Coherent-state Langevin equations for canonical quantum systems with applications to the quantized Hall effect

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The formulation of quantum-statistical-mechanical expectation values as long-time averages involving the solution of associated Langevin equations with complex drift terms is developed. As an example, some applications of this method to study the quantized Hall effect are presented.

I. INTRODUCTION

Statistical methods to study classical and quantum systems are extensively used at the present time with Monte Carlo and Langevin techniques among those more commonly employed.[1] In this paper we elaborate on a method to analyze canonical quantum systems through the study of associated Langevin equations the form of which has been previously announced.[2] As an application we examine some aspects of the two-dimensional (anomolous) quantized Hall effect.[3]

Section II is devoted to a careful and rather complete derivation of the Langevin equation for canonical quantum systems, including a treatment of the exclusion principle for fermions. Essentially all quantum-statistical problems may be formulated as path integrals, most generally expressed in a phase-space formulation. To relate this formulation to Langevin equations it is necessary that all the variables in the path integral assume continuous values, and this is ensured only for coherent-state, phase-space path integrals. Consequently, the first part of Sec. II is devoted to a review and discussion of canonical coherent-state path integrals. Given a path-integral representation of a problem, the introduction of an associated Fokker-Planck equation and a corresponding system of Langevin equations then follows rather standard lines here extended to the case of complex expressions.

In Sec. III the application to the two-dimensional quantized Hall effect is discussed, along with results obtained by a computer solution of the corresponding Langevin equations. While these numerical results generally support the applicability of these methods to complex systems they, unfortunately, do not shed any special new light on the physics of the anamolous quantized Hall effect. Interpretation of our results is aided by an analogous study of a harmonic oscillator, in which the concept of improved approximations naturally arises, and in the study of two uncoupled harmonic oscillators that obey the exclusion principle.

II. DERIVATION OF THE LANGEVIN EQUATIONS

A. Coherent-state and operator properties

For a single degree of freedom, let

$$|p,q\rangle \equiv e^{i(pQ-qP)}|0\rangle\ ,\quad (Q+iP)|0\rangle=0\ ,\quad \langle 0|0\rangle=1\ , \tag{2.1}$$

for all $p,q\in\mathbb{R}^2$ denote the canonical coherent states, where Q and P constitute an irreducible Heisenberg pair, $[Q,P]=i$. These states enjoy a number of interesting and useful properties,[4] the most important of which is the resolution of unity given by

$$1=\int |p,q\rangle\langle p,q|\,\frac{dp\,dq}{2\pi}\ . \tag{2.2}$$

The overlap of two such states reads

$$\langle p_2,q_2|p_1,q_1\rangle=\exp\{\tfrac{1}{2}i(q_2p_1-p_2q_1) -\tfrac{1}{4}[(q_2-q_1)^2+(p_2-p_1)^2]\}\ ,$$

an expression which never vanishes, and as a consequence an operator $\mathscr{H}$ is uniquely determined by its diagonal coherent-state matrix elements.[5] To see this connection assume that $\mathscr{H}$ is expressed in Weyl form,

$$\mathscr{H}=\int \tilde{h}_w(x,k)e^{i(kQ-xP)}\frac{dx\,dk}{2\pi}\ ,$$

where $\tilde{h}_w$ is a uniquely defined distribution associated with $\mathscr{H}$. It follows that

$$\begin{aligned}H(p,q)&\equiv\langle p,q|\mathscr{H}|p,q\rangle\\&=\int \tilde{h}_w(x,k)\langle p,q|e^{i(kQ-xP)}|p,q\rangle\frac{dx\,dk}{2\pi}\\&=\int \tilde{h}_w(x,k)e^{i(kq-xp)}\langle 0|k,x\rangle\frac{dx\,dk}{2\pi}\\&=\int \tilde{h}_w(x,k)e^{i(kq-xp)}e^{-(x^2+k^2)/4}\frac{dx\,dk}{2\pi}\ ,\end{aligned}$$

which shows that H and $\tilde{h}_w$ (hence $\mathscr{H}$) are uniquely correlated. If $h_w(p,q)$ denotes the Fourier transform of $\tilde{h}_w(x,k)$, then it follows that

$$H(p,q)=\int e^{-(p-r)^2-(q-s)^2}h_w(r,s)\frac{dr\,ds}{2\pi}\ ,$$

or stated alternatively,

$$H(p,q)=\exp\left[\frac{1}{4}\left[\frac{\partial^2}{\partial p^2}+\frac{\partial^2}{\partial q^2}\right]\right]h_w(p,q)\ .$$

The operator $\mathscr{H}$ also admits the representation[4]

$$\mathscr{H}=\int h(r,s)|r,s\rangle\langle r,s|\,\frac{dr\,ds}{2\pi}\ , \tag{2.3}$$

where the weight h may be determined from the relation

$$H(p,q)=\langle p,q\,|\,\mathscr{H}\,|\,p,q\rangle$$
$$=\int h(r,s)\,|\,\langle p,q\,|\,r,s\rangle\,|^2\frac{dr\,ds}{2\pi}$$
$$=\int e^{-[(p-r)^2+(q-s)^2]/2}h(r,s)\frac{dr\,ds}{2\pi}\ .$$

Thus we find that

$$H(p,q)=\exp\left[\frac{1}{2}\left[\frac{\partial^2}{\partial p^2}+\frac{\partial^2}{\partial q^2}\right]\right]h(p,q)\ ,$$

which leads to the relations

$$h(p,q)=\exp\left[-\frac{1}{2}\left[\frac{\partial^2}{\partial p^2}+\frac{\partial^2}{\partial q^2}\right]\right]H(p,q)\ , \tag{2.4a}$$

$$h(p,q)=\exp\left[-\frac{1}{4}\left[\frac{\partial^2}{\partial p^2}+\frac{\partial^2}{\partial q^2}\right]\right]h_w(p,q)\ . \tag{2.4b}$$

For later use these relations are conveniently summarized with respect to the basic Weyl representation

$$\mathscr{H}=\frac{1}{2\pi}\int\tilde{h}_w(x,k)e^{i(kQ-xP)}dx\,dk\ , \tag{2.5a}$$

in the form

$$h_w(p,q)=\frac{1}{2\pi}\int\tilde{h}_w(x,k)e^{i(kq-xp)}dx\,dk\ , \tag{2.5b}$$

$$H(p,q)=\frac{1}{2\pi}\int\tilde{h}_w(x,k)e^{i(kq-xp)-1/4(x^2+k^2)}dx\,dk\ , \tag{2.5c}$$

$$h(p,q)=\frac{1}{2\pi}\int\tilde{h}_w(x,k)e^{i(kq-xp)+1/4(x^2+k^2)}dx\,dk\ . \tag{2.5d}$$

B. Path-integral representation

If we combine (2.2) and (2.3) we learn that

$$1-\epsilon\mathscr{H}=\int[1-\epsilon h(p,q)]\,|\,p,q\rangle\langle p,q\,|\,\frac{dp\,dq}{2\pi}$$

and thus, with an error which is $O(\epsilon^2)$, that

$$e^{-\epsilon\mathscr{H}}=\int e^{-\epsilon h(p,q)}\,|\,p,q\rangle\langle p,q\,|\,\frac{dp\,dq}{2\pi}\ . \tag{2.6}$$

