# Dynamics of coherent states

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We apply displacement operator coherent-state (DOCS) methods to calculate coherent states of systems subject to the potential $V(x,t)=g_2(t)x^2+g_1(t)x$. We call states which are initially coherent and which preserve their coherence with time nondispersive, in contrast to those coherent states which lose their coherence which we denote as dispersive or dissipative. We report necessary and sufficient conditions for the time-dependent coefficients $g_1(t)$ and $g_2(t)$ for the DOCS to be nondispersive. We also show that there is no necessary connection between the nondispersive character of coherent states and the ability of the DOCS to provide the quantum analog of the corresponding classical dynamical problem. A number of simple examples are treated.

## I. INTRODUCTION

By a coherent state[1,2] one normally means a state of a physical system described by a wave packet which preserves shape in time, which remains localized in position and momentum about their classical quantities, and indeed, which is the quantum analog of the corresponding classical particle. Coherent states have been the focus of intense study over the past two decades.[1–14] At the root of this activity has been the desire to provide for them a general theoretical framework within quantum mechanics. Different methods of generating coherent states[8] such as displacement operators,[3,4] annihilation operators,[3,4] minimum uncertainty conditions,[2,5,6] etc., have been applied to a number of Hamiltonian systems. Examples include the harmonic[2–5,8,11] and Morse oscillators,[6,9] and the Rosen-Morse[6,10] and Poschel-Teller[5,10] potentials.

The bulk of the work relating to coherent states has been done in the time-independent domain (for exceptions see Refs. 1, 2, 12, and 13). To investigate the time-dependent behavior of coherent states, it would be advantageous to work with a Schrödinger equation which has an explicitly time-dependent potential and which is exactly solvable. We have chosen a potential of the form $V(x,t)=g_2(t)x^2+g_1(t)x$. This is a generalization of the one employed by Hartley and Ray.[12] Solutions of the resulting Schrödinger equation may be obtained[15] independently of the particular form of $g_2(t)$ and $g_1(t)$. Another important feature of the Schrödinger equation with $V(x,t)$ is the existence of a six-dimensional space-time symmetry algebra consisting of first-order differential operators,[16] from which we can fashion ladder operators[15] which depend explicitly on time, which form a Heisenberg algebra, and which step eigenvalues of the corresponding number operator (not necessarily the Hamiltonian). An outline of the construction of these operators is provided in Sec. II. These raising and lowering operators are ideal for the construction of displacement operator coherent states (DOCS) and annihilation operator coherent states (AOCS), as shown in Sec. III. We demonstrate the equivalence of these two representations.

Our main objectives in this research are to formulate time-dependent wave functions of coherent states and to determine under what conditions the wave function describes a nondispersive or dispersive wave packet. In addition, we see whether, in either case, the state is the quantum analog of the classical system. More specifically, we obtain necessary and sufficient conditions for the coefficients $g_2(t)$ and $g_1(t)$ so that the state coheres forever, and when the wave function describes a dispersive or dissipative wave packet, we show further that the classical equations of motion and the classical trajectories are still obtained. To argue and illustrate our points, we make use of five examples. They include the harmonic oscillator (HO), the harmonic oscillator subject to a uniform driving force (HOUDF), and the everywhere nonconfining potentials:[17] free particle (FP), linear potential (LP), and the repulsive oscillator (RP). Finally, we end Sec. III by comparing our results to predictions of the minimum uncertainty coherent states (MUCS) and the number operator states (NOS). We present our conclusions in Sec. IV.

## II. KINEMATICAL SYMMETRY

Details of the calculation of the symmetry algebra[16] and the number operator states[15] can be found elsewhere. We include salient features here to define terms and to make this paper relatively self-contained.

Let $G$ be a Lie group with group element $g$ which is specified by a set of parameters.[18] Let $\vec{x}=(x,t)$, and $F(\vec{x})$ a function of $\vec{x}$. We define a representation $T$ on $G$ by the action of the group element:

$$[T(g)F](\vec{x})=\nu(\vec{x};g^{-1})F(\vec{f}(\vec{x};g^{-1})) \ . \tag{2.1}$$

Such a representation is called a multiplier representation[18] with multiplier $\nu(\vec{x};g^{-1})$. The function $\vec{f}(\vec{x};g^{-1})$ is a two-component, vector-valued, analytic function of its arguments.

