot{P}_r=P_\phi^2/mr^3 \tag{33}$$

and from Eq. (27)

$$\frac{\partial f_1}{\partial r}=\frac{P_\phi^2}{mr^3kT}f_1\,,\qquad \frac{\partial f_1}{\partial P_r}=-\frac{P_r}{mkT}f_1\,. \tag{34}$$

Using these equations in Eq. (31) results in the identity

$$D_0f_1=0\,. \tag{35}$$

Note that Eq. (31) invokes the gradient in phase space with the six phase variables r,Φ,z,P_r,P_ϕ,P_z.

To determine D_1f_1 we need

$$\frac{\partial f_1}{\partial\rho}=\frac{f_1}{\rho}$$

and

$$\frac{\partial f_1}{\partial T}=\left[\frac{H_1-P_\phi(\Omega-\Omega_f)}{kT^2}+\frac{d\ln C}{dT}\right]f_1\,. \tag{36}$$

Using Eq. (6) this can be written

$$\frac{\partial f_1}{\partial T}=\left[\frac{H_0-\Omega P_\phi}{kT^2}+\frac{d\ln C}{dT}\right]f_1\,. \tag{37}$$

Finally we have

$$D_1f_1=\frac{P_r}{m}\left[\left[\frac{H_0-\Omega P_\phi}{kT}+\frac{d\ln C}{d\ln T}\right]\left[\frac{d\ln T}{dr}\right]+\frac{d\ln\rho}{dr}\right]f_1\,. \tag{38}$$

Using the approximation Eq. (28) we can now write down the "perturbed" distribution function to first order:

$$f=f_1-\tau\frac{P_r}{m}\left[\frac{H_0-\Omega P_\phi+\frac{1}{4}m\Omega^2(R^2+r_0^2)}{kT}\left[\frac{d\ln T}{dr}\right]+\frac{d\ln\rho}{dr}\right]f_1\,. \tag{39}$$

Write $H_0-\Omega P_\phi=P^2/2m-\frac{1}{2}m\Omega^2r^2$, where

$$P^2=P_r^2+P_z^2+(P_\phi-\Omega mr^2)^2/r^2\,. \tag{40}$$

Equation (27) becomes

$$f_1 = C_1 \exp(-P^2/2mkT + \tfrac{1}{2}m\Omega^2 r^2/kT) \tag{41}$$

and Eq. (39) reads

$$f = f_1 - \tau(P_r/m)\{[G(r) + P^2/2mkT](d\ln T/dr) + d\ln\rho/dr\} f_1 \,, \tag{42}$$

where

$$G(r) = -\tfrac{1}{2}m\Omega^2[r^2 - \tfrac{1}{2}(R^2 + r_0^2)] \,. \tag{43}$$

The currents of interest are as follows. Total radial particle current at radius r:

$$J(r)_{\rm rad} = 2\pi \int_{-\infty}^{\infty}\int\int (P_r/m) f\, dP_r dP_\phi dP_z \,. \tag{44}$$

Total tangential particle current [see Eq. (18)] at any given Φ:

$$J_{\rm tang} = \int_{r_0}^{R} dr \int_{-\infty}^{\infty}\int\int (P_\phi/mr - \Omega_f r) f\, dP_r dP_\phi dP_z \,. \tag{45}$$

Total radial flow of kinetic energy [see Eq. (15)] at any given r:

$$W(r)_{\rm rad} = 2\pi \int_{-\infty}^{\infty}\int\int (P_r/m)(P^2/2m) f\, dP_r dP_\phi dP_z \,. \tag{46}$$

Total tangential flow of kinetic energy at any given Φ:

$$W_{\rm tang} = \int_{r_0}^{R} dr \int_{-\infty}^{\infty}\int\int (P_\phi/mr - \Omega_f r)(P^2/2m) \times f\, dP_r dP_\phi dP_z \,. \tag{47}$$

It is convenient to write

$$P_\phi/r - m\Omega r = P_t \tag{48}$$

when we have $dP_\phi = r\,dP_t$

$$P^2 = P_r^2 + P_t^2 + P_z^2$$

and

$$\int_{-\infty}^{\infty}\int\int dP_r dP_\phi dP_z = 4\pi r \int_0^{\infty} P^2 dP \,, \tag{49}$$

while the factor in the integrands of Eqs. (45) and (47) becomes

$$(P_\phi/mr - \Omega_f r) = P_t/m + (\Omega - \Omega_f) r \,. \tag{50}$$

Provided we exclude the line $r=0$, ($r_0 > 0$), the integrals are all well behaved and elementary.