If N is an integer chosen such that $\beta\equiv N\epsilon$, then by multiplying (2.6) by itself N times and taking the trace we find that

$$Z\equiv\mathrm{Tr}(e^{-\beta\mathscr{H}})$$
$$=\lim_{N\to\infty}\int\cdots\int\prod_{l=1}^{N}\langle p_{l+1},q_{l+1}\,|\,p_l,q_l\rangle e^{-\epsilon h(p_l,q_l)}\times\frac{dp_l dq_l}{2\pi}\ ,$$

where $p_{N+1},q_{N+1}\equiv p_1,q_1$, and we have taken the $N\to\infty$ limit to eliminate the $O(\epsilon^2)$ terms. This expression yields a (discrete form of) path-integral representation of the partition function Z, which we may approximate by keeping N large but fixed as

$$Z=\int\cdots\int e^{S(p,q)}d\mu\ ,$$

where

$$d\mu\equiv\prod_{l=1}^{N}\frac{dp_l dq_l}{2\pi}\ ,$$

and

$$S(p,q)\equiv\sum_{l=1}^{N}[\ln\langle p_{l+1},q_{l+1}\,|\,p_l,q_l\rangle-\epsilon h(p_l,q_l)]$$
$$=\sum_{l=1}^{N}\{\tfrac{1}{2}i(q_{l+1}p_l-p_{l+1}q_l)$$
$$-\tfrac{1}{4}[(q_{l+1}-q_l)^2+(p_{l+1}-p_l)^2]$$
$$-\epsilon h(p_l,q_l)\}\ . \tag{2.7}$$

Improved actions: With some extra effort one may improve the accuracy of (2.6) [and hence of (2.7)] by using the analog of "improved actions."[6] The idea is to replace (2.6) with the expression

$$e^{-\epsilon\mathscr{H}}=\int e^{-\epsilon h_{(n)}(p,q,\epsilon)}\,|\,p,q\rangle\langle p,q\,|\,\frac{dp\,dq}{2\pi}$$

which is designed to be valid apart from terms of order $O(\epsilon^{n+1})$ for some $n>1$. It is straightforward to determine an expression, at least formally, to give our version of an improved action to any order. First recall the Moyal product formula,

$$h_w^{AB}(p,q)\equiv h_w^A(p,q)e^D h_w^B(p,q)\ ,$$

to find the Weyl representation for the operator product AB in terms of those for A and B separately. Here

$$D\equiv\tfrac{1}{2}i\left[\frac{\partial}{\partial\overleftarrow{q}}\frac{\partial}{\partial\overrightarrow{p}}-\frac{\partial}{\partial\overleftarrow{p}}\frac{\partial}{\partial\overrightarrow{q}}\right]\ ,$$

with the arrows signifying an operation on either the left or right factor. Then it follows that

$$e^{-\epsilon\mathscr{H}}=\sum_{m=0}^{\infty}(m!)^{-1}(-\epsilon)^m\mathscr{H}^m$$
$$\equiv\int\sum_{m=0}^{\infty}(m!)^{-1}(-\epsilon)^m h_{[m]}(p,q)\,|\,p,q\rangle\langle p,q\,|\times\frac{dp\,dq}{2\pi}\ ,$$

where

$$h_{[0]}(p,q)\equiv 1\ ,$$
$$h_{[1]}(p,q)\equiv h(p,q)\ ,$$
$$h_{[m]}(p,q)\equiv\exp\left[-\frac{1}{4}\left[\frac{\partial^2}{\partial p^2}+\frac{\partial^2}{\partial q^2}\right]\right]h_{w[m]}(p,q)\ ,$$

and

$$h_{w[m+1]}(p,q)=h_w(p,q)e^D h_{w[m]}(p,q)$$

for $m=1,2,\ldots$. Thus to determine an expression correct to $O(\epsilon^n)$, i.e., with an error which is $O(\epsilon^{n+1})$, it suffices to choose

$$-\epsilon h_{(n)}(p,q,\epsilon)\equiv\ln\left[\sum_{m=0}^{n}(m!)^{-1}(-\epsilon)^m h_{[m]}(p,q)\right],$$

or any equivalent expression that differs by terms $O(\epsilon^{n+1})$. By using $h_{(n)}$ rather than h alone in numerical studies, a comparable accuracy should be obtained with fewer thermal-time steps (of order $N^{1/n}$ rather than N). In what follows we shall generally use h, but the formulas hold as well if one substitutes $h_{(n)}$ in place of h.

The usual (Metropolis) importance sampling Monte Carlo procedure to estimate Z is inapplicable since S is not real, i.e., e^S is not everywhere positive as needed if it is to be interpreted as a probability density.

C. Fokker-Planck equation

With the eventual goal of circumventing the nonpositivity of e^S in mind, we next introduce a function $G(p,q,\tau)$ that satisfies a Fokker-Planck-type equation[7,2] given by

$$\frac{\partial G(p,q,\tau)}{\partial\tau}=\frac{1}{2}\sum_{l=1}^{N}\left[\frac{\partial}{\partial q_l}\left[-\frac{\partial S}{\partial q_l}+\frac{\partial}{\partial q_l}\right]+\frac{\partial}{\partial p_l}\left[-\frac{\partial S}{\partial p_l}+\frac{\partial}{\partial p_l}\right]\right]G(p,q,\tau)\,. \tag{2.8}$$

This equation has been chosen so that

$$G(p,q,\tau)=Ce^{S(p,q)}$$

is a stationary solution for any C. Moreover, if $G_0(p,q)=G(p,q,0)$ denotes a smooth, general initial condition for (2.8) normalized so that

$$\int G_0(p,q)d\mu\equiv Z_0\neq 0\,,$$

then we require that the solution to (2.8) satisfies

$$G(p,q,\tau)\to Ce^{S(p,q)}\text{ as }\tau\to\infty\,, \tag{2.9}$$

where $C=Z_0/Z$ is a finite, nonzero proportionality factor. This asymptotic criterion is of central importance to our approach, and we now embark on a detailed analysis of it.

If S were real and locally bounded, then the validity of (2.9) follows from the fact that $\exp[S(p,q)/2]$ is the nondegenerate, zero-energy ground state of a Hamiltonian (see below).[7,2] When S is complex there simply is no general argument of this nature to draw on. Since the harmonic-oscillator Hamiltonian is explicitly soluble, we shall initially discuss this case fully. At the end of this analysis we shall see how the harmonic-oscillator results can be extended to a general Hamiltonian.