Generators of these space-time or kinematic symmetries can be obtained by expansion of the right-hand side of (2.1) in a Taylor series at the identity. A generator is a first-order differential operator of the form

$$L=A(x,t)\partial_t+B(x,t)\partial_x+C(x,t) \ , \tag{2.2}$$

 

where the coefficients $A$, $B$, and $C$ are yet to be determined. The set of all such generators, $L$, forms a Lie algebra, $\mathcal{G}$. Exponentiation of elements of $\mathcal{G}$ will return us to a subgroup of the Lie group, $G$ according to standard Lie theory.[18]

The Schrödinger equation of interest in one dimension may be written

$$Q\Psi(x,t)=[\partial_{xx}+2i\partial_t-2g_2(t)x^2-2g_1(t)x]\Psi(x,t)=0\ , \tag{2.3}$$

where $Q$ is the differential operator,

$$Q=\partial_{xx}+2i\partial_t-2g_2(t)x^2-2g_1(t)x\ . \tag{2.4}$$

If $\Psi(x,t)$ lies in the solution space $\mathcal{F}Q$ of (2.3), then it can be shown that $L\Psi$, $L$ given by (2.2), is also a solution,[19] provided that $L$ satisfies the commutation relation

$$[Q,\ L]=\lambda(x,t)Q\ , \tag{2.5}$$

where $\lambda(x,t)$ is a function of the independent variables $x$ and $t$.

Substituting $Q$ from (2.4) and $L$ from (2.2) into (2.5), we obtain a set of coupled partial differential equations for the coefficients $A$, $B$, and $C$.[16] We can solve these equations in general, without the knowledge of the specific forms of the time-dependent functions $g_1(t)$ or $g_2(t)$ [except that $g_1(t)$ and $g_2(t)$ must be real].

Instrumental in that analysis are the ordinary differential equations

$$\ddot{b}+2g_2(t)b=0 \tag{2.6}$$

and

$$\dddot{A}+8g_2(t)\dot{A}=4\dot{g}_2(t)A=0\ . \tag{2.7}$$

Let $\mathcal{X}_1(t)$ and $\mathcal{X}_2(t)$ be two nontrivial, real solutions of (2.6) with Wronskian[16]

$$W(\mathcal{X}_1,\mathcal{X}_2)=\mathcal{X}_1\dot{\mathcal{X}}_2-\dot{\mathcal{X}}_1\mathcal{X}_2=1\ . \tag{2.8}$$

It is advantageous[12] to define the complex solution

$$\xi(t)=\frac{1}{\sqrt{2}}[\mathcal{X}_1(t)+i\mathcal{X}_2(t)]\ . \tag{2.9}$$

Then $\bar{\xi}(t)$, the complex conjugate, will also be a linearly independent solution and the Wronskian will have the form[20]

$$W(\xi,\bar{\xi})=\xi\dot{\bar{\xi}}-\dot{\xi}\bar{\xi}=-i\ . \tag{2.10}$$

We can use the solutions $\xi,\bar{\xi}$ to (2.6) to construct solutions to (2.7) in the following manner:

$$\varphi_1(t)=\xi^2(t),\quad \varphi_2(t)=\bar{\xi}^2(t),\quad \varphi_3=2\xi(t)\bar{\xi}(t)\ . \tag{2.11}$$

With these functions and the auxilary functions

$$\mathcal{A}_1=-\xi\mathcal{C},\quad \mathcal{A}_2=-\overline{\xi\mathcal{C}},\quad \mathcal{A}_3=-\xi\overline{\mathcal{C}}-\bar{\xi}\mathcal{C}\ , \tag{2.12a}$$

$$\mathcal{D}_1=-\tfrac{1}{2}\mathcal{C}^2,\quad \mathcal{D}_2=-\tfrac{1}{2}\bar{\mathcal{C}}^2,\quad \mathcal{D}_3=-\mathcal{C}\overline{\mathcal{C}}\ , \tag{2.12b}$$

