Coulomb half-shell t matrix

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The lth—partial-wave Coulomb half-off-shell t matrix $t_l^c(k,q,k^2)$ is expressed in terms of hypergeometric functions which are polynomials. The expression is used to compute $t_l^c(k,q,k^2)$ as a function of the off-shell momentum q for two laboratory energies $E_{\text{lab}}(=2k^2)=10$ and 20 MeV. Results are presented for $l=0$ to 10. Our numbers refer to repulsive Coulomb potential encountered in the case of p-p scattering. It is found that in addition to the on-shell discontinuity $t_l^c(k,q,k^2)$ exhibits a singularity as $q \to 0$ for higher partial waves.

I. INTRODUCTION

The concept of off-shell Jost functions originally introduced by Fuda and Whiting[1] in the context of scattering by short-range potentials has been extended by van Haeringen[2] to deal with problems involving the Coulomb interaction. For short-range potentials the off-shell Jost function $f_l(k,q)$ is a continuous function of the off-shell momentum q in that $f_l(k,q)$ goes over to the ordinary Jost function[3] $f_l(k)$ as q approaches the on-shell momentum k. In contrast to this the Coulomb off-shell Jost function $f_l^c(k,q)$ exhibits a discontinuity at the energy shell.[2] However for short-range and Coulomb potentials the half-off-shell t matrix $t_l(k,q,k^2)$ can be expressed in terms of appropriate on- and off-shell Jost functions. For the Coulomb potential $2\eta k/r$ (η is the Sommerfeld parameter) we have

$$t_l^c(k,q,k^2)=\left[\frac{k}{q}\right]^l \frac{f_l^c(k,q)-f_l^c(k,-q)}{i\pi q f_l^c(k)} . \quad (1)$$

From Eq. (1) and the explicit expression[4,5] for the off-shell Coulomb Jost function it is easy to see that the singularity in $f_l^c(k,q)$ as $q \to k$ is exactly the discontinuity of the physical half-off-shell t matrix studied by Okubo and Feldman,[6] Mapleton,[7] and Ford.[8]

Recently Maximon[9] has revisited the problem of developing efficient methods to include Coulomb interaction in the final state of an amplitude which already contains the strong interaction pair scattering. In order to investigate how the Coulomb interaction modulates the non-Coulomb amplitude to produce the final state, Maximon proceeds by writing the following integral:

$$A_c(\vec{k})=\int K(\vec{k},\vec{q})A(\vec{q})\,d^3q . \quad (2)$$

Here $A(\vec{q})$ is the amplitude associated with scattering by the strong interaction alone, while $A_c(\vec{k})$ represents the Coulomb-distorted amplitude. The transformation function $K(\vec{k},\vec{q})$ depends on the relative momentum variables $\vec{k}$ and $\vec{q}$ of the particles involved and on their Coulomb parameter. We observe that $K(\vec{k},\vec{q})$ is in fact related to the Coulomb half-off-shell t matrix. Thus knowledge of the Coulomb half-off-shell t matrix is of great practical value in those cases where the amplitude A is known from numerical or analytical calculations.

The object of the present paper is to study in some detail the behavior of $t_l^c(k,q,k^2)$ as a function of q as well as of l. In Eq. (1) $t_l^c(k,q,k^2)$ is written in terms of $f_l^c(k,\pm q)$ and $f_l^c(k)$. The expression for $f_l^c(k)$ is quite simple[10] and can easily be evaluated. In contrast to this the general expression for $f_l^c(k,\pm q)$ is quite complicated[4,5] although the s-wave result is extremely simple. Thus rather than using the values of $f_l^c(k,\pm q)$ in Eq. (1), one would like to have in the literature a noncomplicated expression for $t_l^c(k,q,k^2)$ for easy numerical evaluation of the Coulomb physical half-off-shell t matrix. In Sec. II we achieve this by using some of the results given in Ref. 9. We analyze in Sec. III the numerical results for $t_l^c(k,q,k^2)$ and find that in addition to its singularity at the on-shell point, $t_l^c(k,q,k^2)$ exhibits a singularity as $q \to 0$, particularly for large l.