Harmonic oscillator: For the oscillator example we choose $\mathscr{H}=\frac{1}{2}(P^2+Q^2-1)$, and thus $h(p,q)=\frac{1}{2}(p^2+q^2-2)$. We initially adopt

$$G_0(p,q)=(2\pi)^N\delta^{(N)}(p-p_0)\delta^{(N)}(q-q_0)\,, \tag{2.10}$$

for arbitrary $p_0=(p_{01},\ldots,p_{0N})$ and $q_0=(q_{01},\ldots,q_{0N})$, as the boundary condition at $\tau=0$. More specifically we choose

$$G_0(p,q,0^+)=\tau^{-N}\exp[-(\underline{r}-\underline{r}_0)^2/2\tau]$$

for $0<\tau\ll 1$, where $\underline{r}\equiv(q,p)$ denotes a $2N$-component vector. Given this initial condition, then the solution to (2.8) appropriate to the harmonic oscillator is necessarily of the form

$$G(p,q,\tau)=D(\tau)\exp[-\tfrac{1}{2}\underline{r}^T\underline{C}(\tau)\underline{r}-\underline{r}^T\underline{B}(\tau)]\,,$$

where $\underline{C}$ is a symmetric matrix, $\underline{B}$ a vector, and D a scalar factor. To satisfy (2.8) these quantities obey the equations of motion

$$\dot{\underline{C}}(\tau)=-\underline{C}^2(\tau)+\tfrac{1}{2}[\underline{A}\,\underline{C}(\tau)+\underline{C}(\tau)\underline{A}]\,, \tag{2.11a}$$

$$\dot{\underline{B}}(\tau)=[\tfrac{1}{2}\underline{A}-\underline{C}(\tau)]\underline{B}(\tau)\,, \tag{2.11b}$$

$$\dot{D}(\tau)=\tfrac{1}{2}\mathrm{Tr}[\underline{A}-\underline{C}(\tau)+\underline{B}(\tau)\underline{B}^T(\tau)]D(\tau)\,, \tag{2.11c}$$

where $\underline{A}$ is a $2N\times 2N$ symmetric matrix determined by the relation

$$S=S_0\equiv-\tfrac{1}{2}\underline{r}^T\underline{A}\,\underline{r}+\text{const}\,.$$

Thus $\underline{A}=\frac{1}{4}\underline{A}_0+\epsilon\underline{I}$, which apart from a multiple (ϵ) of the identity $\underline{I}$, is determined by $\underline{A}_0$, where

$$\underline{A}_0=\left[\begin{array}{ccccc|ccccc}
2&-1&&&-1&0&i&&&-i\\
-1&2&-1&&&-i&0&i&&\\
&-1&2&\ddots&&&-i&0&\ddots&\\
&&\ddots&\ddots&-1&&&\ddots&\ddots&i\\
-1&&&-1&2&i&&&-i&0\\
\hline
0&-i&&&i&2&-1&&&-1\\
i&0&-i&&&-1&2&-1&&\\
&i&0&\ddots&&&-1&2&\ddots&\\
&&\ddots&\ddots&-i&&&\ddots&\ddots&-1\\
-i&&&i&0&-1&&&-1&2
\end{array}\right].$$

The solution to the equations (2.11) is given by

$$\underline{C}(\tau)=\underline{A}(\underline{I}-e^{-\underline{A}\tau})^{-1} ,$$

$$\underline{B}(\tau)=\exp\left[\int_1^\tau [\tfrac{1}{2}\underline{A}-\underline{C}(\sigma)]d\sigma\right]\underline{B}_1 ,$$

$$D(\tau)=\exp\left[\tfrac{1}{2}\int_1^\tau \mathrm{Tr}[\underline{A}-\underline{C}(\sigma)+\underline{B}(\sigma)\underline{B}^T(\sigma)]d\sigma\right]D_1 .$$

As $\tau\to 0^+$ it follows that

$$\underline{C}(0^+)=\underline{I}/\tau ,$$

$$\underline{B}(0^+)=\underline{B}_0/\tau ,$$

$$D(0^+)=D_0\tau^{-N}\exp(-\underline{B}_0^T\underline{B}_0/2\tau) .$$

On comparison with the initial condition it is clear that $\underline{B}_1$ and D_1 should be chosen so that $\underline{B}_0=\underline{r}_0$ and $D_0=1$; this is certainly possible. On the other hand, as $\tau\to\infty$ the solution converges to a finite, nonzero multiple of $\exp(S_0)$ if and only if

$$e^{-\underline{A}\tau}\to\underline{0} \text{ as } \tau\to\infty . \tag{2.12}$$

Like S_0, the scale factor is independent of $\underline{r}_0$ as determined by the fact that $\int G(p,q,\tau)d\mu$ is independent of τ and has the value unity, its value as $\tau\to 0^+$, independently of $\underline{r}_0$.

Although $\underline{A}$ is not a real symmetric matrix it does have a complete set of eigenvectors and eigenvalues. Specifically, for $\underline{A}_0$, the eigenvalues are given by

$$\lambda_n=1-e^{2\pi i(n-1)/N}, \quad n=1,2,\ldots,N ,$$

each of which is doubly degenerate. Since $\mathrm{Re}\lambda_n\geq 0$, the convergence criterion (2.12) is in fact satisfied, although the rate of convergence is as slow as $e^{-\epsilon\tau}$.

The preceding argument can be extended to any smooth initial condition $G_0(p,q)$ simply by superimposing the results for sharp initial conditions. Consequently, the convergence of a general solution $G(p,q,\tau)$ to a multiple of $\exp[S(p,q)]$ has been established in the case of the harmonic oscillator. We now deduce the same conclusion for a general Hamiltonian.

General Hamiltonians: If we introduce the quantity

$$Y(p,q,\tau)\equiv G(p,q,\tau)e^{-S(p,q)/2} ,$$

then it follows from (2.8) that Y satisfies the equation

$$\frac{\partial Y(p,q,\tau)}{\partial\tau}=\tfrac{1}{2}(\vec{\nabla}+\tfrac{1}{2}\vec{\nabla}S)\cdot(\vec{\nabla}-\tfrac{1}{2}\vec{\nabla}S)Y(p,q,\tau) , \tag{2.13}$$

where $\vec{\nabla}\equiv(\partial/\partial q,\partial/\partial p)$. Stated otherwise, we find that

$$\frac{\partial Y(p,q,\tau)}{\partial\tau}=-\mathsf{H}Y(p,q,\tau) ,$$

$$\mathsf{H}\equiv-\tfrac{1}{2}\nabla^2+\mathsf{V} ,$$

where

$$\mathsf{V}=\tfrac{1}{8}(\vec{\nabla}S)^2+\tfrac{1}{4}\nabla^2 S .$$

It follows from (2.13) that

$$Y_0(p,q)\equiv e^{S(p,q)/2}$$

is a zero-energy eigenstate of the Hamiltonian H, i.e., $\mathsf{H}Y_0=E_0Y_0$, with $E_0=0$. If S were real and $Y_0\in L^2$, then it follows that Y_0 is the ground state (since it is nowhere vanishing), and the solutions to

$$\mathsf{H}Y_n(p,q)=E_nY_n(p,q) ,$$

assumed discrete for simplicity, have the property that $E_n>0$, for $n>0$. Thus the solution to (2.13) given by

$$Y(p,q,\tau)=\sum_{n=0}^{\infty}a_nY_n(p,q)e^{-E_n\tau}$$

satisfies the limiting relation

$$\lim_{\tau\to\infty}Y(p,q,\tau)=a_0Y_0(p,q) ,$$

or alternatively stated, that

$$\lim_{\tau\to\infty}G(p,q,\tau)=a_0e^{S(p,q)} ,$$

as desired.

