Quantum dynamics of a nonintegrable system

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The quantum motion of a periodically kicked rotator is shown to be related to Anderson's problem of motion of a quantum particle in a one-dimensional lattice in the presence of a static-random potential. Classically, the first problem is nonintegrable and, for certain values of the parameters, exhibits chaos and diffusion in phase space; in the second problem, diffusion takes place in configuration space. Quantum phase interference, however, is known to suppress diffusion in Anderson's problem and to produce quasiperiodic motion. By establishing a mapping between the two systems we show that a similar effect determines the dynamics of the quantum rotator. As a result its wave functions are localized in phase space and their time evolution is quasiperiodic. This result explains the quantum recurrences and boundedness of the energy found in recent numerical work.

I. INTRODUCTION

In recent years the study of the effects of quantization on the properties of classically nonintegrable systems has attracted increasing attention.[1] The understanding of the nature of quantum behavior of these systems is not only of fundamental importance but it is also a problem of experimental relevance in fields as diverse as photochemistry,[2] electron dynamics in microstructures,[3] and other contexts. Understanding the relation between quantum problems and their classical limit may also shed light on the zero wavelength approximation to other wave equations, for example the eikonal approximation to the magnetohydrodynamic equations that are of great interest in plasma physics.[4] In the context of applications to photochemistry various model systems described by time-independent Hamiltonians have been recently studied.[5–7] Some of the evidence from these studies suggests that changes in the quantum behavior may take place when the corresponding classical system undergoes a transition to chaos. It is still unclear, however, whether these changes are only quantitative or if there is a sharp qualitative change in the nature of the long-time behavior as is the case in the corresponding classical system. Considerable effort[8–11] has also been made in order to develop semiclassical quantization rules for systems that are chaotic in their classical limit. These calculations involve many mathematical subtleties and their physical conclusions are still unclear to us.

As with the classical case, much insight into the properties of quantum nonlinear systems can be obtained from the study of simple maps. These are recursion relations defining the coordinates and momenta of the system at discrete time steps. The most studied area-preserving map is perhaps the Chirikov or standard map.[12,13] This map can be generated from a Hamiltonian that describes a planar rotator kicked at regular time intervals with a position-dependent force. It depends classically on a single parameter K, the dimensionless strength of the kick. For each value of K the motion is chaotic or periodic depending on the initial conditions. For small K the chaotic regions are isolated and are separated by Kolmogorov-Arnol'd-Moser (KAM) trajectories[12,13] and consequently the motion is bounded. For $K=K_c\simeq 0.97164$ the last of these trajectories disappears[13] and for $K>K_c$ diffusion in p space takes place, namely, $p^2\propto n$, for large n.

Starting with the early work of Casati *et al.*[14] the quantized version of this map has been studied by several authors.[14–17] These investigations showed drastic differences between the classical and quantum motions. In particular, Chirikov *et al.*[15] found that (except in the special case of the quantum resonances[16]) the energy remains bounded and does not increase with time even for $K>K_c$. Later, Hogg and Huberman[17] showed numerically that the energy is quasiperiodic in time. This is in sharp contrast with the behavior in the classical case as described above.

Surprising as it is, this situation is quite reminiscent of the one encountered when studying a seemingly unrelated problem in solid-state physics, that of finding the motion of wave packets of electrons in a random lattice. The physics of this system is, by now, well understood. It has been known for a long time[18–20] that, in one dimension, all the eigenfunctions of a one-electron random Hamiltonian are exponentially localized in space and, consequently, are normalizable. As a result the electronic diffusion coefficient and the electron mobility vanish at zero temperature. The energy spectrum is pure point. It is, however, dense, so that in the thermodynamic limit the total density of states is continuous. Nevertheless, the local density of states, i.e., the density of states weighted by the wave function, is discrete. As a result, an electron initially localized around a point in the lattice undergoes quasiperiodic motion. These are pure quantum effects caused by destructive interference: If the electron were classical it would diffuse through the lattice.

It appears therefore natural to ask whether quantum interference effects can also destroy classical diffusion in *momentum* space and produce quasiperiodic motion. In

this paper we will present evidence that the answer to this question is affirmative. We show that each of the two problems, the quantum rotator in a periodic time-dependent external field and a quantum particle moving in a static aperiodic potential, can be mapped into the other. By use of this mapping we show that the quantum dynamical system is, in effect, localized in angular momentum space and can therefore only reach a limited number of momentum states in the course of its time evolution. This in turn implies quasiperiodicity and thus boundedness and recurrence of the energy in time.[14–17] This is of course very different from the recurrence one would obtain ignoring these interference effects and considering the motion as an unrestricted random walk in momentum space.[21]

The organization of the rest of this paper is as follows. In Sec. II we introduce the quantum dynamical model studied in this work. The connection between it and the problem of electronic conduction in one-dimensional lattices is derived in Sec. III. In Sec. IV we exploit this connection to discuss the dynamics of the quantum system and present the results of numerical calculations. This is followed by a summary and concluding remarks in Sec. V. A preliminary account of part of this work was reported earlier.[22]

II. THE MODEL

In this section the specific quantum dynamical model to be used in the rest of the paper will be defined. We study the quantum motion of a system defined by the Hamiltonian

$$H=-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2}+\hat{k}V(\theta)\Delta(t)\ . \tag{2.1}$$

This is the Hamiltonian of a planar rotator with moment of inertia I driven by a time-dependent potential assumed to be factorizable into angle- and time-dependent parts. We shall assume that V is a periodic function of θ with period 2π and Δ has period t_0 (and dimensions of inverse time). Choosing t_0 as the unit of time the Hamiltonian depends on two dimensionless parameters $\tau=\hbar t_0/I$ and $k=\hat{k}/\hbar$. In these units the time-dependent Schrödinger equation is

$$i\frac{\partial}{\partial t}\psi(\theta,t)=\left[-\frac{1}{2}\tau\frac{\partial^2}{\partial\theta^2}+kV(\theta)\Delta(t)\right]\psi(\theta,t)\ . \tag{2.2}$$

