Thermal noise effects on the microwave-induced steps of a current-driven Josephson junction

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We study the effects of thermal noise on the microwave-induced steps of a current-driven Josephson junction using the resistively shunted junction model for the junction. We generalize Stephen's method for calculating the width of the steps to the case of an underdamped junction, and to the case of a subharmonic step, provided that there is no overlapping of different steps. In order to treat the case of overlapping steps we introduce a new approach which enables us to calculate the distribution of fluctuations about as well as the transition rates out of the steps. All our results are usually in the form of explicit analytical expressions that facilitate a comparison with experiments.

I. INTRODUCTION

According to the Josephson tunneling theory,[1,2] microwave radiation will induce steps in the I-V characteristic of the junction at voltages $n\hbar\omega_{\text{ex}}/2e$ (where n is an integer and ω_{ex} is the frequency of the incident radiation). These steps, that were first observed by Shapiro,[3] suggest that the ac Josephson effect may be used not only for generation, but also for detection of microwave radiation.[4] The existence of subharmonic steps [namely, steps at voltages $(n/m)\hbar\omega_{\text{ex}}/2e$] is important because it would extend the high-frequency limit for these applications upwards from $2\Delta/\hbar$ to $2m\Delta/\hbar$, i.e., into the far-infrared region. The nonlinear response of the junction to an oscillating driving source also suggests additional applications of the junction[5–7] as a mixer, modulator, and ac amplifier of frequencies in the spectral range mentioned above.

Here we will use the RSJ (resistively shunted junction) model[8,9] for the junction with a purely sinusoidal current-phase relation, and consider the case where the microwave radiation acts as an ac current source.[10,11]

It turns out that the various types of measured I-V characteristics (e.g., those with and without subharmonic steps,[10–14] those of zigzaglike structure,[15] with hysteresis between steps,[16] etc.), as well as chaotic behavior,[17–19] can be obtained from the same model that we have used, by merely changing the parameters of the junction and the incident radiation.[17–19] In Sec. II we present the model and a summary of its dynamics in the absence of noise.

The sensitivity of the Josephson junction used as a microwave detector and the equivalent temperature of mixers, modulators, or parametric amplifiers based on a Josephson junction are limited by noise. Some of the effects of thermal noise on the fundamental step (at the voltage $\hbar\omega_{\text{ex}}/2e$), i.e., the rounding or voltage width of the step, have already been studied both theoretically[20] and experimentally[21–23] for overdamped junctions. In Sec. III we present an approach similar to that of Stephen,[20] which enables us to generalize those results to the case of an underdamped junction as long as there is no overlapping of different steps in the I-V characteristic. This method enables us to evaluate the width of the subharmonic steps approximately. We compare these results with a numerical solution of the Langevin equation,[24] both for the harmonic and for the subharmonic steps. We find that the thermal noise has a stronger effect on the smaller subharmonic steps.

In Sec. IV we consider the effect of thermal noise when there is overlapping of different steps in the I-V characteristic. We derive approximate expressions for the distribution of fluctuations about the zero voltage step and the fundamental step, and for transition rates out of these steps. The method we use is an extension of a method that has recently been developed and has already been applied successfully to the problem of a dc-driven hysteretic junction.[25,26] Section V is a summary of the results.

II. THE MODEL AND THE DYNAMICS IN THE ABSENCE OF NOISE

Assuming that the impedance of the wave guide for the microwave-induced radiation is much larger than the effective impedance of the junction, the induced radiation acts as an ac current source.[10,11] If we further assume the RSJ model[8,9] for the junction, the response of the junction to a microwave-induced radiation, while biased by a dc-current source I_{dc}, is given by

$$C\frac{dV}{dt}+\frac{V}{R}+I_J\sin\theta=I_{\text{dc}}+I_{\text{ac}}\sin(\omega_{\text{ex}}t)\ , \tag{2.1}$$

where C, R, and I_J are the capacitance, resistance, and critical current of the junction, respectively.

Following Refs. 17 and 19, we rewrite this equation in the following dimensionless units:

$$\ddot{\theta}+G\dot{\theta}+\sin\theta=I+a\,\sin(\omega t)\ , \tag{2.2}$$

where

$$G \equiv (\omega_J RC)^{-1}, \quad \omega_J^2 \equiv \frac{2eI_J}{\hbar C} ,$$
$$I \equiv I_{dc}/I_J, \quad a \equiv I_{ac}/I_J , \tag{2.3}$$
$$\omega \equiv \omega_{ex}/\omega_J ,$$

and time is measured in units of ω_J^{-1} (ω_J is the Josephson plasma frquency). This model can be used for low and high frequencies as long as the RSJ model is applicable. For very high frequencies, which become comparable to the gap frequency $\Delta/\hbar$, further effects such as frequency dependence of I_J will come in.[27,28]