where

$$\mathcal{C}=\int_{t_0}^{t}g_1(s)\xi(s)ds \tag{2.13}$$

we then obtain the generators,

$$M_j=i[\varphi_j\partial_j+(\tfrac{1}{2}\dot{\varphi}_jx+\mathcal{A}_j)\partial_x-\tfrac{1}{4}i\ddot{\varphi}_jx^2-i\dot{\mathcal{A}}_jx+\tfrac{1}{4}\dot{\varphi}_j+i\mathcal{D}_j],\quad 1\le j\le 3\ ; \tag{2.14a}$$

$$J_-=\xi\partial_x-ix\dot{\xi}+i\mathcal{C},\ J_+=-\bar{\xi}\partial_x+ix\dot{\bar{\xi}}-i\overline{\mathcal{C}},\quad I=1\ . \tag{2.14b}$$

Their commutation relations have the structure

$$[J_-,\ J_+]=I\ , \tag{2.15a}$$

$$[M_1,\ M_2]=M_3,\quad [M_3,\ M_1]=-2M_1,\quad [M_3,M_2]=2M_2\ , \tag{2.15b}$$

$$[M_1,\ J_-]=0,\quad [M_2,\ J_-]=J_+,\quad [M_3,\ J_-]=-J_-,\quad [M_1,\ J_+]=-J_-,\quad [M_2,J_+]=0,\quad [M_3,\ J_+]=J_+\ . \tag{2.15c}$$

Our motivation for going to the complex solutions of (2.6) and (2.7) will become clear in a moment.

Because of (2.5) and (2.14), we have the identities

$$[Q,\ I]=[Q,\ J_-]=[Q,\ J_+]=0 \tag{2.16a}$$

and

$$[Q,\ M_j]=\dot{\varphi}_jQ,\quad j=1,2,3\ . \tag{2.16b}$$

Since

$$Q=-2H+2i\partial_t\ ,$$

on a suitable subspace of the solution space, $\mathcal{F}_Q$, we have

$$i\frac{d\langle L\rangle}{dt}=i\left\langle\frac{\partial L}{\partial t}\right\rangle+\langle[L,H]\rangle=0\ , \tag{2.17}$$

where $L$ is any of $M_j$, $j=1,2,3$ or $J_+$, $J_-$, or $I$. Hence these generators are constants of the motion;[21] unusual perhaps, in the sense that they have an explicit time dependence. Furthermore, on this subspace of $\mathcal{F}_Q$, which forms a Hilbert space, the operators $M_3,I$ are Hermitian, but $M_1^\dagger=M_2$ and $J_-^\dagger=J_+$ are Hermitian conjugates.

The motivation for the complex transformation will now become clear. If we choose the subalgebra $\{M_3,J_-,J_+,I\}$ of $\mathcal{S}_1$, having the Casimir operator

$$C=J_+J_--M_3I=-\tfrac{1}{2}(\varphi_3Q+1)\ , \tag{2.18}$$

then we can construct[15] an irreducible representation space, a Hilbert space of square-integrable functions $h_n(x,t)$, such that

$$Ch_n=-\tfrac{1}{2}h_n,\quad M_3h_n=(n+\tfrac{1}{2})h_n,\quad Ih_n=h_n\ ,\qquad J_+h_n=(n+1)^{1/2}h_{n+1},\quad J_-h_n=n^{1/2}h_{n-1}\ , \tag{2.19}$$

for $n$ a non-negative integer. This representation we shall denote by $\uparrow-\tfrac{1}{2},1$ as in Miller.[22] Since $M_3$ is self-adjoint, its eigenvalues are real and its eigenfunctions are orthogonal. In normalized form they can be written

$$h_n(x,t)=(\pi\varphi_3)^{-1/4}(n!)^{-1/2}2^{-n/2}\exp\{i[x^2\dot{\varphi}_3/4\varphi_3+x\mathscr{A}_3/\varphi_3-\Lambda-(n+\tfrac{1}{2})\Phi]\}$$
$$\times H_n(x/\varphi_3^{1/2}-\mathscr{B})\exp[-\tfrac{1}{2}(x/\varphi_3^{1/2}-\mathscr{B})^2]\ , \tag{2.20}$$

where

$$\Lambda=\int_{t_0}^{t}ds\left[\frac{\mathscr{A}_3^2(s)}{\varphi_3^2(s)}+\frac{\mathscr{D}_3(s)}{\varphi_3(s)}\right],\quad \Phi=\int_{t_0}^{t}\frac{ds}{\varphi_3(s)}=\frac{i}{2}\ln\left[\frac{\bar{\xi}}{\xi}\right]$$

and (2.21)

$$\mathscr{B}=\int_{t_0}^{t}ds\frac{\mathscr{A}_3(s)}{\varphi^{2/3}(s)}=i\frac{\xi\overline{\mathscr{C}}-\bar{\xi}\mathscr{C}}{\varphi_3^{1/2}}\ .$$