One can readily generalize this class of Hamiltonians by replacing the kinetic energy $-\frac{1}{2}\tau\partial^2/\partial\theta^2$, by a general function $K(p)$ where $p=-i\partial/\partial\theta$ and deal with the operators

$$H=K(p)+kV(\theta)\Delta(t)\ . \tag{2.3}$$

A simple case is that in which $\Delta(t)$ is a sequence of δ-function kicks

$$\Delta(t)=\sum_{n=-\infty}^{\infty}\delta(t-n)\ . \tag{2.4}$$

The reason that (2.4) is a particularly simple choice is that, between kicks, the time evolution is that of free rotation. At the kicks, the force is so strong that the kinetic energy is unimportant and Eq. (2.2) is readily integrated. The time evolution of the wave function between kicks is most easily expressed in the angular momentum representation. We call the wave function in this representation $\psi_n(t)$, where $p\psi_n=n\psi_n$. The relation of the angular momentum representation to the angle representation, $\psi(\theta,t)$ is $\psi(\theta,t)=\sum e^{in\theta}\psi_n(t)$. Since the time dependence between kicks is trivial, it is only necessary to consider times infinitesimally before or after the tth kick. We thus regard t as an integer and use the subscript $\mp$ to mean before or after the kick. We also absorb k into the definition of V.

The free propagation is thus represented by

$$\psi_n^-(t+1)=e^{-iK(n)}\psi_n^+(t)\ . \tag{2.5}$$

Direct integration of the Schrödinger equation over a kick gives

$$\psi^+(\theta,t)=e^{-iV(\theta)}\psi^-(\theta,t)\ . \tag{2.6}$$

Equations (2.5) and (2.6) are easily combined to obtain

$$\psi_m^+(t+1)=\sum_{n=-\infty}^{\infty}J_{m-n}\exp(-in^2\tau/2)\psi_n^+(t)\ , \tag{2.7}$$

where

$$J_{m-n}=(2\pi)^{-1}\int_0^{2\pi}d\theta\, e^{i(m-n)\theta}e^{-iV(\theta)}\ . \tag{2.8}$$

Note that the matrix J depends only on the difference $m-n$. This feature generally is lost for more general free Hamiltonians than $K(p)$. The recursion relations (2.7) can be solved numerically for various initial conditions. In order that this be feasible, it is necessary that J_r become suitably small for large $|r|$. This in turn requires that $V(\theta)$ be sufficiently smooth and small. The notation of Eq. (2.8) was chosen because J is a generalization of the Bessel function. Indeed, in the best studied case,[14–17] $V=k\cos\theta$, it is essentially just a Bessel function of the first kind.

III. MAPPING OF THE DYNAMICAL MODEL

In this section we will show that the problem defined by (2.7) is equivalent to a tight binding model for electronic conduction known as the Anderson model. Since the Hamiltonian is periodic in time, namely, $H(t)=H(t+1)$, we can classify the solutions of (2.7) according to the way the wave function transforms under translations in time. This leads to the introduction of a new quantum number, the quasienergy,[23] which is the only good quantum number in this problem. The states of fixed quasienergy ω have the form

$$\psi_\omega(\theta,t)=e^{-i\omega t}u_\omega(\theta,t)\ , \tag{3.1}$$

where $u_\omega(\theta,t)=u_\omega(\theta,t+1)$, and we have momentarily reverted to continuous time. Notice that this is an analog of the well known Bloch-Floquet theorem familiar in the case where potential is periodic in space. States of different quasienergies are orthogonal.[23] It is also believed that they form a complete set, but we know of no rigorous proof. In what follows we will expand arbitrary functions

in quasienergy states. Due to the periodicity of the quasienergy states in time it is sufficient to study these states just before or after the kick, which we denote as before by $u^{\mp}$. We suppress, except when necessary, the dependence of the wave functions on ω. Again we use both the angle and angular momentum representations. Substitution of Eq. (3.1) into Eqs. (2.6) and (2.7) with t integer gives

$$u_m^- = e^{i\omega} e^{iK(m)} u_m^+ , \tag{3.2}$$

$$u^+(\theta) = e^{-iV(\theta)} u^-(\theta) , \tag{3.3}$$

and

$$u_m^+ = e^{i\omega} \sum J_{m-n} e^{iK(n)} u_n^+ . \tag{3.4}$$

The transformation we wish to make uses an alternative representation of the operator V in terms of an Hermitian operator W, namely, we define

$$e^{-iV(\theta)} = \frac{1+iW(\theta)}{1-iW(\theta)} \tag{3.5}$$

or equivalently

$$W(\theta) \equiv -\tan[V(\theta)/2] . \tag{3.6}$$

Defining

$$\bar{u}(\theta) = [u^+(\theta) + u^-(\theta)]/2 \tag{3.7}$$

we reformulate (3.3) as

$$\frac{u^+(\theta)}{1+iW(\theta)} = \bar{u}(\theta) = \frac{u^-(\theta)}{1-iW(\theta)} . \tag{3.8}$$

Substitution of (3.8) in (3.4) yields the equations for the Fourier components

$$u_m + i \sum W_{m-r} u_r = \left[u_m - i \sum W_{m-r} u_r \right] e^{-iE_m} , \tag{3.9}$$

where u_m (written without the bar for convenience) is the angular momentum representation of $\bar{u}(\theta)$. Here W_m is the Fourier transform of $W(\theta)$, and $E_m = \omega - K(m)$. Finally, (3.9) can be rewritten in the form

$$T_m u_m + \sum_{r(\neq 0)} W_r u_{m+r} = E u_m \tag{3.10}$$

with

$$T_m = \tan\left[\frac{E_m}{2}\right] = \tan\left[\frac{\omega - K(m)}{2}\right] \tag{3.11}$$

and

$$E = -W_0 . \tag{3.12}$$

Equation (3.10) describes a one-dimensional tight-binding model, with hopping W_r to the rth neighbor and "diagonal" potential T_m. This equation establishes therefore the correspondence between the quantum dynamical problem and the solid-state problem with the angular momentum in the quantum problem corresponding to the lattice sites in the solid-state problem. We will refer to (3.4) as the quantum dynamical problem or the rotator problem while (3.10) we will refer to as the tight-binding problem or the lattice problem.