Derivation of Eq. (1) is implicit from some of the equations given by van Haeringen.[11] It is of interest to note that the expression for $t_l^c(k,q,k^2)$ in terms of Jost functions is formally similar to that for a short-range potential. In the Appendix we express the Coulomb physical (outgoing wave) off-shell wave function $\psi_l^{c(+)}(k,q,r)$ in terms of appropriate Jost functions and solutions by the use of Green's-function technique. As in the case of the Coulomb half-off-shell t matrix, we find that $\psi_l^{c(+)}(k,q,r)$ is also formally similar to the physical off-shell wave function for short-range potentials.

II. HALF-OFF-SHELL T MATRIX

The lth—partial-wave physical half-off-shell t matrix which exhibits a singularity at the energy shell is written as[7,8]

$$t_l^c(k,q,k^2)=\frac{2}{\pi k q}\int_0^\infty \hat{j}_l(qr)\frac{2\eta k}{r}\psi_l^{c(+)}(k,r)\,dr , \quad (3)$$

where $\hat{j}_l(x)$ $[=xj_l(x)]$ is a Riccati-Bessel function and $\psi_l^{c(+)}(k,r)$, the outgoing wave solution for the Coulomb potential.

In terms of the regular solution[10] $\phi_l^c(k,r)$, Eq. (3) reads

$$t_l^c(k,q,k^2)=\frac{4\eta k^{l+1}}{\pi(2l+1)!!f_l^c(k)}\int_0^\infty j_l(qr)\phi_l^c(k,r)\,dr \tag{4}$$

with the ordinary Coulomb Jost function

$$f_l^c(k)=\frac{e^{\pi\eta/2}\Gamma(l+1)}{\Gamma(l+1+i\eta)}\ , \tag{5}$$

and

$$j_l(qr)=\frac{2^l\Gamma(l+1)}{\Gamma(2l+2)}(qr)^l e^{-iqr}\,{}_1F_1(l+1,2l+2;2iqr)\ , \tag{6}$$

$$\phi_l^c(k,r)=r^{l+1}e^{-ikr}\,{}_1F_1(l+1-i\eta,2l+2;2ikr)\ . \tag{7}$$

Using Eqs. (6) and (7) in Eq. (4) we have

$$t_l^c(k,q,k^2)=\frac{2^{l+2}k^{l+1}q^l\Gamma(l+1)\eta}{\pi(2l+1)!!\Gamma(2l+2)f_l^c(k)}I_l(k,q)\ , \tag{8}$$

where

$$I_l(k,q)=\lim_{\epsilon\to 0}\int_0^\infty e^{-(\epsilon+ik+iq)r}r^{2l+1}\times{}_1F_1(l+1-i\eta,2l+2;2ikr)\times{}_1F_1(l+1,2l+2;2iqr)\,dr\ . \tag{9}$$

To evaluate $I_l(k,q)$ we make use of the integral[12]

$$\int_0^\infty e^{-st}t^{c-1}\,{}_1F_1(a,c;t)\,{}_1F_1(a',c;\lambda t)\,dt$$
$$=\Gamma(c)(s-1)^{-a}(s-\lambda)^{-a'}s^{a+a'-c}\times{}_2F_1\left[a,a';c;\frac{\lambda}{(s-1)(s-\lambda)}\right],$$
$$\mathrm{Re}c>0\ ,\quad \mathrm{Re}s>\mathrm{Re}\lambda+1\ . \tag{10}$$

We now transform the variables in Eq. (10) by substituting

$$t=2ku\ ,\quad \lambda=q/k\ ,\quad s=\sigma/2k \tag{11}$$

and arrive at

$$\int_0^\infty e^{-\sigma u}u^{c-1}\,{}_1F_1(a,c;2ku)\,{}_1F_1(a',c;2qu)\,du$$
$$=\Gamma(c)(\sigma-2k)^{-a}(\sigma-2q)^{-a'}\times\sigma^{a+a'-c}\,{}_2F_1\left[a,a';c;\frac{4kq}{(\sigma-2k)(\sigma-2q)}\right],$$
$$\mathrm{Re}c>0\ ,\quad \mathrm{Re}\sigma>(2k+2q)\ . \tag{12}$$

Further substitution $u=ir$ reduces Eq. (12) to

$$\int_0^\infty e^{-i\sigma r}r^{c-1}\,{}_1F_1(a,c;2ikr)\,{}_1F_1(a',c;2iqr)\,dr$$
$$=i^{-c}\Gamma(c)(\sigma-2k)^{-a}(\sigma-2q)^{-a'}\sigma^{a+a-c}\times{}_2F_1\left[a,a';c;\frac{4kq}{(\sigma-2k)(\sigma-2q)}\right]. \tag{13}$$