A current step at the voltage $V_{n/m} \equiv (n/m)V_1$, where V_1 is the voltage of the fundamental step

$$V_1 \equiv \frac{\hbar}{2e}\omega_{ex} , \tag{2.4}$$

is due to an n/m subharmonic solution in which θ advances by $2\pi n$ during m periods of the microwave field. Such digital numerical results[17,19] of harmonic ($n/m=1$ and $n/m=3$) and subharmonic ($n/m=\frac{3}{4}$) solutions are shown in Fig. 1. It can be proved that every periodic (in $\dot\theta$) solution is either harmonic or subharmonic, although the solution is not necessarily unique, and overlapping of different steps may occur.[17,19] Here we shall discuss only the periodic solutions [the aperiodic (chaotic) solutions are discussed, e.g., in Refs. 17—19]; in general, the n/m solution can be written in the following way:[17]

$$\theta(t) = \theta_0 + \frac{n}{m}\omega t + \sum_{l=1}^{\infty} a_l \sin\left[\frac{l}{m}\omega t + \varphi_l\right] , \tag{2.5}$$

where the sum can usually be approximated by the first few terms. In Refs. 17 and 19 it was shown that when the amplitude of the microwave field is small, i.e.,

$$A \equiv a[\omega(\omega^2+G^2)^{1/2}]^{-1} < 1 \tag{2.6}$$

then $a_l \ll 1$ for all the terms in the sum in Eq. (2.5). When A is large and the external frequency is high, $\theta(t)$ is given by

$$\theta(t) \cong \theta_0 + \frac{n}{m}\omega t + A\sin(\omega t+\varphi_m) + \sum_{l\,(\neq m)} a_l \sin\left[\frac{l}{m}\omega t + \varphi_l\right] , \tag{2.7}$$

where $a_l \ll 1$ for all l. The sum in this equation may be ignored to leading order, and the first correction is the term $l=1$ in the sum. For additional analytical approximations see Refs. 16 and 29.

For high frequencies ($\omega_{ex} > \omega_J$) the height of the fundamental step, ΔI_1, agrees quite well with the Bessel function expression

$$\Delta I_1 \cong 2J_1(A) \tag{2.8}$$

obtained in the model of an ac-voltage source. In this

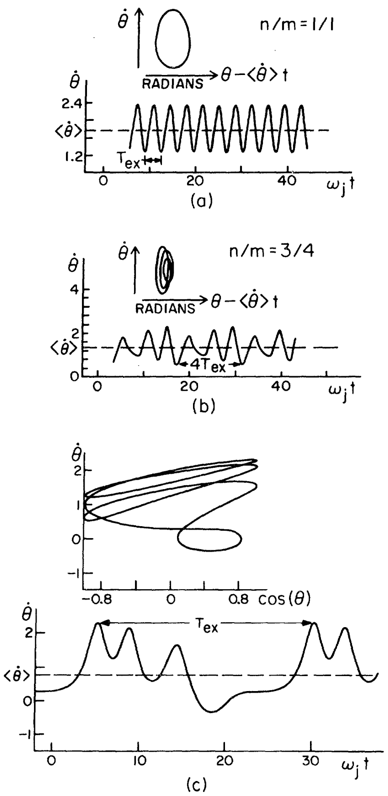

FIG. 1. Some typical steady-state solutions for $\dot\theta$ as a function of time. The insets show the phase-space trajectories. (a) The fundamental step ($n=m=1$) for $I=1.35$, $G=0.7$, $a=0.8$, and $\omega=1.76$. (b) Subharmonic solution ($n=3, m=4$) for $I=1.024$, $G=0.7$, $a=0.8$, and $\omega=1.76$. (c) The third harmonic solution ($n=3, m=1$) for $I=0.95$, $G=0.7$, $a=0.4$, and $\omega=0.25$.

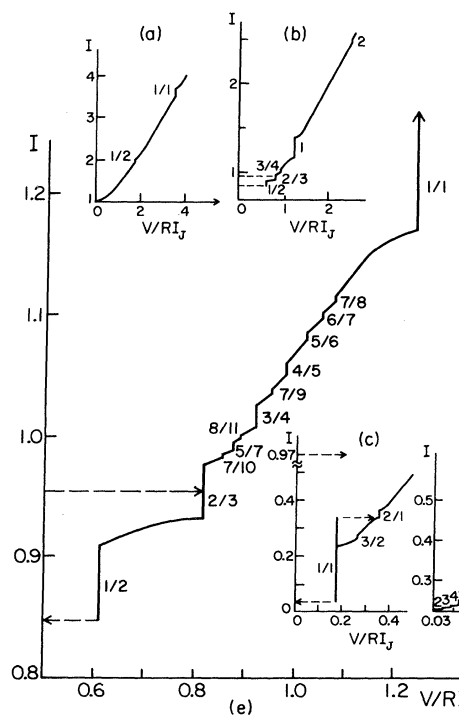

FIG. 2. Some typical noiseless I-V characteristics. (a) For $G=2, \omega=1.76, a=1$. (b) For $G=0.7, \omega=1.76, a=1$. (c) For $G=0.1, \omega=1.76, a=1$. (d) For $G=0.3, \omega=0.25, a=0.8$. (e) The same as (b) on enlarged scale.