In the tight-binding problem Eq. (3.10) is an eigenvalue equation for the energy E. In contrast, in the dynamical problem, E is a fixed parameter determined by the perturbation V [cf. (3.12) and (3.6)]. The quasienergy, on the other hand, which in the lattice problem simply determines which of several more or less equivalent sequences T_m is used, is the eigenvalue in the rotator problem. However, the relation between them is simple. Indeed, by applying the Feynman-Hellman theorem to Eq. (3.10) we find that $dE/d\omega = \sum \langle \sec^2 E_m \rangle /2$ is positive implying that E is a monotonic function of ω. Second, eigenvalues $E_\nu(\omega)$ belonging to distinct states ν will not become degenerate as ω is varied because the low symmetry of the system will, in general, prevent level crossing. Thus the functions $E_\nu(\omega)$ giving the lattice eigenvalues for fixed ω can in principle be inverted to give $\omega_\nu(E)$, the rotator quasienergies for fixed E, i.e., for fixed $V(\theta)$, and vice versa. The wave functions are, of course, essentially the same in the two pictures. The character of the spectrum is the same, as well.

The properties of the solutions of the lattice problem depend on the nature of the sequence T_m. Three cases are of great interest. The simplest of them occurs if T_m is periodic in m. Then the corresponding eigenstates are Bloch states. These states are extended and unnormalizable giving rise to electronic propagation, thus diffusion and conductivity in solids. In the dynamical problem this case corresponds to the quantum resonances.[16] Next in complexity is the case in which T_m is periodic, but with a period incommensurate with the (unit) lattice period. We discuss this case in detail elsewhere.[24] Finally, if $\{T_m\}$ constitutes a random sequence, each of its elements being chosen independently from a given fixed distribution, Anderson's model of localization in a one-dimensional random potential is obtained.

The properties of the solutions in the random case follow from rigorous results on the asymptotic behavior of products of random matrices.[25] It is known[18-20] that all the eigenstates of (3.10) are localized around some lattice site and decay exponentially away from that site with a characteristic length $\gamma^{-1}(E)$ which is solely determined by the probability distribution of the potential. Eigenstates with nearly identical energies are, however, generally localized around centers which are far apart. Conversely, two eigenstates localized at centers close compared with γ^{-1} will typically have eigenenergies separated by a finite energy spacing which is of order $\gamma \langle | W_r | \rangle$. The local density of states (i.e., as weighted by the square of the wave function) is discrete and, at a given site n, it consists of about γ^{-1} δ-function peaks at certain energies ϵ_n. The existence of a discrete local spectrum implies quasiperiodicity of the motion and absence of diffusion.

One can certainly arrive at the Anderson model by choosing for $\{K_m\}$ a random sequence. However, there is yet another possibility that, to our knowledge, has not been studied before in any detail and is of importance to understand the dynamics of the quantum rotator, namely, the one in which the sequence $\{K_m\}$ is *pseudorandom*, i.e., $\{K_m\}$ has some but not all the properties of a truly random sequence. This is of relevance for the problem at hand because, as will be seen shortly, the sequence

$K_m = \alpha m^2 \pmod 1$ that corresponds to the rotator problem is, indeed, pseudorandom.

As an example of a pseudorandom sequence consider the case of $\{K_m = \alpha 2^m \pmod 1\}$ where α is a given irrational number. This sequence reflects the random properties of the sequence of zeros and ones that occurs in the binary representation of α. It can be easily shown that, if α is irrational, the elements of $\{K_m\}$ are uniformly distributed in [0,1]. They are not independent, however, as the correlation function $\langle K_m K_{m+r}\rangle \cong 2^{-r}$ in this case. The existence of correlations of this type is a main difference between random and pseudorandom sequences. Little is known about the effect of these correlations on localization because the theorems mentioned above apply only to the case of independent random variables. While independence is certainly a sufficient condition for localization in one dimension, it is unlikely that it is necessary. For example, it is obvious on physical grounds that the fundamental results should still remain valid in the presence of weak, short-range correlations between the elements of $\{T_m\}$. The existence of localized states in a related quasiperiodic potential[24] can be taken as an indication that the conditions as stated in the relevant theorems[25] are too restrictive. These considerations are important because the numerical evidence that we present in Sec. III indicates that the sequence $\{T_m\}$ is "random enough" to localize all the solutions of Eq. (3.10).

The only rigorous result known to us which bears on this question has been provided recently by Bellissard[26] who shows that if W_r is of short range and small enough, and K_m is of the form $\tau(m+k)^2$, then for all k $(0<k<1)$ in a certain (Cantor) set C of measure approaching unity as W approaches zero, the spectrum of (3.10) is pure point, and the wave functions are exponentially localized. The theorem says nothing for larger W's or for k not in C, in particular for our case, $k=0$. Bellissard conjectures that the spectrum in this case is singular continuous, or, at least, has a singular continuous component. It is almost surely true, however, that there is no absolutely continuous part of the spectrum. Although the mathematical problems posed are interesting and it is important that these issues be cleared up we expect, from a physical point of view, that it will be next to impossible to detect, in this case, any singular continuous part of the spectrum if finite size, finite time, finite temperature, dissipation, small random deviations from the m^2 law, etc., are taken into account. The main reason for this belief is that we always find numerically that the wave function drops off by 20 or 30 powers of e with no sign of comeback at large distances from the ostensible center. A singular continuous spectrum would imply that the apparent δ-function spikes in the local density of states (see Sec. IV) are really almost infinitely degenerate with the largest splittings of relative order e^{-30} or less. It must be a good approximation to neglect this if any other source of broadening is present. The numerical results may be an artifact of roundoff error, but this error probably mimics crudely the finite temperature effects mentioned above. In other words, we believe that the computer version of the model is more in accord with possible real systems than the idealized model is. Further, Bellissard's results provide evidence for this view, since with the finite smearing effects taken into account, the physical results surely cannot depend sensitively on whether k is zero or some small number in the set C.