For our case $a=l+1-i\eta$, $a'=l+1$, $c=2l+2$, and $\sigma=(k+q-i\epsilon)$. Thus we have

$$I_l(k,q)=i^{-c}\Gamma(2l+2)\lim_{\epsilon\to 0}(q-k-i\epsilon)^{-l-1+i\eta}(k-q-i\epsilon)^{-l-1}(k+q-i\epsilon)^{-i\eta}\,{}_2F_1\left[l+1-i\eta,l+1;2l+2;-\frac{4kq}{(q-k)^2+\epsilon^2}\right]. \tag{14}$$

In Eq. (14) we now make use of the transformation formulas[13]

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\,{}_2F_1\left[a,1-c+a;1-b+a;\frac{1}{z}\right]+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,{}_2F_1\left[b,1-c+b;1-a+b;\frac{1}{z}\right], \tag{15}$$

$${}_2F_1(a,b;c;z)=(1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z) \tag{16}$$

and substitute the resulting $I_l(k,q)$ in Eq. (8) to get

$$t_l^c(k,q,k^2)=\frac{\Gamma(l+1)}{i\pi q f^c(k)}\left[\frac{\Gamma(1+i\eta)}{\Gamma(l+1+i\eta)}\left[\frac{q+k}{q-k}\right]^{i\eta}{}_2F_1\left[-l,l+1;1-i\eta;-\frac{(q-k)^2}{4kq}\right]-\frac{\Gamma(1-i\eta)}{\Gamma(l+1-i\eta)}\left[\frac{q+k}{q-k}\right]^{-i\eta}{}_2F_1\left[-l,l+1;1+i\eta;-\frac{(q-k)^2}{4kq}\right]\right]. \tag{17}$$

Equation (17) is very convenient for calculating numerical values for the Coulomb half-off-shell t matrix since the hypergeometric functions appearing here are polynomials. In Sec. III we study the behavior of $t_l^c(k,q,k^2)$ as a function of q and l.

III. RESULTS AND DISCUSSION

Based on Eq. (17) we have calculated numerical results for $t_l^c(k,q,k^2)$ for $E_{\text{lab}}=10$ and 20 MeV for $l=0-10$. Our numbers refer to a repulsive Coulomb potential with $(2k\eta)^{-1}=28.8$ fm. This is the proton Bohr radius. The real and imaginary parts of the t matrix for $l=0$, 1, 2, and 3 are plotted in Figs. 1, 2, 3, and 4 as a function of q. Each figure consists of two parts. The upper part displays the results for $E_{\text{lab}}=10$ MeV and the lower part those for $E_{\text{lab}}=20$ MeV. The variation of real and imaginary parts of $t_l^c(k,q,k^2)$ are represented by solid and dashed curves. Each of the curves exhibits the characteristic discontinuity at the energy shell arising from the fact that the Coulomb potential distorts not only the scattered waves but also the incident plane wave.[8] This discontinuity is shown by two vertical dotted lines. Interestingly, the branches of $\text{Re}t_l^c(k,q,k^2)$ [or $\text{Im}t_l^c(k,q,k^2)$] corresponding to $q>k$ and $q<k$ approach the on-shell discontinuity with gradients of unequal magnitudes. This is in agreement with the observation of Kok *et al.*[14] Looking closely into our curves we see that besides the on-shell discontinuity, the $l=2$ and 3 tend to show singularities at small q values.

In Figs. 5 and 6 we have plotted $t_l^c(k,k/8,k^2)$ and $t_l^c(k,2k,k^2)$ as a function of the angular momentum l for $E_{\text{lab}}=10$ and 20 MeV, respectively. The t matrices $t_l^c(k,k/8,k^2)$ and $t_l^c(k,2k,k^2)$ behave very differently. For