case, for a small effective amplitude ($A \ll 1$) ΔI_1 is approximately equal to A. The heights of the $1/m$ steps with $m>1$, under the above conditions, are much smaller but depend linearly upon A. The heights of the n/m steps with $n \neq m$ and $n>1$ are proportional to A^2. For high frequencies and large effective amplitudes, the heights of the harmonic steps ($m=1$) are given by

$$\Delta I_n \cong 2J_n(A) \tag{2.9}$$

while those of the subharmonic steps can be written in terms of products of Bessel functions of argument A, and this leads to smaller steps. The same results are valid for low frequencies, provided $J_n(A) \ll \omega(\omega^2+G^2)^{1/2}$.

In Fig. 2 we show various types of I-V characteristics obtained for different values of the parameters a, ω, and G.

We have found that the subharmonic steps have the largest heights for medium frequencies ($\omega_{\mathrm{ex}} \sim \omega_J$), $G \sim 1$, and $a \sim 1$. The step size quickly decreases for complicated n/m ratios. With further and larger numerical computations, more small steps are found [see Figs. 2(a) and 2(b)]. It thus appears that the I-V characteristic (for such values of the parameters) contains a dense set of steps which form a "devil's staircase"-like structure. However, we will show in the next section that the smaller steps are easily washed out by thermal noise. Thus these steps (mainly those with a complicated n/m ratio) are usually not observed experimentally. For high frequencies and an overdamped junction ($G>1$), the I-V characteristic has the same structure but only the fundamental step is sizable, as confirmed also in experiments.[30] In the limit of small G and low frequencies, heights of the steps were found to approach zero (Fig. 2). Again the results are consistent with experiments.[3,11–14] We have observed overlapping of steps, and thus hysteresis, for small G and high frequencies (Fig. 2). In this case the thermal noise causes transitions among the overlapping steps as will be discussed in Sec. IV.

III. THE NONHYSTERETIC CASE

In this paper we assume that the dominant noise is Johnson noise due to the resistance R. Thus the dynamics in the presence of noise is described by the following Langevin equation:[24]

$$\begin{gathered}\ddot{\theta}+G\dot{\theta}+\sin\theta=I+a\,\sin(\omega t)+\mathscr{L}(t)\;,\\ \langle \mathscr{L}(t)\mathscr{L}(t+\tau)\rangle=2GT\delta(\tau)\;,\end{gathered} \tag{3.1}$$

where T is the temperature in temperature units of $\hbar I_J/2ek_B$.

In this section we consider the case with no overlapping of steps in the I-$\overline{V}$ characteristic. The thermal noise introduces rounding of the steps via fluctuations about the zero-temperature steady state. For the sake of simplicity, we first calculate the rounding of the fundamental step. (The approach is similar, but not identical, to the approach of Stephen.[20])

We define the slowly fluctuating variable

$$X(t) \equiv \theta-\omega t-\varphi-A\,\sin(\omega t+\alpha) \tag{3.2}$$

which can be shown to satisfy the following Langevin equation (a detailed derivation of this equation is presented in the Appendix):

$$\ddot{X}+G\dot{X}+J_1(A)\sin X=\Delta I+\mathscr{L}(t) \tag{3.3}$$

when the constants φ,α,A are chosen appropriately. The other quantities that appear are

$$\Delta I = I - G\omega \tag{3.4}$$

and $J_1(A) \cong \frac{1}{2}A$, which is a Bessel function of the first kind. The quantity $J_1(A)$ is one-half of the total height of the fundamental step. In obtaining (3.3), use was made of the fact that x fluctuates much more slowly than the driving frequency ω. In terms of $X(t)$, the average voltage measured with respect to the step voltage $\hbar\omega/2e$ is given by

$$\Delta V/RI_J = G\langle \dot{X}\rangle \tag{3.5}$$

(in our units [Eq. (2.3)] the voltage in real units is given by $V=RI_JG\dot{\theta}$) and thus the quantity $\Delta V/(\Delta I)$— the step rounding—can be easily obtained from Eq. (3.3), since this equation is similar to that of a particle moving in a potential