We must note first that, because of the factor (P_r/m) in the perturbed distribution function, the tangential currents, Eqs. (45) and (47), are both identically zero. The only tangential currents are the convective currents arising from the unperturbed function f_1 in the integrands, and these are independent of the temperature and density gradients. The radial currents are obtained easily. From Eqs. (41) and (42) we may write

$$\begin{aligned} J_{\rm rad}(r) &= -2\pi r\tau \int_{-\infty}^{\infty}\int\int (P_r/m)^2[(G + P^2/2mkT)(d\ln T/dr) + d\ln\rho/dr] f_1 dP_r dP_t dP_z \\ &= -(8\pi/3) r\tau C_1 \exp[\tfrac{1}{2}(m\Omega^2 r^2)/kT]\{[G(d\ln T/dr) + d\ln\rho/dr)]I_1 + (d\ln T/dr) I_2\} \,, \end{aligned} \tag{51}$$

where

$$\begin{aligned} I_1 &= \int_0^{\infty} (P^4/m^2) e^{-P^2/2mkT} dP = (3\sqrt{\pi}/8m^2)(2mkT)^{5/2} \,, \\ I_2 &= \int_0^{\infty} (P^6/2m^3kT) e^{-P^2/2mkT} dP = (15\sqrt{\pi}/16m^2)(2mkT)^{5/2} \,. \end{aligned} \tag{52}$$

Using Eq. (28) and retaining only terms up to first order in $m\Omega^2 r^2/kT$ we find

$$J(r)_{\rm rad} = -2\pi r\tau\rho(kT/m)\{1 + m\Omega^2[r^2 - \tfrac{1}{2}(R^2 + r_0^2)]/2kT\}[(G + \tfrac{5}{2})(d\ln T/dr) + d\ln\rho/dr] \,. \tag{53}$$

Using the same procedures we also find easily that

$$W_{\rm rad}(r) = -2\pi r\tau\rho(4k^2T^2/m)\{1 + m\Omega^2[r^2 - \tfrac{1}{2}(R^2 + r_0^2)]/2kT\}(5/8)[(G + \tfrac{7}{2})(d\ln T/dr) + d\ln\rho/dr] \,. \tag{54}$$

Here it must be noted that even if $d\ln\rho/dr$ were zero, there would still be a net particle current $J(r) \neq 0$, induced by the temperature gradient. We must suppose that the boundary is impenetrable to particle flow and that a density gradient must accompany the temperature gradient and adjust itself to cancel the particle current, $J(r) = 0$:

$$d\ln\rho/dr = -(G + \tfrac{5}{2})(d\ln T/dr) \,. \tag{55}$$

Using this relation in Eq. (54) we find the heat current

$$W_{\rm rad}(r) = -5\pi r\tau\rho(k^2T/m)(dT/dr) \times \{1 + m\Omega^2[r^2 - \tfrac{1}{2}(R^2 + r_0^2)]/2kT\} \,. \tag{56}$$

The dependence on r should be adjusted to conform with conservation of energy, but as this is only a first-order perturbation, we shall have to accept the temperature gradient as an average over the radius and write

$$W(R)/2\pi R = -\tfrac{5}{2}\rho\tau(k^2T/m)(dT/dr) \times [1 + m\Omega^2(R^2 - r_0^2)/4kT] \,. \tag{57}$$

The thermal conductivity at $\Omega = 0$ is only one-half the conventional result which is written down from Eq. (54) by equating the density gradient to zero and ignoring the particle current that must then accompany the heat flow. We emphasize that these calculations have been carried out in the generalized dynamical variables belonging to the rotating reference frame and that the results are independent of the rate of rotation Ω_f of that frame.

On the other hand the convective currents are not frame indifferent. As mentioned above, these are found by using the unperturbed distribution function f_1 in place of f in Eqs. (45) and (47). From Eq. (50) we have

$$J_{\text{conv}} = (\Omega-\Omega_f)\int_{r_0}^{R} r^2 dr \int\int\int f_1 dP_r dP_t dP_z = N(\Omega-\Omega_f)\langle r\rangle \tag{58}$$

and similarly

$$W_{\text{conv}} = (\Omega-\Omega_f)\tfrac{3}{2}NkT\langle r\rangle \ . \tag{59}$$

Both these currents vanish in a "comoving" reference frame $\Omega_f=\Omega$.

We now proceed to evaluate terms second order in τ and first order in the temperature gradient. From Eq. (35) $D_1D_0f_0$ is identically zero. It remains to consider $D_0D_1f_0$, and from Eqs. (31) and (32) we have

$$\tfrac{1}{2}D_0D_1f_1 = \tfrac{1}{2}\left[\dot{P}_r\frac{\partial}{\partial P_r}+\dot{r}\frac{\partial}{\partial r}\right]\times\left[\frac{P_r}{m}\left[\frac{dT}{dr}\frac{df_1}{\partial T}+\frac{d\rho}{dr}\frac{df_1}{d\rho}\right]\right], \tag{60}$$

so the terms of interest are

$$(\tfrac{1}{2}D_0D_1f_1)_c = \tfrac{1}{2}\frac{dT}{dr}\left[\left[\dot{P}_r\frac{\partial}{\partial P_r}+\frac{P_r}{m}\frac{\partial}{\partial r}\right]\left[\frac{P_r}{m}\frac{\partial f_1}{d\tau}\right]\right] \tag{61}$$

and using Eqs. (33) and (34) this reduces to

$$(\tfrac{1}{2}D_0D_1f_1)_c = \tfrac{1}{2}\frac{dT}{dr}\frac{\partial f_1}{\partial T}(P_\phi^2/m^2r^3) \ . \tag{62}$$