When S is complex, as is the case of interest, then V is complex and the general theory of self-adjoint operators is not available to us to reach the desired conclusion. The spectrum of $A+iB$ when A and B are both self-adjoint operators is a nontrivial question with diverse answers. For instance, consider the operator $rQ+iP$, which in the Schrödinger representation is given by $rx+\partial/\partial x$. The solution of the eigenvector equation

$$\left[rx+\frac{\partial}{\partial x}\right]\psi(x)=\mu\psi(x)$$

is given by

$$\psi(x)=Ne^{-rx^2/2+\mu x} .$$

For $r>0$, the solutions are all normalizable and yield an eigenfunction for all complex μ (these are just coherent states!), which are not mutually orthogonal. However, if $r<0$, then there are *no* eigenstates, i.e., there is *no* spectrum whatsoever.[8]

In the present case we have [cf. (2.7)]

$$S(p,q)=S_1(p,q)+iS_2(p,q) ,$$

where S_1 and S_2 are real. But instead let us consider

$$S_\sigma(p,q)\equiv S_1(p,q)+\sigma S_2(p,q) ,$$

where σ is a complex number. Correspondingly, we introduce

$$\mathsf{H}_\sigma\equiv-\tfrac{1}{2}\nabla^2+\mathsf{V}_\sigma ,$$

$$\mathsf{V}_\sigma\equiv\tfrac{1}{8}(\vec{\nabla}S_\sigma)^2+\tfrac{1}{4}\nabla^2S_\sigma .$$

It is easy to convince oneself that H_σ does have eigenfunctions and eigenvalues (unlike the case $r<0$ above), and that both are analytic functions of σ in a strip near the real axis that includes $\sigma=i$. Moreover, if σ is real, then S_σ is real and the asymptotic condition (2.9) is fulfilled. That means, for σ real, that $E_0=0$ and $E_1\geq E_{1,\min}>0$ for σ bounded, say $|\mathrm{Re}\sigma|\leq K<\infty$; all other eigenvalues are even higher. Now as σ becomes complex, on its way toward $\sigma=i$, E_0 remains zero, while all E_n, $n>0$, generally become complex. Violations of the desired asymptotic

condition (2.9) occurs whenever any of the inequalities $\mathrm{Re}E_n > 0$, $n > 0$ is not satisfied. Evidently there is some interval $J = J(K)$, where $|\,\mathrm{Im}\sigma\,| \leq J$, for which, by continuity, $\mathrm{Re}E_n > 0$, $n > 0$; whether this interval extends to $J = \infty$ is unknown, but this is not really important. What is surely true and has been borne out in numerical eigenvalue studies carried out by Petersen,[9] is that if the inequalities $\mathrm{Re}E_n > 0$, $n > 0$ are valid for some $\bar{S} \equiv \bar{S}_1 + iS_2$, then these inequalities remain valid for $S = S_1 + iS_2$ whenever $S_1 \leq \bar{S}_1$. Since we have already established the convergence criteria for the harmonic oscillator (or implicitly for a positive multiple thereof), we can choose $\bar{S}_1 = \mathrm{Re}S_0$. Consequently, whenever

$$h(p,q) \geq \alpha(p^2+q^2) + \text{const}\,,$$

for some $\alpha > 0$, it follows that $S_1 \leq \bar{S}_1$ and thus $\mathrm{Re}E_n > 0$, for all $n > 0$. This result establishes the desired convergence criterion (2.9) for a general Hamiltonian and concludes our discussion of this point.

D. Langevin equations

Associated with every Fokker-Planck equation is a set of Langevin equations,[10] which in the present case are given by

$$\dot{q}_l(\tau) = \frac{1}{2}\frac{\partial S}{\partial q_l(\tau)} + \xi_l(\tau)\,, \tag{2.14a}$$

$$\dot{p}_l(\tau) = \frac{1}{2}\frac{\partial S}{\partial p_l(\tau)} + \eta_l(\tau)\,, \tag{2.14b}$$

where $1 \leq l \leq N$, and ξ and η denote two independent sets of standard Gaussian white-noise sources determined by their mean

$$\langle \xi_l(\tau) \rangle = \langle \eta_l(\tau) \rangle = 0$$

and by their variance

$$\langle \xi_l(\tau)\xi_m(\sigma) \rangle = \delta_{lm}\delta(\tau-\sigma)\,,$$
$$\langle \eta_l(\tau)\eta_m(\sigma) \rangle = \delta_{lm}\delta(\tau-\sigma)\,,$$
$$\langle \xi_l(\tau)\eta_m(\sigma) \rangle = 0\,.$$

Here and elsewhere the average $\langle\cdot\rangle$ is with respect to the noise ensemble. We are interested in the solution to these equations for $\tau \geq 0$ subject to the initial conditions

$$q_l(0) = q_{0l}\,,\quad p_l(0) = p_{0l}\,,$$

for $1 \leq l \leq N$. The white-noise sources as well as the initial conditions q_0, p_0 are chosen real, but since S is complex the solution $q(\tau), p(\tau)$ will, in general, be complex, too. For fixed (nonrandom) initial conditions q_0, p_0, the solution of the Langevin equation is related to the solution of the Fokker-Planck equation by the expression

$$\langle F(p(\tau),q(\tau)) \rangle = \int F(p,q)G(p,q,\tau)d\mu\,,$$

where $G(p,q,\tau)$ is the solution subject to the initial condition (2.10). Linearity of this expression in the distribution of initial values then extends its validity to any smooth distribution $G_0(p,q)$ normalized so that $\int G_0 d\mu = 1$. Moreover, since G satisfies (2.9) it follows that

$$\lim_{\tau\to\infty} \langle F(p(\tau),q(\tau)) \rangle = \frac{\int F(p,q)e^{S(p,q)}d\mu}{\int e^{S(p,q)}d\mu}\,. \tag{2.15}$$

In addition, the convergence criterion (2.9) ensures that the ensemble is also ergodic. Ergodicity means that for almost all solutions of the Langevin equations we have the relation

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T F(p(\tau),q(\tau))d\tau = \frac{\int F(p,q)e^{S(p,q)}d\mu}{\int e^{S(p,q)}d\mu}\,. \tag{2.16}$$

If C denotes the right side of this equation and

$$A_T \equiv \frac{1}{T}\int_0^T F(p(\tau),q(\tau))d\tau\,,$$

then this limiting behavior may be understood as

$$\lim_{T\to\infty} \left\langle \exp\left[i\int_0^\infty (up - rq)dt \right](A_T - C) \right\rangle = 0$$

for all smooth functions u and r of compact support.

One degree of freedom: For the specific case at hand, where S is given by (2.7), the Langevin equations (2.14) become

$$\dot{q}_l = \frac{i}{4}(p_{l-1} - p_{l+1}) - \frac{1}{4}(2q_l - q_{l+1} - q_{l-1}) - \frac{\epsilon}{2}\frac{\partial h}{\partial q_l} + \xi_l\,, \tag{2.17a}$$

$$\dot{p}_l = \frac{i}{4}(q_{l+1} - q_{l-1}) - \frac{1}{4}(2p_l - p_{l+1} - p_{l-1}) - \frac{\epsilon}{2}\frac{\partial h}{\partial p_l} + \eta_l\,, \tag{2.17b}$$

for $1 \leq l \leq N$. It is important to assess the stability and sensitivity to the initial conditions of these equations.