$$U(X)=-\Delta I\,X-J_1(A)\cos X\;. \tag{3.6}$$

The forward transition rate r_A (these transitions tend to increase the voltage V) out of the static states of the potential $U(X)$ [for $\Delta I < J_1(A_e)$] is given by[31]

$$r_A=\Omega_e e^{-\Delta U/T} \tag{3.7}$$

where the potential barrier ΔU is given by

$$\Delta U(\Delta I)=2\left[[J_1(A)^2-\Delta I^2]^{1/2} - \Delta I\cos^{-1}\left[\frac{\Delta I}{J_1(A)}\right]\right] \tag{3.8}$$

and the attempt frequency Ω_e is given by

$$\Omega_e=\begin{cases}[J_1(A)^2-\Delta I^2]^{1/2}/2\pi G & (3.9)\\ \left\{\left[\left[\dfrac{G}{2}\right]^2+[J_1(A)^2-\Delta I^2]^{1/2}\right]^{1/2}-\dfrac{G}{2}\right\} & (3.10)\\ G\Delta U/T\;, & (3.11)\end{cases}$$

for the overdamped, intermediate, and underdamped states, respectively. The backward transition rate (these transitions tend to decrease V) r_B is given by

$$r_B=r_Ae^{-2\pi\Delta I/T}\;. \tag{3.12}$$

The total rounding of the step is then given by

$$\Delta V(\Delta I)=2\pi(r_A-r_B)\;. \tag{3.13}$$

A comparison of this analytical approximation with the results of numerical simulations of the Langevin equation, Eq. (3.1),[24] is shown in Fig. 3. For comparison of a similar approach of Stephen[20] with experiments see Ref. 23.

In experiments the subharmonic steps are rarely observed. It has been stated in the past[24] that this may be

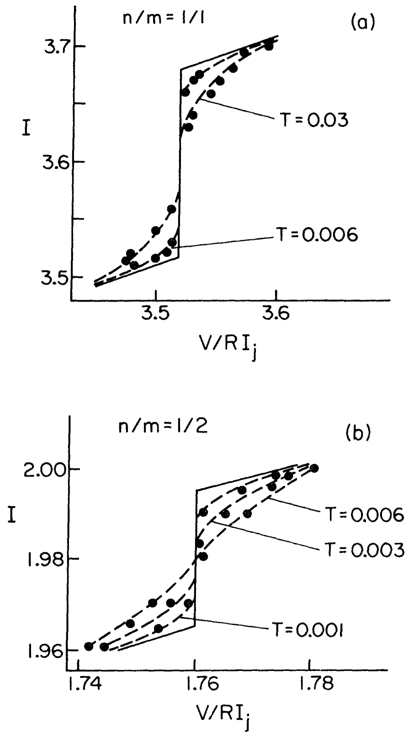

FIG. 3. (a) Comparison of the numerical simulations (solid circles) with the analytical approximation discussed in text (dashed lines) for the broadening of the fundamental (1/1) step and the $\frac{1}{2}$ step by noise. In calculating the analytical results we used Eq. (3.9) for the prefactor Ω_e, replacing $J_1(A_e)$ by the numerically evaluated half-height of the $\frac{1}{2}$ step in the absence of noise. The solid curve is the noise-free I-$\bar{V}$ characteristic. The parameters are $G=2$, $a=1$, and $\omega=1.76$. (a) The fundamental step. (b) Step $\frac{1}{2}$.

due to a stronger effect of the thermal noise on the subharmonic steps. We now proceed to give a qualitative argument to substantiate that claim. In principle, we would try to calculate the rounding of any step by analogy with the treatment of the fundamental step. However, the rigorous derivation of an equation analogous to Eq. (3.3) for an n/m subharmonic step is rather complicated, and the effective potential $U(X)$ apparently has additional structure. Therefore we argue qualitatively as follows: The amplitude of the $\cos X$ term in $U(X)$ of (3.6) is $J_1(A_e)$, which is half the height of the fundamental step. It is reasonable to assume that in the analogous equation for the n/m subharmonic step, a similar term will appear, i.e., a $\cos X$ term with an amplitude proportional to the height of the step $\Delta I_{n/m}$. Because the amplitude is smaller for smaller steps, the potential barrier will also be smaller. Thus the transition rates will be greater, and the resulting voltage rounding $\langle \dot{X} \rangle$ will be larger.

In Fig. 3 we can in fact see that the thermal noise has a stronger effect on the subharmonic step $\frac{1}{2}$ than on the fundamental step: The fundamental step can still be observed at a noise level where the step $\frac{1}{2}$ is completely washed out. The thermal noise has an even larger effect on the smaller steps (those with more complicated n/m ratio). Thus at finite temperature most of the small steps are washed out.