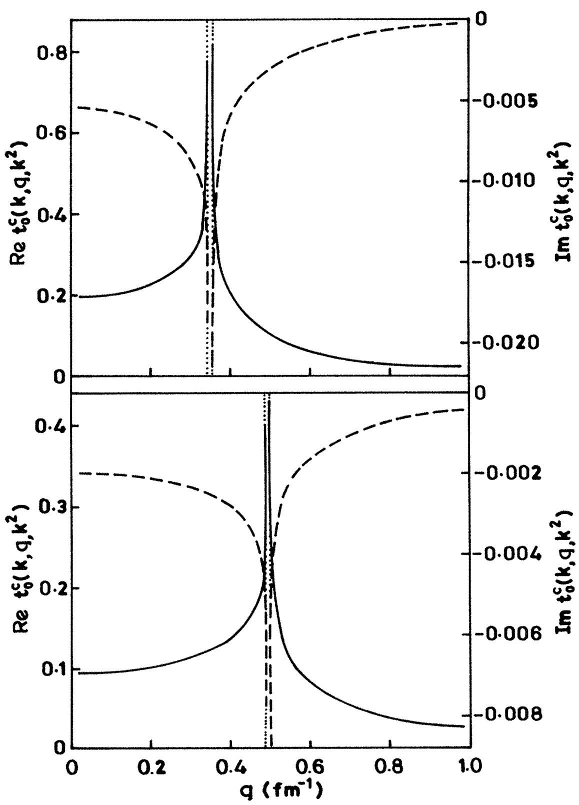

FIG. 1. s-wave Coulomb half-off-shell t matrix as a function of q. Upper part of the figure is for $E_{\text{lab}}=10$ MeV and lower part for $E_{\text{lab}}=20$ MeV. Solid and dashed lines represent the real and imaginary parts of t. This notation is followed in Figs. 2, 3, and 4 for p, d, and f waves.

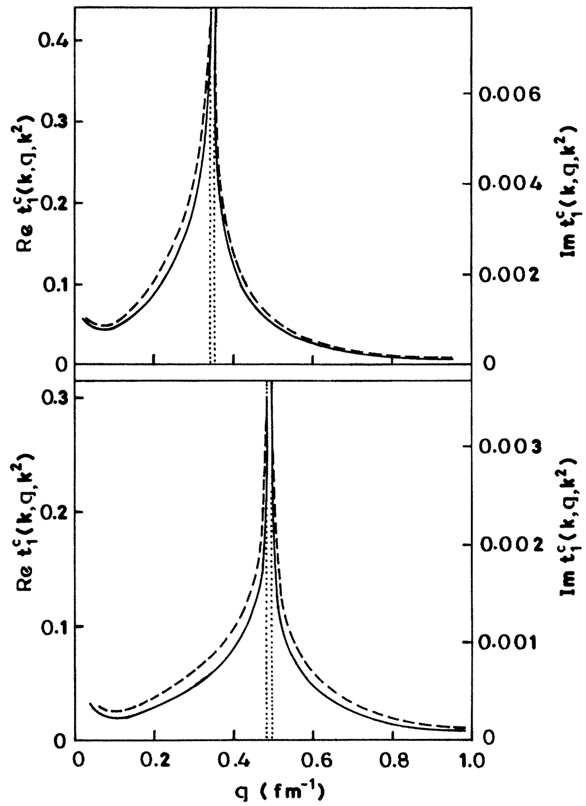

FIG. 2. p-wave t matrix.

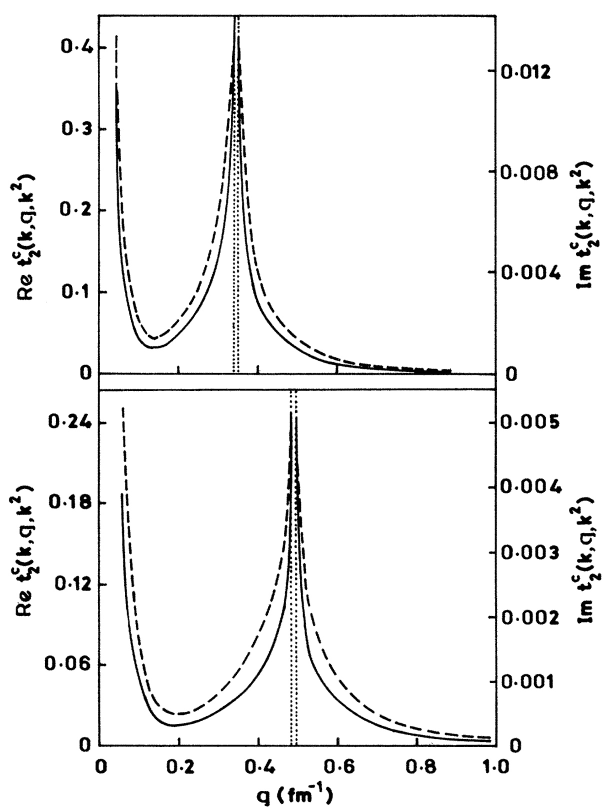

FIG. 3. d-wave t matrix.