Stability is guaranteed whenever the forces $\partial h/\partial q$ and $\partial h/\partial p$ all act attractively toward the origin, at least for large enough arguments. To study sensitivity to the initial conditions let us introduce the quantity

$$R(\tau) \equiv \sum_{l=1}^N [\,|\,q_l(\tau) - \bar{q}_l(\tau)\,|^2 + |\,p_l(\tau) - \bar{p}_l(\tau)\,|^2\,]\,,$$

where q, p, and $\bar{q}, \bar{p}$ denote two solutions of (2.17) determined by different initial conditions, q_0, p_0, and $\bar{q}_0, \bar{p}_0$, but with the *same* noise histories for each solution. It follows from the Langevin equations that

$$\dot{R} = -\tfrac{1}{2}\sum_{l=1}^N [\,|\,(q_l - \bar{q}_l) - (q_{l+1} - \bar{q}_{l+1})\,|^2 + |\,(p_l - \bar{p}_l) - (p_{l+1} - \bar{p}_{l+1})\,|^2] - \epsilon\sum_{l=1}^N \mathrm{Re}\left[(q_l - \bar{q}_l)^* \left[\frac{\partial h}{\partial q_l} - \frac{\partial h}{\partial \bar{q}_l} \right] + (p_l - \bar{p}_l)^* \left[\frac{\partial h}{\partial p_l} - \frac{\partial h}{\partial \bar{p}_l} \right] \right]. \tag{2.18}$$

Like the large-time behavior of the solution of the

Fokker-Planck equation it is difficult to make definitive statements regarding (2.18) for a general Hamiltonian. However, in the case of the harmonic oscillator, where $h=\frac{1}{2}(p^2+q^2-2)$, it follows that

$$\dot{R}=-\tfrac{1}{2}\sum_l [\,|(q_l-\bar{q}_l)-(q_{l+1}-\bar{q}_{l+1})|^2+|(p_l-\bar{p}_l) -(p_{l+1}-\bar{p}_{l+1})|^2]-\epsilon\sum_l[\,|q_l-\bar{q}_l|^2 +|p_l-\bar{p}_l|^2]\,.$$

From the general fact

$$0\le|A-B|^2\le 2(|A|^2+|B|^2)\,,$$

we learn that

$$-\epsilon R\ge\dot{R}\ge-(2+\epsilon)R\,,$$

with the solution

$$e^{-\epsilon\tau}R(0)\ge R(\tau)\ge e^{-(2+\epsilon)\tau}R(0)\,.$$

Thus we see that R decays, $R(\tau)\to 0$, establishing the desired asymptotic independence on initial condition, but as noted previously this decay can be rather slow.

Several degrees of freedom: Up to this point we have largely worked as if there was only a single degree of freedom under consideration. If instead there are M distinguishable degrees of freedom, then we need only add a "particle" label and interpret the coherent states as direct products over the M variables as in

$$|p,q\rangle\equiv\bigotimes_{m=1}^{M}|p^m,q^m\rangle\,.$$

It follows directly that the modified form of the Langevin equations (2.17) is given by

$$\dot{q}_l^m=\frac{i}{4}(p_{l-1}^m-p_{l+1}^m)-\frac{1}{4}(2q_l^m-q_{l+1}^m-q_{l-1}^m) -\frac{\epsilon}{2}\frac{\partial h}{\partial q_l^m}+\xi_l^m\,, \tag{2.19a}$$

$$\dot{p}_l^m=\frac{i}{4}(q_{l+1}^m-q_{l-1}^m)-\frac{1}{4}(2p_l^m-p_{l+1}^m-p_{l-1}^m) -\frac{\epsilon}{2}\frac{\partial h}{\partial p_l^m}+\eta_l^m\,, \tag{2.19b}$$

where $1\le l\le N$ and $1\le m\le M$, and all white noises are mutually independent of one another. Here $h=h(p,q)$, which is given by an evident generalization of (2.4) as

$$h(p,q)=\exp\left[-\frac{1}{2}\sum_{m=1}^{M}\left[\frac{\partial^2}{\partial(p^m)^2}+\frac{\partial^2}{\partial(q^m)^2}\right]\right]H(p,q)\,,$$

$$=\exp\left[-\frac{1}{4}\sum_{m=1}^{M}\left[\frac{\partial^2}{\partial(p^m)^2}+\frac{\partial^2}{\partial(q^m)^2}\right]\right]h_w(p,q)\,.$$

Finally we must discuss the case of M nonrelativistic electrons, indistinguishable particles that obey the exclusion principle. A description of electrons may be accomplished by the replacement of the distinguishable particle coherent-state overlap factor

$$\langle p_2,q_2\,|\,p_1,q_1\rangle=\prod_{m=1}^{M}\langle p_2^m,q_2^m\,|\,p_1^m,q_1^m\rangle$$

by the associated Slater determinant

$$(M!)^{-1}\det(\langle p_2^m,q_2^m\,|\,p_1^n,q_1^n\rangle)\,, \tag{2.20}$$

where the determinant is of the $M\times M$ matrix of conventional coherent-state inner products. The factor $(M!)^{-1}$ ensures that this kernel satisfies the integral equation of a projection operator onto the space of antisymmetric functions. To incorporate the exclusion principle it is adequate to replace just one overlap factor in the multiparticle path integral. However, it is analytically equivalent to replace all overlap factors by determinants, and this is the approach we have generally followed in our numerical studies. Consequently the revised form of the function S is given by

$$S=\sum_{l=1}^{N}\{\ln[(M!)^{-1}\det(\langle p_{l+1},q_{l+1}\,|\,p_l,q_l\rangle)]-\epsilon h(p_l,q_l)\}\,. \tag{2.21}$$

Guided by our previous discussion we are led to reconsider the set of Langevin equations (2.14) based on the new choice for S, which then becomes

$$\dot{q}_l^m=\frac{1}{2}\frac{\partial}{\partial q_l^m}\ln(\det\langle p_{l+1},q_{l+1}\,|\,p_l,q_l\rangle\det\langle p_l,q_l\,|\,p_{l-1},q_{l-1}\rangle)-\frac{\epsilon}{2}\frac{\partial h(p_l,q_l)}{\partial q_l^m}+\xi_l^m\,, \tag{2.22a}$$

$$\dot{p}_l^m=\frac{1}{2}\frac{\partial}{\partial p_l^m}\ln(\det\langle p_{l+1},q_{l+1}\,|\,p_l,q_l\rangle\det\langle p_l,q_l\,|\,p_{l-1},q_{l-1}\rangle)-\frac{\epsilon}{2}\frac{\partial h(p_l,q_l)}{\partial p_l^m}+\eta_l^m\,, \tag{2.22b}$$

for $1\le l\le N$ and $1\le m\le M$. Although our intuition suggests that the long-time average of functions of the solution of these Langevin equations is equivalent to an average of such functions in the normalized distribution based on S as given by (2.21), we have not succeeded in directly proving this result as we did for the case of distinguishable particles. However, a model problem of two uncoupled harmonic oscillators that obey the exclusion principle does give the right results (see Sec. III) and supports the validity of these equations for more general problems.

A new feature appears in the Langevin equations (2.21) that we have not previously encountered. Since the determinant (2.20) can vanish when two (or more) electrons are in the same state, this can lead to singularities in the drift terms of the Langevin equations. However, the nature of these singularities is such as to drive the electrons apart.

An algorithm explaining how to deal with such singularities which is also suitable for numerical studies, has been discussed elsewhere[2] and will not be repeated here. In practice, for the examples studied in Sec. III, no incidence of a singularity or of a near singularity occurred and no special algorithm was required.