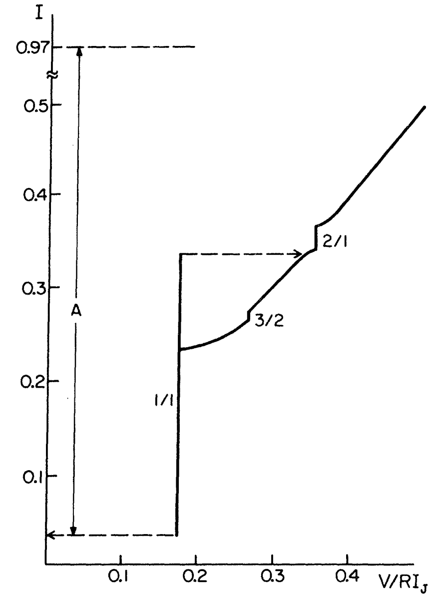

FIG. 4. A typical noiseless I-V characteristic with overlapping steps. The parameters are $G=0.1$, $a=1$, and $\omega=1.76$.

IV. THE HYSTERETIC CASE

We now turn to discuss the case where the noise-free I-$\bar{V}$ characteristic has overlapping steps, i.e., $G \ll 1$, $\omega_{\rm ex} > \omega_J$, and $a < I_J$. A typical I-$\bar{V}$ characteristic is shown in Fig. 4. In the presence of thermal noise each of the steps in the overlap region (regime A in Fig. 4) has a finite lifetime.[32] Outside this regime the thermal noise will only cause fluctuations about the steady state. We are interest-

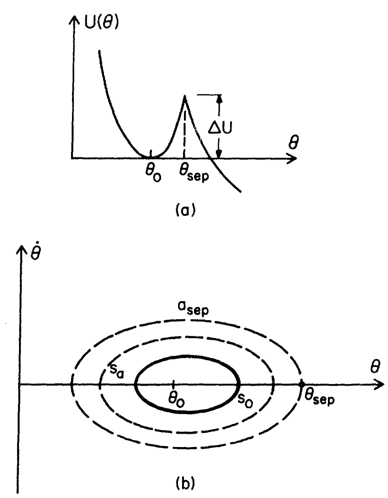

FIG. 5. (a) The truncated harmonic oscillator potential. (b) Schematic picture of the phase-space trajectories. S_0 is the steady-state trajectory for amplitude a of the driving force. The dashed curve is the trajectory for $a=a_{\rm sep}$, which is the one that passes through $\theta_{\rm sep}$.

ed here in the regime of overlap of steps. We start by calculating the mean lifetime of the zero voltage step.[33] In this branch the particle oscillates at the frequency ω ($=\omega_{ex}/\omega_J$) in one of the wells of the potential $U(\theta)$. First, we approximate the potential well by a harmonic oscillator potential of unit frequency, as shown in Fig. 5. For a given amplitude a_0 the steady-state solution is then

$$\theta_0(t)=\frac{1-\omega^2}{\omega_r^2}a_0\sin(\omega t)-\frac{G\omega}{\omega_r^2}a_0\cos(\omega t)\ ,$$
$$\omega_r^2\equiv(1-\omega^2)^2+G^2\omega^2\ . \tag{4.1}$$

In the presence of thermal noise, the probability density of fluctuations about the steady-state trajectory in phase space S_0 can be obtained by solving the following Fokker-Planck equation

$$\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial\theta}(\dot\theta\rho)-\frac{\partial}{\partial\dot\theta}\{[a_0\sin(\omega t)-\theta-G\dot\theta]\rho\}+GT\frac{\partial^2\rho}{\partial\dot\theta^2}\ . \tag{4.2}$$

The stationary solution of this equation ρ_S which corresponds to a vanishing diffusion current in phase space is[33]

$$\rho_S=\frac{1}{2\pi T}\exp\left[-\frac{[\theta-\theta_0(t)]^2+[\dot\theta-\dot\theta_0(t)]^2}{2T}\right]. \tag{4.3}$$

However, this stationary distribution is not sufficient for a calculation of the escape rate. For that purpose, we need a solution with a nonvanishing diffusion current. In order to find such a solution, we first derive a one-dimensional Smoluchowski-type diffusion equation for ρ following Refs. 25, 26, and 33.