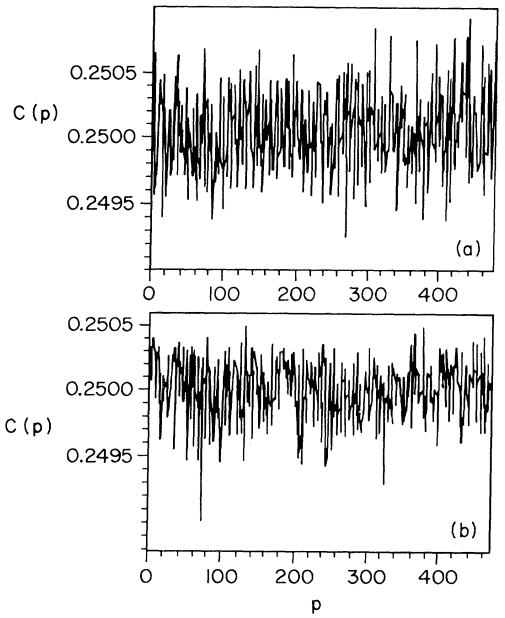

FIG. 1. Power spectrum of (a) the sequence $K_n = \sqrt{5} n^2 \pmod 1$ and (b) a sequence of random numbers with uniform distribution in [0,1].

IV. THE PSEUDO ANDERSON MODEL

We start this section by examining some statistical properties of $\{K_m = \tau m^2 \pmod 1\}$ for irrational τ. This sequence is ergodic with uniform distribution in the interval [0,1]. This is a rigorous result of a theorem by Weyl.[27] We have checked numerically that, for finite sequences of N elements, the deviations from uniformity follow the large-number behavior of random sequences, i.e., the fluctuation ΔN in the number of elements of the sequence that fall on any interval of fixed length contained in [0,1] is $\Delta N \cong \sqrt{N}$. The pair-correlation function is defined as

$$C(r) = \frac{1}{N} \sum_{m=1}^{N} K_m K_{m+r} \,. \tag{4.1}$$

Its Fourier transform,

$$C(p) = \sum_{r=1}^{N} e^{-i(2\pi r/N)p} C(r) \,, \tag{4.2}$$

is the power spectrum of the sequence. In Fig. 1(a) we plot $C(p)$ for a sequence of about 10^5 elements with $\tau = \sqrt{5}$. For comparison we show in Fig. 1(b) a plot of the Fourier transform of Eq. (4.1) with K_m replaced by "random" numbers as generated by a standard computer algorithm The general structure of the two plots is quite similar, the most important feature being the absence of correlations at any particular length scale. The statistical prediction for random numbers is that $C(p)$ is, on the average, independent of p with fluctuations of size $\sqrt{N}$. We observe this behavior in both the random and pseudoran-

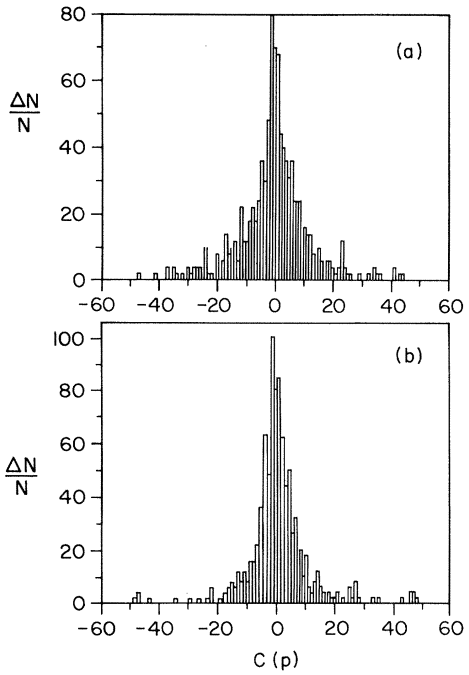

FIG. 2. Distribution of pair correlations for the potential $T_m=\tan(x_n)$ with (a) $x_n=\sqrt{5}n^2$ and (b) $\{x_n\}$ a sequence of random numbers with uniform distribution in [0,1].

dom sequences. Notice, however, that the peaks in the pseudorandom case are more numerous than those of the random one which shows that there are more correlations in the former case.

The statistical properties of $\{T_m\}$ follow from those of $\{K_m\}$. In particular, since $\{K_m\}$ is uniformly distributed, then $\{T_m\}$ will follow the Cauchy or Lorentzian distribution:

$$P(T_m)=\frac{1}{\pi}\frac{1}{1+T_m^2}\ . \tag{4.3}$$

The correlation function for the potential can be computed by an expression like Eq. (4.1). In Fig. 2(a) we plot the distribution of the values taken by the correlation function for the same sample as used before. Figure 2(b) shows the same function in the case in which the argument of the tangent is a random number. In one sense both plots are similar: the correlations are of statistical nature (as opposed to systematic) and the widths of the distributions follow the law of large numbers for large sequences. The distribution function is, however, wider in the pseudorandom case, which shows again the presence of larger correlations.