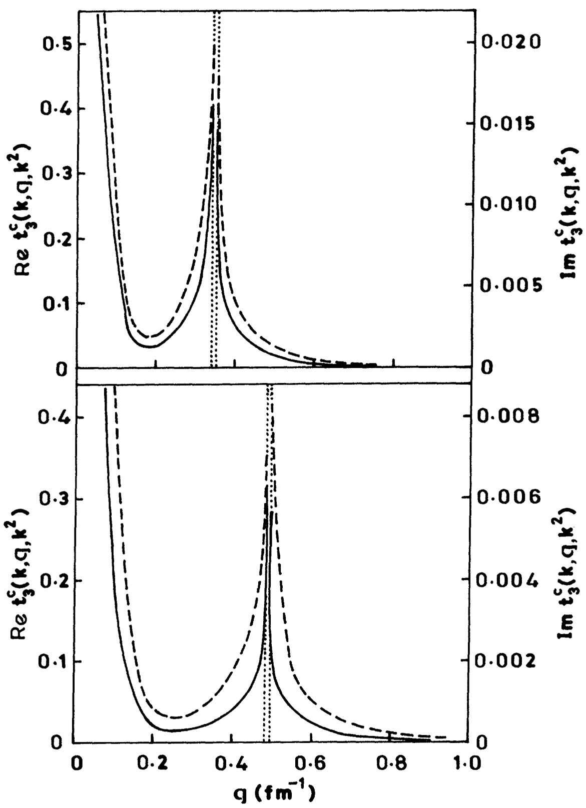

FIG. 4. f-wave t matrix.

example, $t_l^c(k,k/8,k^2)$ diverges logarithmically as l increases and we had to plot $\ln t_l^c(k,k/8,k^2)$ against l. The appropriate curves are denoted by set a with the ordinate

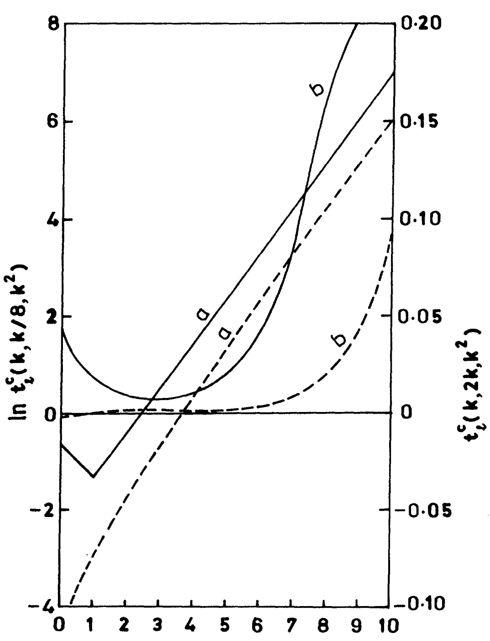

FIG. 5. Coulomb half-off-shell t matrix as a function of l. Set a represents the appropriate scaled real and imaginary parts of $t_l^c(k,k/8,k^2)$ for $E_{\text{lab}}=10$ MeV. Set b denotes similar variation for $t_l^c(k,2k,k^2)$ at the same on-shell energy.

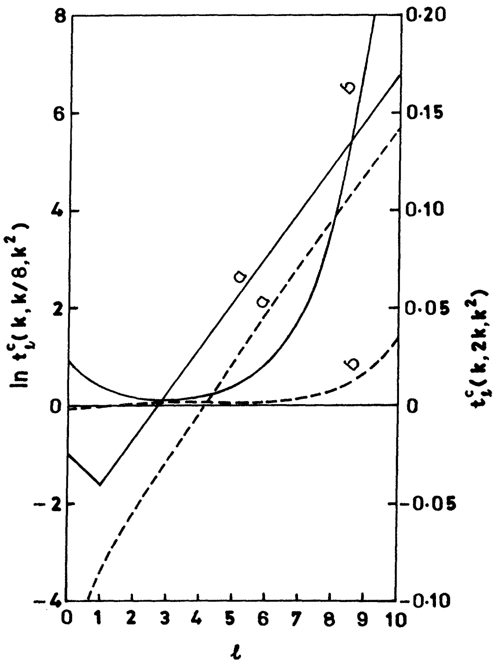

FIG. 6. Coulomb half-off-shell t matrix as a function of l for $E_{\text{lab}}=20$ MeV. Sets a and b have the same meaning as in Fig. 5.