We start with the observation that through each point $(\theta,\dot\theta)$ in the basin of attraction of S_0 there passes a trajectory S_a which is the steady-state solution for a different value of the amplitude of the external force, namely $a=a_0+\Delta a$. In the absence of noise, a system that was initially at $(\theta,\dot\theta)$ will decay to S_0, albeit very slowly if $G\ll 1$ (since a^2 is proportional to the energy which decays slowly at this limit). Consequently, we can use the steady-state trajectories S_a as the basis of a new coordinate system in phase space, in which $a(\theta,\dot\theta)\equiv a_0+\Delta a$ is the slowly varying coordinate, while the other coordinates are fast. We now assume that ρ depends only on a, and we rewrite the Fokker-Planck equation (4.2) in such a way that the streaming terms will vanish when averaged over S_a

$$\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial\theta}(\dot\theta\rho)-\frac{\partial}{\partial\dot\theta}\{[(a_0+\Delta a)\sin(\omega t)-\theta-G\dot\theta]\rho\}+\frac{\partial}{\partial\dot\theta}[\Delta a\sin(\omega t)\rho]+GT\frac{\partial^2\rho}{\partial\dot\theta^2}\ , \tag{4.4}$$

where we still have to use the following relations, obtained from (4.1), to replace $\sin(\omega t)$ and $\partial/\partial\dot\theta$):

$$a\sin(\omega t)=G\dot\theta+(1-\omega^2\theta)\ ,$$
$$\theta^2+\dot\theta^2/\omega^2=a^2/\omega_r^2\ , \tag{4.5}$$
$$\left.\frac{\partial}{\partial\dot\theta}\right|_\theta\cong\frac{\omega_r^2}{\omega^2}\frac{\dot\theta}{a}\frac{\partial}{\partial a}\ .$$

When the last two terms on the right-hand side of (4.4) are averaged over S_a, we obtain a Smoluchowski-type one-dimensional diffusion equation in a

$$\frac{\partial\bar\rho}{\partial t}=\frac{G}{2}\frac{\partial}{\partial a}\left[(a-a_0)\bar\rho+\left[\frac{\omega_r}{\omega}\right]^2 Ta\frac{\partial(\bar\rho/a)}{\partial a}\right]. \tag{4.6}$$

We can check this equation by solving it for the stationary distribution with zero diffusion current

$$\bar\rho_S\cong\text{const}\,a\exp\left[-\frac{(\Delta a)^2\omega^2}{2T\omega_r^2}\right] \tag{4.7}$$

This agrees with the form obtained earlier in (4.3).[33] Following Refs. 31 and 33 we can obtain the transition rate τ_0^{-1} by solving Eq. (4.6) using the stationary diffusion approximation.[31] We thus obtain

$$\tau_0^{-1}\cong\frac{G\omega}{4(2\pi T\omega_r)^{1/2}}\frac{a_{sep}(a_{sep}-a_0)}{a_0}\times\exp\left[-\frac{(a_{sep}-a_0)^2\omega^2}{2T\omega_r^2}\right], \tag{4.8}$$

where a_{sep} is the value of the amplitude whose trajectory passes through θ_{sep}, the position of the top of the barrier (see Fig. 5). The value of θ_{sep} is given by a $a_{sep}=\omega_r\theta_{sep}$.

So far, we have calculated the transition rate in the truncated harmonic oscillator approximation. In the real potential $[U(\theta)=-I\theta-\cos\theta]$ the fluctuations in the vicinity of the steady-state S_0 are as calculated before, but farther away deviations will appear. The most serious difficulty is that the exit point from the basin of attraction of S_0 is no longer the point θ_{sep} of Fig. 5. In general, the exit should occur for the smallest value of a (again called a_{sep}) for which S_a becomes unstable in the absence of noise. We can calculate $a_{sep}(I)$ by noting that the height of the zero voltage step $I_0(a)$ is given approximately by[15,16]

$$I_0(a)\cong J_0\left[\frac{a}{\omega(\omega^2+G^2)^{1/2}}\right]\quad\text{for } \omega^2\gg a\ . \tag{4.9}$$

Therefore, a_{sep} is determined by taking $I_0=I$, i.e., by solving the following equation:

$$I=J_0(a_{sep}/\omega(\omega^2+G^2)^{1/2})\ . \tag{4.10}$$

In cases when this approximation cannot be used, we can still determine $a_{sep}(I)$ by a numerical solution of (2.2)—the noiseless equation of motion—to find where the zero-voltage steady state becomes unstable. We can now use this value of $a_{sep}(I)$ in (4.8) to determine the transition rate out of the zero voltage state.

We now turn our attention to the fundamental nonzero voltage step. Again we derive a one-dimensional Smoluchowski-type diffusion equation. Here we must choose a slow variable of a different type. The reason is that for most values of I within this step, if we only change the value of the ac current amplitude a, the average voltage remains unchanged. By contrast, if we change the magnitude of the driving frequency to $\omega+\Delta\omega$, we get a series of steady-state solutions that do not intersect, and which appear to fill up all the relevant regions in the basin of attraction of the steady state for ω.