If one is now willing to make the assumption that this degree of randomness is sufficient to localize all the solutions of Eq. (3.10) then, by virtue of the correspondence between the two problems established in Sec. III, the following picture of the quasienergy eigenstates emerges. For a given potential $V(\theta)$ [fixed E and W_r in Eq. (3.10)], each of the solutions of Eq. (2.12) is localized around some value of the angular momentum. Away from the center of localization, the solutions decay exponentially with an exponent $\gamma(E)$ which is independent of both the

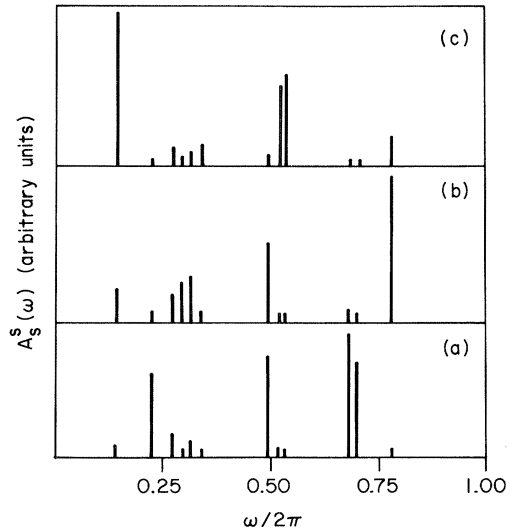

FIG. 3. Local quasienergy spectra as obtained from the time evolution of states with initial angular momenta (a) $l=46$, (b) $l=48$, and (c) $l=50$. The potential is the one shown in Eq. (4.8) and the parameters are $\kappa=2.8$, $\tau=4.867$.

values of the quasienergy and of τ and is uniquely determined by $V(\theta)$. In contrast, the center of localization depends upon the quasienergy. This dependence is not smooth, with states belonging to nearby quasienergies being centered, in general, around angular momenta that are far apart and states centered at nearby angular momenta having, in general, quite different quasienergies.

To see the effect of these properties on the time evolution of the quantum rotator, consider the expansion of the time-dependent wave function in terms of the quasienergy eigenstates. We have

$$\psi_n^+(t)=\sum_\nu C_\nu u_{n\nu}^+ e^{-i\omega_\nu t}\ . \tag{4.4}$$

The expansion coefficients are determined by the initial conditions. If, for simplicity, we start with an eigenstate of the angular momentum, $\psi_n^+(0)=\delta_{ns}$, the time-

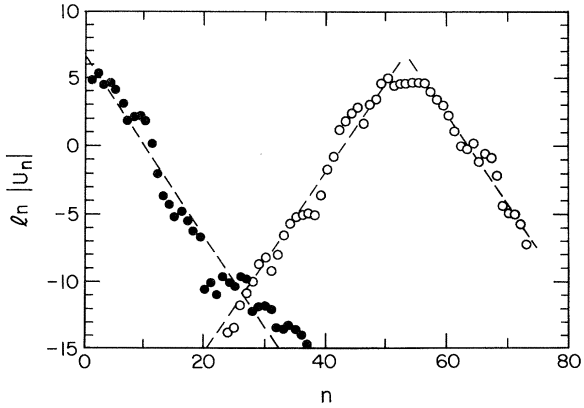

FIG. 4. Two quasienergy eigenstates for the same potential as in the previous plot. Quasienergies are $\omega=2\pi j/2^{10}$ with $j=323$ (solid circles) and $j=621$ (open circles).

dependent wave function is

$$\psi_n^+(t)=\sum_\nu (u_{s\nu}^+)^* u_{n\nu}^+ e^{-i\omega_\nu t} \tag{4.5}$$

and its spectrum,

$$A_n^s(\omega)\equiv \int_{-\infty}^{\infty}\frac{dt}{2\pi}\psi_n^+(t)e^{i\omega t}$$

$$=\sum_\nu (u_{s\nu}^+)^* u_{n\nu}^+\delta(\omega-\omega_\nu)\ . \tag{4.6}$$

A quantity of particular importance is $A_s^s(\omega)$, the projection of ψ on the initial state. It is determined by the local density of quasienergy states,

$$A_s^s(\omega)=\sum |u_{s\nu}^+|^2\delta(\omega-\omega_\nu)\ . \tag{4.7}$$

Although the sum in Eq. (4.7) is over all quasienergies the exponential decay of the wave functions implies that only a few states will effectively contribute to the sum: those centered on angular momenta within a distance of order γ^{-1} from the initial state. Since there is one quasienergy state for each angular momentum state, $A_s^s(\omega)$ will consist of about γ^{-1} peaks. The structure of Eq. (4.7) implies that $A_s^s(t)$ is an almost-periodic function.[28] Thus the state vector returns infinitely often to any given neighborhood of the initial state during the course of its evolution. Similar considerations show that, if the localization picture is correct, the expectation values of all observables are almost periodic functions. Notice that in these arguments no use is made of any properties of $V(\theta)$ other than that it produces a W_r of finite range.

To support these ideas we present results of numerical calculations. They consist of direct iteration of Eq. (2.7) for two different potentials and various initial conditions. The most efficient method to solve these equations that we found takes advantage of the translation invariance of the kernel in Eq. (2.8) and uses a forward-backward fast Fourier transform (FFT) technique. In this way one can easily include up to about 10^3 angular momentum states. Once a time series for the state vector is obtained, the FFT can be used again to calculate $A_s^s(\omega)$. From the latter, the spectrum and the eigenstates may be computed from Eqs. (4.6) and (4.7). In applications we took 2^{10} time steps.

The two potentials that we considered are

$$V_L(\theta)=-2\arctan(\kappa\cos\theta-E) \tag{4.8}$$

and

$$V_c(\theta)=k\cos\theta\ . \tag{4.9}$$

$V_c(\theta)$ corresponds to the familiar standard map. $V_L(\theta)$ has no obvious motivation in the rotator problem. It has, however, the advantage that it corresponds to the simplest case of Eq. (3.10), namely, a tight-binding model with hopping limited to nearest neighbors [see Eq. (3.6)]. For a diagonal potential with Lorentzian distribution this is the so-called Lloyd model of disorder. Many exact results are known for this model. In particular, the localization exponent $\gamma(E)$ (Ref. 29) is

$$2\kappa\cosh\gamma(E)=[(E-\kappa)^2+1]^{1/2}+[(E+\kappa)^2+1]^{1/2}\ . \tag{4.10}$$

Since this provides a quantitative test of our ideas we will refer to this case in most of the remainder of this section.