III. STUDY OF TWO-DIMENSIONAL QUANTIZED HALL EFFECT

A fascinating development of the past several years has been the discovery and explanation of the ordinary and anamolous two-dimensional quantized Hall effects.[11] The explanation of these effects involves the deformation of the free-electron energy levels into highly degenerate Landau levels as modified by the Coulomb interaction between electrons. We shall (i) concentrate on the low temperature, large magnetic field case in which only the first Landau level need be considered, (ii) eliminate the fast component of electronic motion, and (iii) retain only the Coulomb interaction between electrons within the first Landau level. As shown by Fukuyama and Yoshioka[12] the Hamiltonian operator under these circumstances reduces to

$$\mathscr{H}=\frac{e^2}{2\pi}\sum_{\alpha>\beta}\int\frac{d^2k}{|k|}\exp\left[ilk_1(Q^\alpha-Q^\beta)+ilk_2(P^\alpha-P^\beta)-\frac{k^2l^2}{2}\right].$$

Here $l=\sqrt{c/eH}$ denotes the Larmor radius and α,β refer to particle labels $1\le\alpha,\beta\le M$. By definition of the Weyl representation we have

$$h_w(p,q)=\frac{e^2}{2\pi}\sum_{\alpha>\beta}\int\frac{d^2k}{|k|}\exp\left[ilk_1(q^\alpha-q^\beta)+ilk_2(p^\alpha-p^\beta)-\frac{k^2l^2}{2}\right].$$

Consequently the function h needed for the functional description of Sec. II is given by [cf. (2.4b) and (2.5d)]

$$\begin{aligned}h(p,q)&=\exp\left[-\frac{1}{4}\sum_\gamma\left[\frac{\partial^2}{\partial(p^\gamma)^2}+\frac{\partial^2}{\partial(q^\gamma)^2}\right]\right]h_w(p,q)\\&=\frac{e^2}{2\pi}\sum_{\alpha>\beta}\int\frac{d^2k}{|k|}\exp[ilk_1(q^\alpha-q^\beta)+ilk_2(p^\alpha-p^\beta)]\\&=\frac{e^2}{l}\sum_{\alpha>\beta}\frac{1}{[(q^\alpha-q^\beta)^2+(p^\alpha-p^\beta)^2]^{1/2}},\end{aligned}$$

which is just the Coulomb potential again.

So as to deal with only a finite number of electrons as well as a finite electron density, we replace this potential by a periodic one

$$h(p,q)=\frac{e^2}{l}\left[\sum_{\alpha>\beta}\sum_m\frac{1}{[(q^\alpha-q^\beta+m_1)^2+(p^\alpha-p^\beta+m_2)^2]^{1/2}}+\sum_\alpha w(p^\alpha,q^\alpha)+c\right]. \tag{3.1}$$

Here m_i is an integral multiple of B,

$$m_i=\ldots,-2B,-B,0,B,2B,\ldots,$$

where B is the periodic size in both the q and p directions, and w and c denote contributions from the jellium background of positive charge necessary to ensure that the sum defining the periodic potential converges. It may seem natural in this case to replace the sum defining the periodic potential by an Ewald sum,[13] however, since our coordinates q and p can become *complex* (as they appear in the Langevin equation) this procedure is not advisable. Instead we truncate the sum at some reasonable value $\bar{M}$ to approximate the results of an infinite sum so that in practice

$$m_i=-\bar{M}B,-(\bar{M}-1)B,\ldots,\bar{M}B.$$

For most of the Langevin calculations the value $\bar{M}=4$ has been chosen; this choice leads to 81 terms in the "periodic" sum and yields energies within 1% of those given by the full sum.

Closed form expressions for $w(p,q)$ and c are given by [$L\equiv(2\bar{M}+1)B$]

$$\begin{aligned}w(p,q)=\left[\frac{M}{B}\right]\Bigg[&(L+p)\sinh^{-1}\left[\frac{L+q}{L+p}\right]+(L+q)\sinh^{-1}\left[\frac{L+p}{L+q}\right]+(L+p)\sinh^{-1}\left[\frac{L-q}{L+p}\right]\\&+(L-q)\sinh^{-1}\left[\frac{L+p}{L-q}\right]+(L-p)\sinh^{-1}\left[\frac{L+q}{L-p}\right]+(L+q)\sinh^{-1}\left[\frac{L-p}{L+q}\right]\\&+(L-p)\sinh^{-1}\left[\frac{L-q}{L-p}\right]+(L-q)\sinh^{-1}\left[\frac{L-p}{L-q}\right]\Bigg],\end{aligned}$$

$$c=\frac{4M^2L}{B^2}\left[\sinh^{-1}(1)-\frac{\sqrt{2}-1}{3}\right].$$

In practice the equation for w was expanded out to terms of order L^{-3}, an approximation which contributed no more than a 1% error. The so-expanded relations complete the characterization of the energy expression entering the Langevin equations (2.19) and (2.22) used to study this problem. Most of our effort has been directed toward the four-electron problem ($M=4$), while we have also made a few studies for nine electrons ($M=9$, no data presented). We principally have calculated the mean energy

$$\langle\langle \mathscr{H} \rangle\rangle \equiv \mathrm{Tr}(\mathscr{H} e^{-\beta \mathscr{H}})/\mathrm{Tr}(e^{-\beta \mathscr{H}}) \tag{3.2a}$$

according to the rule [cf. (2.16)]

$$\frac{1}{T}\int_0^T h(p(\tau),q(\tau))d\tau = \langle\langle \mathscr{H} \rangle\rangle \tag{3.2b}$$

which holds for large T. Here $p(\tau),q(\tau)$ denote solutions of the $2NM$ coupled, complex Langevin equations (2.19) or (2.22).

The numerical solution of coupled Langevin equations (i.e., of coupled stochastic differential equations) is a problem that has been studied for some time. We have utilized the second-order, Runge-Kutta scheme systematically derived by Helfand and by Greenside and Helfand.[14] This approach has the virtue that not only is the deterministic part of the solution given correctly apart from terms $O(h^3)$, but the first two moments of the stochastic part of the solution are correct save for terms $O(h^3)$ as well. Here h denotes the discrete time step $\Delta\tau$, and we have used both $h=0.1$ and 0.01 in our work. The value of β chosen has varied between 1 and 16, while ϵ has generally been chosen as $\frac{1}{16}$. This means that N, the number of thermal-time steps, has been taken as large as 256. For four particles this has resulted in as many as 2048 coupled, complex Langevin equations to deal with in a single run. The value of T, the upper limit of the τ variable, was frequently taken as large as 100. The most involved runs took up to 40 min of Cray-1 time. A good fraction of this time was spent in evaluating the determinants and the lattice sums involved in computing the forces and energy pertinent to this problem.

We have studied the quantized Hall effect for both distinguishable and indistinguishable electrons. The energy levels are typically lower for the former case, as expected. The energy has been studied as a function of the filling factor ν, which in our notation is given by $\nu=2\pi M/B^2$; when $0\leq\nu\leq 1$ it represents the fraction of the first Landau level that is filled. Experimentally[3,11] one observes a plateau in the Hall conductivity at $\nu=\frac{1}{3}$ (and certain other fractional values) due to a special ground state and enhanced excitation gap for that value of ν. This behavior is now interpreted as the consequence of a novel quantum liquid which is analytically described by an approximate ground state proposed by Laughlin.[11] At the level our studies were made we are unable to shed any interesting light on the special phenomena that take place at $\nu=\frac{1}{3}$. We shall comment further on this point below.