Using this idea, we rewrite the Fokker-Planck equation as follows[32]

$$\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial\theta}(\dot\theta\rho)-\frac{\partial}{\partial\dot\theta}\{[I+a\sin(\omega t+\Delta\omega\, t)-\sin\theta-G\dot\theta]\rho\}+a\frac{\partial}{\partial\dot\theta}\{[\sin(\omega t+\Delta\omega\, t)-\sin(\omega t)]\rho\}+GT\frac{\partial^2\rho}{\partial\dot\theta^2}\ , \quad (4.11)$$

where $\Delta\omega=\Delta\omega(\theta,\dot\theta)$ is chosen so that the steady-state trajectory of the fundamental step for external frequency $\omega+\Delta\omega$ passes through $(\theta,\dot\theta)$. According to Sec. II, these steady-state trajectories are given approximately by

$$\theta(t)\cong\omega t+\frac{a}{\omega^2}\sin(\omega t+\varphi_1)+\varphi_2\ , \quad (4.12)$$

and consequently we can write $\dot\theta$ in the form

$$\dot\theta\cong\omega+f(\theta) \quad (4.13)$$

where $f(\theta)$ is periodic with period 2π. Moreover, the average of $f(\theta)$ is very small compared to ω for $\omega^2\gg a$. As before, we now assume that ρ depends only on the slow variable ω, and average (4.11) over the "fast coordinate" (in this case $\omega\cong\int_0^{2\pi}\dot\theta\, d\theta$ which is the action of the trajectory and thus decays slowly for small dissipation). This can be chosen in various ways, but the most convenient of those seems to be θ, because then the Jacobian of the transformation to the new coordinates is simply

$$\frac{\partial(\theta\dot\theta)}{\partial(\theta\omega)}=\left.\frac{\partial\dot\theta}{\partial\omega}\right|_\theta\cong 1\ . \quad (4.14)$$

This average makes the streaming terms of (4.11) vanish, and from (4.14) we see that, in the remaining terms, $\partial/\partial\dot\theta$ gets replaced by $\partial/\partial\omega$. It remains to calculate the average of $a\{\sin[(\omega+\Delta\omega)t]-\sin(\omega t)\}$. This is accomplished by noting that we can write

$$a\sin(\omega t)=\ddot\theta+G\dot\theta+\sin\theta-I\ . \quad (4.15)$$

Using (4.13), we can easily evaluate the average over θ of the right-hand side of (4.15) to get

$$\frac{1}{2\pi}\int_0^{2\pi}a\sin(\omega t)d\theta=G\omega-I=-\Delta I\ . \quad (4.16)$$

Consequently, the averaging procedure applied to (4.11) results in the following Smoluchowski-type equation for ρ:

$$\frac{\partial\rho(\Delta\omega,t)}{\partial t}\cong G\frac{\partial}{\partial\Delta\omega}\left[\Delta\omega\,\rho+T\frac{\partial\rho}{\partial\Delta\omega}\right]\ . \quad (4.17)$$

The distribution of fluctuations about the steady state is now easily obtained from this equation

$$\rho_S\cong \text{const}\times\exp-(\Delta\omega)^2/2T\ . \quad (4.18)$$

Similarly, the transition rate out of the fundamental nonzero voltage branch is given approximately by

$$\tau_1^{-1}\cong\frac{G}{2}\left[\frac{\Delta U_1}{\pi T}\right]^{1/2}e^{-\Delta U_1/T}\ , \quad (4.19)$$

where

$$\Delta U_1\cong\tfrac{1}{2}(\omega_{\rm sep}-\omega)^2\ . \quad (4.20)$$

The value of ω at which the steady state becomes unstable, $\omega_{\rm sep}$, is determined by equating ΔI to one-half the size of the fundamental step in the absence of noise [see the discussion following (3.4)]

$$\Delta I\equiv I-G\omega_{\rm sep}\cong\pm J_1\left[\frac{a}{\omega_{\rm sep}^2}\right]\ . \quad (4.21)$$

This is an approximate relation, valid for $\omega_{\rm sep}^2\gg a$. When this approximation is inapplicable, $\omega_{\rm sep}$ may be found by numerical solution of the noiseless equation of motion.