shown in the left. As before, the solid and dashed lines stand for the real and imaginary parts of t. Save the anomalous behavior of $\ln \text{Re}t_l^c(k,k/8,k^2)$ at $l=1$, the gradient of both solid and dashed lines are quite appreciable. In contrast to this, the results for $t_l^c(k,2k,k^2)$ remain constant or slightly decrease in their values for few lower partial waves and then increases linearly. Set b represents the curves for $t_l^c(k,2k,k^2)$ with the ordinate in the right. In view of this we conclude the following.

Maximon[9] has suggested the use of Eq. (2) by making partial wave decomposition of the amplitudes which occur there. The on-shell discontinuity of the kernel K will present the first difficulty in calculating the Coulomb distorted amplitude. In addition to this there may be some problem arising out of the low-energy behavior of the integrand involved.

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APPENDIX

The off-shell Jost and physical (outgoing wave) solutions $f_l(k,q,r)$ and $\psi_l^{(+)}(k,q,r)$ satisfy[1,15] the differential equations

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-V(r)\right]f_l(k,q,r)=(k^2-q^2)e^{i\pi l/2}w_l^{(+)}(qr) \quad \text{(A1)}$$

and

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-V(r)\right]\psi_l^{(+)}(k,q,r)=(k^2-q^2)\hat{j}_l(qr) \tag{A2}$$

with the Riccati-Bessel function[16]

$$\hat{j}_l(qr)=\frac{1}{2i}[w_l^{(+)}(qr)-w_l^{(-)}(qr)] \ . \tag{A3}$$

The functions $w_l^{(+)}(qr)$ and $w_l^{(-)}(qr)$ are related by

$$w_l^{(-)}(qr)=[w_l^{(+)}(qr)]^* \ . \tag{A4}$$

Equations (A1) and (A2) hold good both for short-range and Coulomb-type potentials.[2] The full Green's functions for Eqs. (A1) and (A2) corresponding to irregular (Jost) and outgoing-wave boundary conditions are given by[17]

$$G_l^I(r,r')=-\frac{\phi_l(k,r)f_l(k,r')-\phi_l(k,r')f_l(k,r)}{\mathscr{F}_l(k)} \tag{A5}$$

and

$$G_l^{(+)}(r,r')=-\frac{\phi_l(k,r_<)f_l(k,r_>)}{\mathscr{F}_l(k)} \ . \tag{A6}$$

Here $\mathscr{F}_l(k)$ is related to the Jost function $f_l(k)$ as

$$\mathscr{F}_l(k)=-(2l+1)!!k^{-l}e^{i\pi l/2}f_l(k) \ . \tag{A7}$$

In Eqs. (A5) and (A6) the functions $\phi_l(k,r)$ and $f_l(k,r)$ are the regular and irregular solutions of the Schrödinger equation for our general potential $V(r)$. The particular solutions of Eqs. (A1) and (A2) in terms of the Green's functions in Eqs. (A5) and (A6) are the off-shell wave functions $f_l(k,q,r)$ and $\psi_l^{(+)}(k,q,r)$, while the complementary functions of the complete primitive refer to appropriate solutions of the Schrödinger equation. Thus we write

$$f_l(k,q,r)=\frac{(k^2-q^2)e^{i\pi l/2}}{\mathscr{F}_l(k)}\left[f_l(k,r)\int_r^\infty \phi_l(k,r')w_l^{(+)}(qr')\,dr'-\phi_l(k,r)\int_r^\infty f_l(k,r')w_l^{(+)}(qr')\,dr'\right] \tag{A8}$$

and

$$\psi_l^{(+)}(k,q,r)=\frac{(k^2-q^2)}{2i\mathscr{F}_l(k)}\left[f_l(k,r)\int_0^r \phi_l(k,r')[w_l^{(+)}(qr')-w_l^{(-)}(qr')]\,dr' +\phi_l(k,r)\int_r^\infty f_l(k,r')[w_l^{(+)}(qr')-w_l^{(-)}(qr')]\,dr'\right] . \tag{A9}$$

Equations (A8) and (A9) are quite formal and side remarks about their relations with other established results are in order. For example, the integral representation[18] of $f_l(k,q)$ can be obtained from (A8) as follows.