As a preliminary to a full presentation of our data we first list some selected data for distinguishable particles and discuss the results. In Table I we show typical results of an evaluation (here and elsewhere in units of e^2/l) of the energy per particle

$$\frac{1}{MT}\int_0^T h(p(\tau),q(\tau))d\tau$$

based on (2.19) for several T values. Over the time interval shown there is remarkably little statistical fluctuation in the data. This is probably to be interpreted as due to some very slow relaxation times in the problem. As noted in Sec. II the time constant for the decay of correlations may be as long as ϵ^{-1}, or 16 in our case. From this point of view 20 is not a very large value for T. Hence the resultant value at 20 is still influenced by the initial value at $\tau=0$. Recognizing this fact we have opted to choose our initial configuration at or near to minimum energy classical configurations thus avoiding the prohibitively long relaxation time that starting from a random configuration entails. In Table II we show the results for a run similar to that in Table I except for a random choice of the initial conditions. Here the results are dominated by a large amplitude, under damped component, which is largely absent in the case of an ordered start as in Table I.

TABLE I. Energy per particle for various averaging times T for the distinguishable particle case. (Parameters chosen so that $\beta=8$, $\epsilon=\frac{1}{16}$, $\nu=\frac{1}{3}$, $h=0.01$, along with an ordered start.)

Averaging time T	Energy per particle
10	−0.354
11	−0.351
12	−0.348
13	−0.345
14	−0.346
15	−0.347
16	−0.349
17	−0.351
18	−0.355
19	−0.359
20	−0.362

TABLE II. Energy per particle for various averaging times T for the distinguishable particle case. (Parameters chosen as in Table I save for a random start rather than an ordered start.)

Averaging time T	Energy per particle
10	−0.324
11	−0.328
12	−0.333
13	−0.340
14	−0.346
15	−0.352
16	−0.358
17	−0.365
18	−0.371
19	−0.378
20	−0.384

TABLE III. Energy per particle for different β and averaging times T for the distinguishable particle case. Compare also Table I. (Parameters as in Table I save for change of thermal time β.)

Averaging time T	Energy per particle $\beta=1$	$\beta=4$	$\beta=12$	$\beta=16$
10	−0.353	−0.354	−0.354	−0.353
11	−0.350	−0.350	−0.351	−0.350
12	−0.347	−0.347	−0.348	−0.347
13	−0.345	−0.345	−0.345	−0.344
14	−0.345	−0.345	−0.346	−0.344
15	−0.346	−0.346	−0.347	−0.344
16	−0.348	−0.348	−0.349	−0.346
17	−0.351	−0.351	−0.351	−0.348
18	−0.355	−0.354	−0.355	−0.352
19	−0.358	−0.358	−0.358	−0.356
20	−0.361	−0.362	−0.362	−0.360

TABLE IV. Average energy for harmonic oscillator for several different β and averaging times T. Correct values are also listed. (Parameters chosen so that $\epsilon=\beta$, $h=0.001$, and vanishing initial values.)

Averaging time T	Energy $\beta=1$	$\beta=2$	$\beta=4$
25.0	0.565	0.106	0.0121
27.5	0.472	0.0978	0.0188
30.0	0.434	0.0879	0.0273
32.5	0.521	0.109	0.0274
35.0	0.612	0.108	0.0285
37.5	0.546	0.0818	0.0196
40.0	0.476	0.0878	0.0282
42.5	0.491	0.0971	0.0280
45.0	0.460	0.111	0.0338
47.5	0.428	0.107	0.0334
50.0	0.379	0.0791	0.0275
Correct value	0.582	0.157	0.0187

In all cases the long time average excludes about 25% of the run in an effort to minimize the effects of the initial conditions.

Another and more striking feature of our results for the average energy is their dependence, or better, their *lack* of dependence on the parameter β in (3.2). Table III shows the results of a run similar in all respects to that of Table I except that the values of β are 1, 4, 12, and 16 (rather than 8 as in Table I). This remarkable insensitivity to changes in β over the range studied could arise for a suitably special density of states. In fact it was hoped to probe the density of states by studying the β dependence of the average energy (for another example, see below). However, it is unlikely that the density of states for the problem at hand is so special as to cause the insensitivity to β that was observed.[15] A more plausible explanation is that the averaging time T was simply inadequate to permit the various features of the Hamiltonian to leave an imprint on the solution to the coupled Langevin equations. Moreover, for small ϵ (large N) the effect of the Hamiltonian may simply get lost in the numerical solution at our level of precision.

Harmonic oscillator: To study further the question of β dependence we have also investigated the elementary example of a single harmonic oscillator for which $M=1$ and $h=\frac{1}{2}(p^2+q^2-2)$. In Table IV we show the average energy at $\beta=1$, 2, and 4 as a function of the averaging time T. The correct value of $1/(e^{\beta}-1)$ is indicated for each case. While these numbers show significant fluctuations they do illustrate that the method we propose here is indeed sensitive to the variable β, at least in principle. In order to obtain more accurate results much longer averaging times are needed. Table V lists the average energy at $\beta=1$ over much longer averaging times and shows a tendency for the values to settle down to the correct value of 0.582. For a harmonic oscillator with unit frequency the relaxation time is approximately unity. In order to achieve an accuracy of 1% in the average energy it is necessary to average over roughly 10^4 independent time units. Thus it is not surprising that it takes $T\simeq 5000$ in order for the average energy to be given to about two significant figures.

It is important to remark that the harmonic-oscillator examples were studied with an improved action. Since it

TABLE V. Average energy for harmonic oscillator for a variety of averaging times T for $\beta=1$ (correct answer equals 0.582). Compare also the first column of Table IV. (Parameters chosen so that $\epsilon=\beta=1$, $h=0.001$ for $T\leq 300$ while $h=0.01$ for $T>300$, and vanishing initial values.)

Averaging time T	Energy	Averaging time T	Energy	Averaging time T	Energy
150	0.514	500	0.562	2500	0.592
165	0.520	550	0.566	2750	0.591
180	0.549	600	0.563	3000	0.592
195	0.527	650	0.555	3250	0.586
210	0.523	700	0.541	3500	0.581
225	0.509	750	0.574	3750	0.583
240	0.526	800	0.594	4000	0.588
255	0.550	850	0.587	4250	0.591
270	0.552	900	0.601	4500	0.589
285	0.559	950	0.593	4750	0.586
300	0.537	1000	0.584	5000	0.582

TABLE VI. Average energy for harmonic oscillator for various thermal-time steps N and averaging times T for $\beta=1$ (correct answer equals 0.582). Compare also the last column of Table V. (Parameters chosen so that $\epsilon=\beta/N=1/N$, $h=0.01$ and vanishing initial values.)