We note that there are usually two solutions to (4.21), the plus sign leading to

$$\omega_{\rm sep}=\omega_+\cong\frac{1}{G}\left[I-J_1\left[\frac{a}{\omega^2}\right]\right]<\omega \quad (4.22)$$

and the minus sign leading to

$$\omega_{\rm sep}=\omega_-\cong\frac{1}{G}\left[I+J_1\left[\frac{a}{\omega^2}\right]\right]>\omega\ . \quad (4.23)$$

When the fluctuating frequency $\omega+\Delta\omega$ reaches either one of these values, a transition takes place to a new steady state, the general character of which can be discerned by considering a different type of experiment, where the external driving frequency itself is varied. In that case, when ω is decreased down to ω_+, the system jumps to a *higher* step (i.e., some subharmonic $n/m>1$), while when ω is increased up to ω_-, the system jumps to a *lower* subharmonic step—usually to the zero-voltage step. We thus conclude, somewhat paradoxically, that an exit by a $\Delta\omega>0$ fluctuation will take the system to a *lower* step—in our case (where we assume that there are no *overlapping* low-lying subharmonic steps) to the zero-voltage step—while an exit by a $\Delta\omega<0$ fluctuation will take the system to a *higher* step.

A similar approach can be used to calculate the transition rate out of any other nonzero voltage step.

V. SUMMARY

We considered the effect of thermal noise on the microwave induced steps of a current-driven Josephson junction.

For the case of no overlapping of steps in the I-V characteristic we generalized Stephen's approach to account for underdamped junctions and the subharmonic steps. The analytical results were compared with a numerical solution of the Langevin equation. We explained why the thermal noise has a much stronger effect on the smaller subharmonic steps.

Our main contribution is the calculation of fluctuations about and transition rates from overlapping steps.

First, we consider the zero voltage step, which is analogous to the problem of a particle in a potential well in the presence of an external oscillating field—a problem which is related to many phenomena, e.g., multiphoton dissociation, laser assisted desorption, and chemical reactions in the presence of radiation.

We found that the activation energy for escape over the potential barrier is given not by the height of the potential, as in the absence of radiation, but accordance with Eqs. (4.8) and (4.10).

The fluctuations about the fundamental finite voltage step were found to be determined by Eq. (4.18), while the transition rate out of that step is determined by Eqs. (4.19)—(4.23).

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APPENDIX

In this appendix we will present a derivation of Eq. (3.3). In the spirit of Ref. 20, we make the following ansatz for $\theta_1(t)$, the fundamental step solution of (3.1):

$$\theta_1(t)=\omega t+\varphi+A\,\sin(\omega t\;\alpha)+X(t)\;, \tag{A1}$$

where φ,α,A are constants to be determined in such a way as to make $x(t)$ slowly varying. We substitute this ansatz in Eq. (3.1), and develop each term that has an explicit ωt dependence in a Fourier series. If we discard all terms that involve frequencies greater than ωt (i.e., $2\omega t$, $3\omega t$, etc.) [this is justified when $A \ll 1$ because the coefficients of the discarded terms are Bessel functions $B_n(A)$ where $n \geq 1$], then we are left with the following form for Eq. (3.1):

$$\Delta I \equiv I-G\omega=\ddot{x}+G\dot{x}-J_1(A)\sin(\varphi-\alpha+x)-\mathscr{L}(t)+J_0(A)\sin(\omega t+\varphi+x)$$
$$-J_2(A)\sin(\omega t+2\alpha-\varphi-x)-(a\,\cos\alpha+\omega^2 A)\sin(\omega t+\alpha)\;, \tag{A2}$$

where J_0,J_1,J_2 are Bessel functions of the first kind. We now choose A and α so as to make the last two terms vanish

$$A=a[\omega(\omega^2+G^2)^{1/2}]^{-1}\;,$$
$$\sin\alpha=-G(\omega^2+G^2)^{-1/2}\;, \tag{A3}$$
$$\cos\alpha=-\omega(\omega^2+G^2)^{-1/2}\;.$$

This leaves us with a Langevin equation for X that still involves the frequency ω. These terms do not resonate with X for $|\Delta I| \leq J_1(A)$, i.e., for I within the fundamental step, but they will cause it to have a small oscillatory component at the frequency ω. This can be taken care of, in principle, by making a different choice of A. However, if $a \gg 1$, the change in A will be very small.

We thus discard the remaining oscillatory terms, and are left with

$$\ddot{X}+G\dot{X}-J_1(A)\sin(\varphi-\alpha+X)=\Delta I+\mathscr{L}(t)\;. \tag{A4}$$

Finally, if we choose $\varphi-\alpha=\pi$ we obtain Eq. (3.3) for $X(t)$.

Apart from the random rapid fluctuations caused by the noise term (t), the natural frequency ω_1 for small oscillations of $X(t)$ satisfies the following inequality:

$$2\omega_1^2 \leq 2J_1(A) \leq A \cong \frac{a}{\omega(\omega^2+G^2)^{1/2}} \cong 2(\Delta I)_{\max} \ll I \cong G\omega\;. \tag{A5}$$

Thus, because we usually have $G<\omega$, we find $\omega_1 \ll \omega$, so that $X(t)$ is indeed slowly varying compared to $\theta(t)$.

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