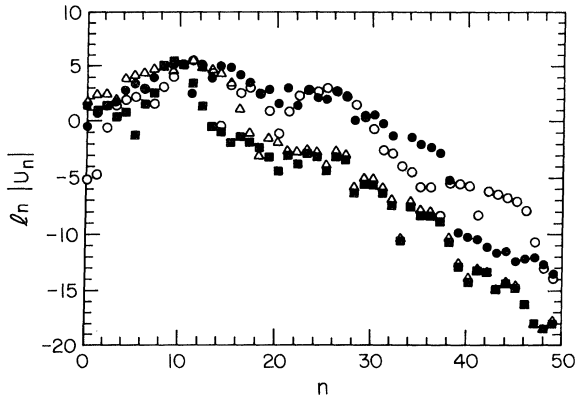

FIG. 5. Quasienergy eigenstates corresponding to $\omega=2\pi j/2^{10}$ with $j=50$, 325, 614, and 810. Parameters are the same as in Fig. 3.

Figure 3 shows three quasienergy spectra obtained from the time series for state vectors that start with well defined angular momentum $l=46$, 48, and 50, respectively. The parameters for this plot are $\kappa=2.8$, $E=0$, and $\tau=4.867$. The main features to notice are that, as predicted above, (i) only a few peaks contribute effectively to the local spectrum, (ii) the same quasienergies appear in all three cases, and (iii) the amplitudes are very sensitive functions of the initial conditions. Although the initial states are closely spaced in angular momentum there are large differences in the weights of the peaks. The inverse of the localization exponent for this example is $\gamma^{-1}(E)\sim 1.5$. Thus quasienergy states that peak at more than a few units from a given one, l (say), make a very small contribution to the evolution of a state that starts at l. Conversely, from the localization of the peak of largest amplitude in the spectrum we see that states with very different quasienergy may be centered at nearby angular momenta.

Figure 4 shows two typical quasienergy eigenstates obtained as described above. These states peak at angular momenta $l=0$ and 50 and rapidly decay away from their

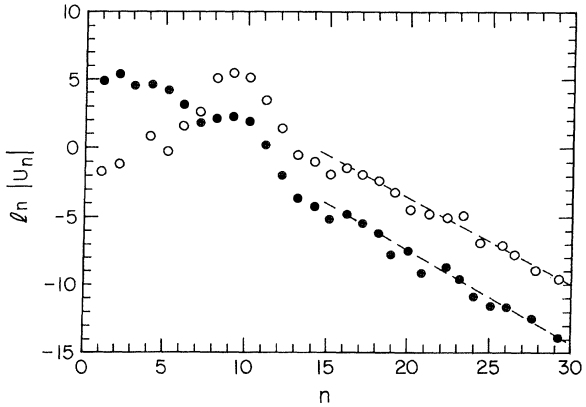

FIG. 6. Two nearby quasienergies with eigenfunctions centered at widely different angular momenta. $\omega=2\pi j/2^{10}$ and $j=323$ (solid circles) and $j=325$ (open circles).

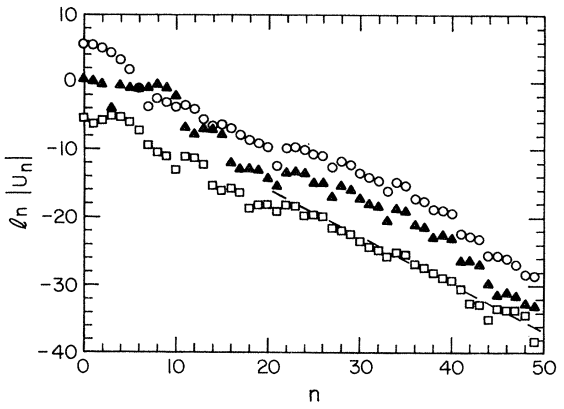

FIG. 7. A plot similar to Fig. 5 but with $\tau=0.1$. Other parameters are as before. $\omega=2\pi j/2^{10}$ with $j=298$, 741, and 858. Slope of these curves is the same as in Fig. 5.

centers. The decay is exponential as predicted. The dashed lines are the theoretical slopes calculated from Eq. (4.10). There is good agreement. Notice that the localization exponent is the same for both quasienergies. To further illustrate the independence of $\gamma(E)$ upon the quasienergy we plot in Fig. 5 four states all of which have appreciable weight at or near $l=10$. Their quasienergies are quite different but they all have the same long-distance behavior. The case of nearby quasienergies is shown in Fig. 6. We plot two eigenstates whose quasienergies differ by only 0.6%. Their centers are displaced by ten units of angular momentum. Thus, in general, the relation between quasienergy and center of localization is not smooth. This situation is familiar in the localization problem.

Results obtained for other values of τ are similar, provided it is not too small. Figure 7 shows three eigenstates corresponding to $\tau=0.1$, the rest of the parameters being fixed at their previous values. The states are localized with the same localization exponent that we found before. If τ is very small the short-distance behavior of the wave function may be considerably modified with respect to the one found for larger values. This is shown in Fig. 8 where we plot several states for $\tau=0.01$. The states are still localized and drop off rapidly as a function of distance but now we find relatively flat regions in between those in which the wave function decays exponentially. This can be understood by noticing that the origin of the pseudorandom behavior of $\{K_n\}$ lies in the discontinuities introduced by the operation of taking the modulum. If τ is small, however $\{K_n\}$ is smooth over distances of the order of the location of the first discontinuity. In the present example this occurs at $n_1\cong 30$. Thus the phase of the wave function is coherent over about n_1 sites before interference effects set in. This coherence shows up as a plateau in a logarithmic plot of the amplitude of the wave function. Similarly, there are additional plateaus over scales corresponding to the distances between successive discontinuities. The phase of the wave function is coherent within each such region, but the correlation between the overall phases in different regions is rapidly lost. As n increases the distance between discontinuities becomes smaller and smaller and hence the width of the plateaus decreases until, at large distances (of order $\tau^{-1/2}$), they disappear completely. These features are clearly seen in Fig. 8.