Averaging time T	Energy $N=2$	$N=3$	$N=4$	$N=5$	$N=25$
2500	0.586	0.560	0.589	0.504	0.402
2750	0.593	0.553	0.555	0.551	0.416
3000	0.606	0.553	0.545	0.572	0.446
3250	0.614	0.530	0.553	0.602	0.432
3500	0.611	0.538	0.562	0.597	0.422
3750	0.609	0.527	0.576	0.603	0.412
4000	0.606	0.540	0.584	0.602	0.444
4250	0.607	0.525	0.575	0.592	0.457
4500	0.600	0.540	0.560	0.589	0.456
4750	0.592	0.559	0.567	0.584	0.448
5000	0.596	0.565	0.573	0.575	0.438

follows that

$$e^{-\beta\mathscr{H}}=\int e^{\beta}e^{-(e^{\beta}-1)(p^2+q^2)/2}\,|p,q\rangle\langle p,q\,|\,\frac{dp\,dq}{2\pi} \tag{3.3}$$

holds as an exact relation[16] when $\mathscr{H}=\frac{1}{2}(P^2+Q^2-1)$, it follows if we use (3.3) in place of (2.6) that there are no $O(\epsilon^2)$ errors and consequently no need to decompose the thermal time β into a large number N of small steps ϵ. The results quoted above for the harmonic oscillator were all obtained for $N=1$. Table VI illustrates the N dependence of the average energy where (3.3) has been used N times for a β value of β/N. These results tend to show decreasing accuracy as N increases, as could be expected. In each solution the initial values were taken as $q_0=p_0=0$, which may help explain why the average energy for $N=25$ lags well behind that for $N\leq 5$, not only because the relaxation time is longer but because the Hamiltonian has a harder time making itself felt.

Indistinguishable harmonic oscillators: As another test case we have examined the example of two uncoupled harmonic oscillators ($M=2$), $h=\frac{1}{2}(p_1^2+q_1^2+p_2^2+q_2^2-4)$, that obey the exclusion principle. This example permits us to study the validity of the set of equations (2.22) for a soluble problem. The exact average energy in this case is given by

$$1+\frac{1}{e^{\beta}-1}+\frac{2}{e^{2\beta}-1}\,.$$

In our numerical study of this problem we have again used the exact relation (3.3), and therefore all choices of thermal-time steps N should be exact, at least in principle. In Table VII we present the resulting average energy obtained for two β values and two N values as a function of averaging time T along with the correct values. The approximate validity of the results for this model lend credence to the applicability of (2.22) to more general problems.

Quantized Hall effect: Based on the results of our study of the harmonic-oscillator test cases we are forced to conclude that our present study of the two-dimensional quantized Hall effect suffers from too short an averaging time, a problem aggrevated even further by the need to divide the thermal time β into many N factors of short duration ϵ. This latter feature could be helped by an improved action, but this has yet not been done. Also additional computer time could be devoted to this problem, but we have chosen for the present not to do so. The need for extensive computation to achieve reliable answers is well recognized in usual Monte Carlo studies as well.[17]

Having stated the shortcomings in our results, we proceed to enumerate in Table VIII a list of the energy values for different ν values for both the distinguishable and indistinguishable cases. For the distinguishable case the data all refer to an averaging time $T=80$, $\beta=2$, $\epsilon=\frac{1}{16}$, and $h=0.01$. For the indistinguishable case the data all refer to an averaging time $T=100$, $\beta=8$, $\epsilon=\frac{1}{16}$, and $h=0.1$. In each run the same set of random numbers was used (results were not too sensitive to this choice). The overall statistical accuracy of these data is less than

TABLE VII. Average energy for two uncoupled harmonic oscillators that satisfy the exclusion principle for various thermal-time steps N and averaging times T. (Parameters chosen so that $N\epsilon=\beta=0.5$ or $N\epsilon=\beta=0.3$ and $h=0.1$.)

Averaging time T	Energy ($\beta=0.5$) $N=1$	$N=4$	Energy ($\beta=0.3$) $N=1$	$N=4$
5000	3.65	3.94	6.06	6.53
5500	3.64	3.88	6.09	6.44
6000	3.60	3.74	6.03	6.26
6500	3.57	3.81	5.98	6.26
7000	3.59	3.86	6.00	6.28
7500	3.62	3.80	6.05	6.17
8000	3.60	3.73	6.00	6.02
8500	3.61	3.73	6.02	6.03
9000	3.63	3.74	6.06	6.04
9500	3.64	3.75	6.09	6.00
10000	3.64	3.76	6.10	6.00
Correct value	3.71		6.29	

TABLE VIII. Energy per particle for various filling factors ν for the classical, distinguishable (dis), exact, and indistinguishable (indis) cases. (Parameters for the dis case are $T=80$, $\beta=2$, $\epsilon=\frac{1}{16}$, $h=0.01$, and an ordered start; for the indis case the parameters are $T=100$, $\beta=8$, $\epsilon=\frac{1}{16}$, $h=0.1$, and an ordered start.)

Filling factor ν	Energy per particle			
	Classical	dis	exact	indis
0.3125	−0.435	−0.408	−0.398	−0.283
0.3333	−0.450	−0.430	−0.413	−0.299
0.3542	−0.463	−0.445	−0.420	−0.309
0.4167	−0.503	−0.487	−0.442	−0.341
0.5000	−0.550	−0.530	−0.468	−0.379

2%, a fortuitous feature of this model. For comparison we have also included the minimum classical energy per particle values as well as the exact quantum-mechanical ground-state energy per particle values (including the exclusion principle) obtained by Yoshioka *et al.*[18]

Several remarks regarding the data presented are in order. First we observe that the energy results obtained in the distinguishable particle case lie above the classical minimum energy of (3.1) and below the exact quantum-mechanical ground-state results. For example, at $\nu=\frac{1}{3}$ the minimum classical energy per particle is -0.45, our result is -0.43, while the exact quantum-mechanical ground state is given by -0.41. However, our indistinguishable particle data all lie well above the exact ground-state results. This behavior could represent the result of an average over a suitable density of states, however, the lack of dependence of such data on β tends to belie this fact. We believe the discrepancy of our indistinguishable data arises because of two factors: (i) an insufficient averaging time T, and, perhaps more importantly, (ii) the need to use a step size $h=0.1$ in the numerical solution of the coupled Langevin equations. Note that for the distinguishable particle case we were able to choose $h=0.01$, a factor of ten better. It is interesting to add that for relatively short averaging times T ($T\lesssim 10$), the energy per particle values for the distinguishable and indistinguishable particle cases more nearly agree with each other as is to be expected for this example, particularly near $\nu=\frac{1}{3}$. This result suggests that the error in the indistinguishable particle case accumulates, more or less coherently, for larger averaging times T, indeed, due to the larger time step h. This situation could be improved, of course, by using significantly more computer time, but we have chosen not to do so.

To study the behavior at and near $\nu=\frac{1}{3}$, the average energy per particle was measured at $\nu=\frac{15}{48}$, $\frac{16}{48}$, and $\frac{17}{48}$. It is amusing to note that in both the distinguishable and indistinguishable cases the values at $\nu=\frac{1}{3}$ are slightly lower than the average of the two neighboring values (relative energies may possibly be more accurate here than absolute ones). The exact results listed in Table VIII also show a slight dip at $\nu=\frac{1}{3}$. Such a cusp in energies at $\nu=\frac{1}{3}$, not present classically, is exactly what one anticipates on the basis of the Laughlin quantum-mechanical ground state and which provides the energy gap needed to have a plateau in the Hall conductivity. At the very least our data is not inconsistent with that picture.

IV. CONCLUSION

We have presented a first-effort study of the application of coupled, complex Langevin equations to evaluate average energies of an involved system, namely, that of the two-dimensional quantized Hall effect. Our results are not unreasonable given the modest averaging times proffered to the seemingly insatiable beast that is at the heart of all statistical approaches. In the future we intend to apply the experience gained here to the study of quantum spin systems by means of associated Langevin equations, the form of which has already been presented elsewhere.[19,2]

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