Because we are looking for tangential heat flow under conditions of zero convective heat flow, we must set $\Omega_f=\Omega$ in Eq. (47). It then becomes convenient to write, using Eqs. (50), (16), and (25),

$$P_\phi-\Omega mr^2 = P_t r,\quad dP_\phi = r\,dP_t\ ,$$
$$H_1 = (P_r^2+P_z^2+P_t^2)/2m - \tfrac{1}{2}m\Omega^2r^2\ , \tag{63}$$

$$f_1 = c_1e^{-H_1/kT},\quad P^2 = P_r^2+P_z^2+P_t^2\ . \tag{64}$$

Then Eq. (47) simplifies to read

$$W_{\text{tang}} = \tfrac{1}{2}\tau^2\int\int r\,dr\,dP_rdP_zdP_t(P_t/m)(P^2/2m)D_0D_1f_1\ , \tag{65}$$

where

$$D_0D_1f_1 = \frac{dT}{dr}\frac{df_1}{dT}(P_tr+\Omega mr^2)^2/m^2r^3\ . \tag{66}$$

Clearly the only nonzero contribution to the tangential heat flow comes from the term in Eq. (66) that is linear in P_t, which is

$$(D_0D_1f_1)_c = 2\Omega\left[\frac{dT}{dr}\right]\left[\frac{\partial f_1}{\partial T}\right]P_t/m\ , \tag{67}$$

where

$$\frac{df_1}{dT} = \left[\frac{dc_1}{dT}+c_1\frac{P^2}{2mkT^2}\right]e^{-P^2/2mkT}\ . \tag{68}$$

Using these equations we write Eq. (65) as follows:

$$W_{\text{tang}} = \tfrac{1}{2}\tau^2\left[\frac{dT}{dr}\right](\Omega/m^3)\int_{r_0}^{R} re^{m\Omega^2r^2/2kT}dr\int_{-\infty}^{\infty}\int P_t^2P^2\left[\frac{dc_1}{dT}+c_1\frac{P^2}{2mkT^2}\right]e^{-P^2/2mkT}dP_rdP_zdP_t\ . \tag{69}$$

Because we are not interested in terms with higher powers of Ω, we take

$$\int_{r_0}^{R} re^{m\Omega^2r^2/2kT}dr = \tfrac{1}{2}(R^2-r_0^2)\ ,$$
$$\frac{dc_1}{dT} = -\tfrac{3}{2}\frac{c_1}{T},\quad c_1 = \rho(2m\pi kT)^{-3/2} \tag{70}$$

and therefore

$$W_{\text{tang}} = \tfrac{1}{4}\tau^2\left[\frac{dT}{dr}\right](\Omega/m^3)(R^2-r_0^2)(c_1/T)\frac{4\pi}{3}\int_0^{\infty}P^2dP(-\tfrac{3}{2}P^4+P^6/2mkT)e^{-P^2/2mkT}\ . \tag{71}$$

The elementary integrals involved here are

$$-2\pi\int_0^{\infty}P^6e^{-P^2/2mkT}dP+\frac{4\pi}{3}(2mkT)^{-1}\times\int_0^{\infty}P^8e^{-P^2/2mkT}dP = \tfrac{5}{2}(2mkT)^{7/2}\pi^{3/2}\ . \tag{72}$$

The final result is (in three dimensions)

$$W_{\text{tang}} = \tfrac{5}{2}\tau^2\rho(k^2T/m)\Omega(dT/dr)(R^2-r_0^2)\ . \tag{73}$$

If one is interested in a two-dimensional model and suppresses the z dimension completely, $P^2=P_t^2+P_r^2$, and modifies Eqs. (70) accordingly, one finds easily that (in two dimensions)

$$W_{\text{tang}} = \frac{\pi}{2}\tau^2\rho\Omega(k^2T/m)(dT/dr)(R^2-r_0^2)\ . \tag{74}$$

Generally, the mean tangential velocity $\langle r\dot{\phi}\rangle$ per particle at a given radius r using Eqs. (12) and (27) is easily shown to be

$$\langle r\dot{\phi}\rangle(r)=(\Omega-\Omega_f)r \quad (75)$$

which vanishes in a corotating frame; so the vorticity also vanishes there. But using the perturbation determined by Eq. (66) as derived for the corotating frame, we find

$$\langle r\dot{\phi}\rangle(r)=\tau^2\Omega(k/m)(dT/dr)\ ; \quad (76)$$

so the vorticity is now

$$\tfrac{1}{2}\tau^2\Omega(k/mr)(dT/dr)\ . \quad (77)$$

Initiating the heat flow must therefore produce a torque, and the product of this torque tensor with the vorticity tensor is responsible for the tangential component of the heat flow (Ref. 6).

The result found here in Eq. (67) is essentially the same as that given in Ref. 3, Eq. (3), but its interpretation is different. Our final results, Eqs. (73) and (74) should serve as a check on the computer simulations at least in the limit of zero interaction energy.

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