The usual standard map corresponds to the potential of Eq. (4.9). The associated hopping potential may be calculated in closed form for $k<\pi$. For our present purposes it is sufficient to note its form for long distances:

$$W_r=\begin{cases}0 & \text{for even r}\\ \dfrac{2}{\pi}\exp\left[-r\ln\left[\dfrac{2\pi}{k}\right]\right] & \text{for odd r}\,.\end{cases} \tag{4.11}$$

Thus the hopping potential is of finite range and we will find the same behavior as before. Since the localization exponent is unknown for this case we have no quantitative test of the theory. We have obtained numerical results for several values of k and τ. As an example we plot in Figs. 9 and 10 quasienergy spectra and wave functions computed from the time series for state vectors that start out at $l=0$ and 50, respectively. There is no qualitative

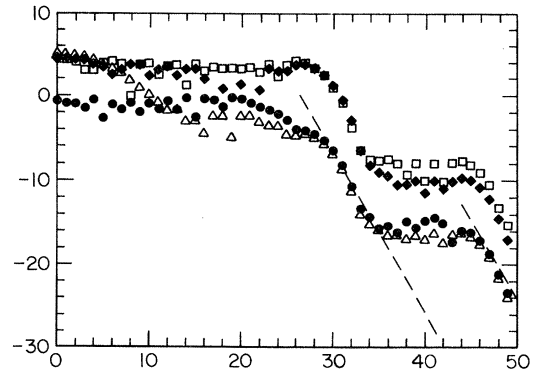

FIG. 8. A plot equivalent to Fig. 7 corresponding to $\tau=0.01$. Notice the presence of plateaus of decreasing extension in between regions of exponential decay.

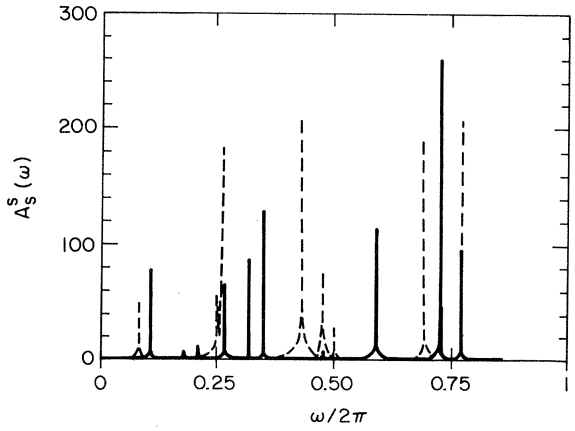

FIG. 9. Quasienergy spectra for the potential of the standard map, Eq. (4.9) for states starting with $l=0$ (dashed line) and $l=50$ (solid line). Parameters are $k=2.8$ and $\tau=4.867$.

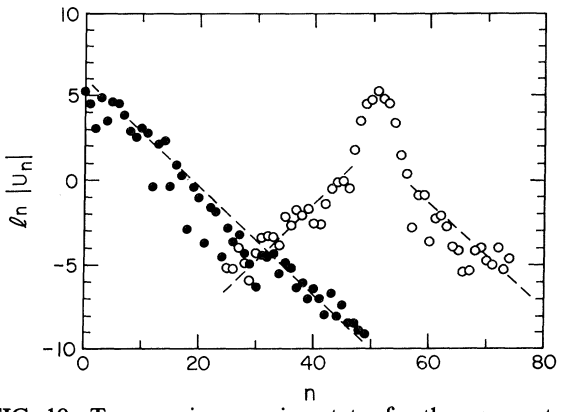

FIG. 10. Two quasienergy eigenstates for the same potential as in Fig. 9. Quasienergies are $\omega=2\pi j/2^{10}$ with $j=511$ (solid circles) and $j=709$ (open circles).

difference between these and our previous results. The same is true for all other sets of parameters as well.

If $\tau/2\pi$ is a rational, p/q, the system is periodic, since $T_{m+q}=T_m$. However, if $q\gamma \gg 1$, the wave function will not realize that the potential is periodic until it has already become very small. The bands in this case can be estimated by the following procedure. Find approximate localized eigenstates, v_m^ν centered at site ν, $\nu=1,\ldots,q$, say, by replacing p/q by a nearby irrational. Then the wave function will be approximated by

$$u_m^{k\nu}=\sum_{-\infty}^{\infty} e^{irqk}v_{m-rq}^{\nu} . \tag{4.12}$$

Here k is the continuous crystal momentum, $-\pi/q<k\leq\pi/q$, and ν is the band index. The energy can be found by taking the expectation of H in this state. The bandwidth will be of order $e^{-\gamma q}$, the spacing between bands of order $1/q$.

V. CONCLUSIONS

We have presented evidence that the motion of the periodically pinged rotator is bounded and almost periodic in phase space if the natural frequency of the system is not rotationally related to that of the external field. The qualitative features of the motion are independent of the detailed form of the pinging potential as long as it is sufficiently smooth. These results are a consequence of pure quantum interference effects with no counterpart in classical mechanics. A mapping of this problem into the Anderson problem of localization of electronic states in random lattices allowed us to identify the mechanism responsible for the absence of diffusion and recurrence phenomena reported earlier in numerical work.[14–17] We have also presented numerical evidence in support of the ideas put forward in this paper.