Irrespective of whether the potential has a simple pole or is analytic there, the off-shell Jost function is defined by

$$f_l(k,q)=\frac{q^l e^{-i\pi l/2}(2l+1)}{(2l+1)!!}\lim_{r\to 0} r^l f_l(k,q,r) \tag{A10}$$

in close analogy with the definition of the on-shell or ordinary Jost function[10] $f_l(k)$ in terms of $f_l(k,r)$. Combining Eqs. (A7), (A8), and (A10) we have

$$f_l(k,q)=\frac{(k^2-q^2)}{(2l+1)!!}q^l\int_0^\infty \phi_l(k,r)w_l^{(+)}(qr)\,dr \ . \tag{A11}$$

In Eq. (A11) we now use

$$k^2\phi_l(k,r)=-\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-V(r)\right]\phi_l(k,r) \tag{A12}$$

and integrate the resulting equation twice by parts to get Fuda's integral representation

$$f_l(k,q)=1+\frac{q^l}{(2l+1)!!}\int_0^\infty w_l^{(+)}(qr)V(r)\phi_l(k,r)\,dr \ . \tag{A13}$$

In deriving Eq. (A13) we have used the fact that[10]

$$\lim_{r\to 0} r^{-l-1}\phi_l(k,r)=1 \tag{A14}$$

and[16]

$$w_l^{(+)}(x)\underset{x\to 0}{\sim} x^{-l}(2l-1)!! \ . \tag{A15}$$

Let us now return to the main topic of this Appendix. Using Eq. (A8) we can write

$$\frac{(k^2-q^2)}{2i\mathscr{F}_l(k)}\phi_l(k,r)\int_r^\infty f_l(k,r')[w_l^{(+)}(qr')-w_l^{(-)}(qr')]\,dr' =\frac{(k^2-q^2)}{2i\mathscr{F}_l(k)}f_l(k,r)\int_r^\infty \phi_l(k,r')[w_l^{(+)}(qr')-w_l^{(-)}(qr')]\,dr'-\frac{1}{2i}[e^{-i\pi l/2}f_l(k,q,r)-e^{i\pi l/2}f_l(k,-q,r)] \ . \tag{A16}$$

We now replace the second integral in Eq. (A9) by Eq. (A16) and arrive at

$$\psi_l^{(+)}(k,q,r)=-\frac{1}{2if_l(k)}\left[\frac{k}{q}\right]^l\left[\frac{(k^2-q^2)q^l}{(2l+1)!!}g_l(k,q)\right]e^{-i\pi l/2}f_l(k,r)+\frac{1}{2i}[e^{-i\pi l/2}f_l(k,q,r)-e^{i\pi l/2}f_l(k,-q,r)]\ , \tag{A17}$$

where

$$g_l(k,q)=\int_0^\infty \phi_l(k,r)[w_l^{(+)}(qr)-w_l^{(-)}(qr)]\,dr\ . \tag{A18}$$

Because of Eqs. (A11) and (A18) the term inside the large parentheses in Eq. (A17) can be written in terms of off-shell Jost functions and we have

$$\psi_l^{(+)}(k,q,r)=-\tfrac{1}{2}\pi q\left[\left[\frac{k}{q}\right]^l\frac{f_l(k,q)-f_l(k,q)}{i\pi q f_l(k)}\right]e^{-i\pi l/2}f_l(k,r)+\frac{1}{2i}[e^{-i\pi l/2}f_l(k,q,r)-e^{i\pi l/2}f_l(k-q,r)]\ . \tag{A19}$$

The term inside the large square brackets of Eq. (A19) stands for the half-off-shell t matrix. Thus we see that in close analogy with the work of Fuda and Whiting[1] for the short-range potentials, the half-off-shell t matrix for the Coulomb and Coulomb-type potentials can also be expressed directly in terms of the Jost function. It is of interest to note that the derivation of Ref. 1 is based on a peculiar attention[19,20] to the asymptotic behavior of $\psi_l^{(+)}(k,q,r)$. In the above we could circumvent this by the use of full Green's functions.

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