Several generalizations of this work are possible. Since the effective randomness in $\{T_m\}$ is produced by the operation of taking the modulus in Eq. (3.11), it is clear that all Hamiltonians $K(p)$ with K an increasing function of p will be associated with pseudorandom diagonal potentials in the lattice representation. We have shown elsewhere[24] that even if $K(p)$ is linear the amount of randomness is enough to ensure localization. Strong nonlinearity in the unperturbed Hamiltonian can only improve upon the randomness of the sequence. If $V(\theta)$ is not smooth the associated hopping potential will have a long-range component. There are no exact results for localization in this case. However, it is believed[30] that $W_r\cong 1/r$ [i.e., a discontinuous $V(\theta)$] constitutes the dividing line between the cases of localized and extended solutions. The simplest Hamiltonian that produces long-range hopping is the quantized version of "Arnol'd's Cat."[31] We have performed[32] numerical calculations of the type reported here for this potential. Our preliminary results indicate that, indeed, the states are extended in this case and there is diffusion in phase space.

It is of interest to consider the case in which the unperturbed spectrum has a continuous component, for this situation is closer to the one present in experimental systems. In these cases either there is a finite number of bound states or the spectrum has an accumulation point at the dissociation threshold. An example of the latter class was recently studied in the classical limit.[33] If one confines the attention to the bound states the same formal manipulations that we used in Sec. II of this paper can be performed to get a new type of equivalent lattice problem. This turns out to be that of the motion of a quantum particle in a semi-infinite chain with a diagonal potential that is nonrandom at long distances from the "surface." Under these circumstances one expects richer behavior than that found here because now, depending on the parameters, the states can be either extended or localized in the "surface region." The existence of extended states would make diffusion possible but quantum effects (i.e., the presence of localized states) will most likely introduce corrections to the classical results. We do not know at this stage whether these corrections are of quantitative or qualitative nature.

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[1]For a review see, e.g., G. Zaslavsky, Phys. Rep. 80, 158 (1981).

[2]*Photoselective Chemisty,* edited by J. Jortner, R. D. Levine, and S. A. Rice, Vol. 47 of *Advances in Chemical Physics* (Wiley, New York, 1981).

[3]See, e.g., *VLSI Electronics, Microstructure Science,* edited by N. G. Eimspruch (Academic, New York, 1981), Vols. 1 and 2.

[4]E. Ott, in *Long Time Prediction in Dynamical Systems,* edited by C. W. Horton, Jr., L. E. Reichl, and V. B. Szebehely (Wiley, New York, 1983), p. 281.

[5]See the article by S. A. Rice in Ref. 11.

[6]M. Bixon and J. Jortner (unpublished).
[7]G. Hose and H. S. Taylor, J. Chem. Phys. 76, 5356 (1982).
[8]M. V. Berry, Lecture Notes of the NORDITA School on Chaos, June 1982 (unpublished).
[9]M. V. Berry, N. L. Balazs, M. Tabor, and A. Voros. Ann. Phys. 122, 26 (1979).
[10]M. Tabor, Physica (Utrecht) 6D, 195 (1983).
[11]M. Gutzwiller, J. Math. Phys. 11, 1791 (1970); 12, 343 (1971); Phys. Rev. Lett. 45, 150 (1980).
[12]B. V. Chirikov, Phys. Rep. 52, 263 (1979).
[13]J. M. Greene, J. Math. Phys. 20, 1183 (1979).
[14]G. Casati, B. V. Chirikov, F. M. Izrailev, and J. Ford in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems,* Vol. 93 of *Lecture Notes in Physics,* edited by G. Casati and J. Ford (Springer, Berlin, 1979).
[15]B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyansky, Sov. Sci. Rev. Sec. C 2, 209 (1981).
[16]F. M. Izrailev and D. L. Shepelyansky, Teor. Mat. Fiz. 43, 417 (1980) [Theor. Math Phys. 43, 553 (1980)]; Dok. Akad. Nauk SSSR 249, 1103 (1979) [Sov. Phys—Dokl. 24, 996, 1979)].
[17]T. Hogg and B.A. Huberman, Phys. Rev. Lett. 48, 711 (1982); Phys. Rev. A 28, 22 (1983).
[18]P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[19]P. W. Anderson, Rev. Mod. Phys. 50, 191 (1978).
[20]For a review, see, e.g., D. J. Thouless, in *Ill Condensed Matter, Proceedings of Les Houches Summer School,* edited by R. Balian, R. Maynard, and G. Toulouse (North-Holland, Amsterdam, 1979), Vol. 31.
[21]A. Peres, Phys. Rev. Lett. 49, 1118 (1982).
[22]S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. 49, 509 (1982).
[23]Ya. B. Zeldovich, Zh. Eksp. Teor. Fiz. 51, 1492 (1966) [Sov. Phys.—JETP 24, 1006 (1967)].
[24]D. R. Grempel, S. Fishman, and R. E. Prange, Phys. Rev. Lett. 49, 833 (1982); R. E. Prange, D. R. Grempel, and S. Fishman (unpublished).
[25]K. Ishii, Prog. Theor. Phys. Suppl. 53, 77 (1973).
[26]J. Bellissard (unpublished).
[27]I. P. Cornfeld, S. V. Forniu, and Ya. G. Sinai, *Ergodic Theory* (Springer, Berlin-Heidelberg-New York, 1982), Chap. 2.
[28]A. S. Besicovich, *Almost Periodic Functions* (Cambridge University, Cambridge, England, 1932).
[29]D. J. Thouless, J. Phys. C 5, 77 (1972).
[30]P. W. Anderson (private communication).
[31]J. H. Hannay and M. V. Berry, Physica (Utrecht) D1, 267 (1980).
[32]D. R. Grempel, S. Fishman, and R. E. Prange (unpublished).
[33]R. V. Jensen, Phys. Rev. Lett. 49, 1365 (1982).