The functions $\varphi_3$, $\mathscr{A}_3$, $\mathscr{D}_3$, $\xi$, and $\mathscr{C}$ are defined by (2.11), (2.12a), (2.12b), (2.9), and (2.13), respectively.

The wave functions $h_n(x,t)$ form a complete set, by hypothesis, of eigenfunctions of the operator $M_3$ and of the number operator $J_+J_-$, i.e.,

$$J_+J_-h_n=nh_n\ .$$

In addition, they satisfy the Schrödinger equation $Qh_n=0$, and because of (2.18), $Ch_n=-\frac{1}{2}h_n$. Note that on $\mathscr{F}_Q$, $M_3=J_+J_-+\frac{1}{2}$, i.e., $M_3$ is the number operator plus a constant.

Different choices of the time-dependent function $g_2(t)$ will yield different solutions $\xi$. The auxiliary time-dependent functions $\mathscr{C}$, $\mathscr{A}_j$, $\mathscr{D}_j$, $\mathscr{B}$, and $\Lambda$ will only be nonzero if $g_1(t)$ is nonzero. In Table I, we have put the values of these functions for the choices of $g_2(t)$ and $g_1(t)$ appropriate for the Hamiltonians of interest. The authors believe that the existence of a set of discrete states for the systems FP, LP, and RO (see Table I) is not well known, and may have computational advantages over conventional bases.

The space-time symmetry algebra (2.14) with commutation relations (2.15) is the complexification[15] of the Schrödinger algebra of Ref. 16. The Schrödinger algebra, denoted $\mathscr{S}_1$, is constructed from the real solutions $\chi_1$ and $\chi_2$ of (2.6) and (2.7). There is some merit at this point to outline properties of $\mathscr{S}_1$ and to provide a further motivation for the complexification step. If we construct the generators of $\mathscr{S}_1$, from the real solutions of (2.6) as in Ref. 16, then we obtain the operators $L_1$, $L_2$, $L_3$, $B_1$, $B_2$, and $E$, where $L_j$, $1\le j\le 3$, generate the sl(2,ℝ) algebra while $B_1$, $B_2$, and $E$ generate the Heisenberg algebra $w_1$. The Schrödinger algebra $\mathscr{S}_1=\mathrm{sl}(2,\mathbb{R})\Box w_1$. The adjoint action of the corresponding Lie group[18] on this algebra[19] partitions the algebra into disjoint classes called orbits. There are five such orbits with representatives $L_1$, $L_1+L_2$, $L_3$, $L_1+B_2$, and $B_1$. To each of the four orbits, represented by $L_1$, $L_1+L_2$, $L_3$, and $L_1+B_2$, there corresponds a separable coordinate system.[18,23] In this paper we have chosen to separate variables in the coordinate system associated with the orbit represented by $L_1+L_2$ where we have set $M_3=i(L_1+L_2)$. Because of the structure of the algebra $\{M_3,J_+,J_-,I\}$ and the calculations of Ref. 15, just outlined above, the spectrum of $M_3$ in the basis (2.20) is discrete. The other four-dimensional algebras of the type $\{O,B_1,B_2,E\}$, i.e., $\{O\}\Box w_1$ where $O$ is an orbit representative, are all of special interest for (2.3) because each of these Lie algebras will characterize a separable coordinate system and solution space for the Schrödinger equation (2.3). However, only in the case that $O=L_1+L_2$ will the complexification of this four-dimensional solvable algebra yield ladder operators (2.14b) which step the eigenvalues of the orbit representative $O$. For all other orbits the spectrum of the orbit representative is continuous (cf. Hartley and Ray[24]). The energy operator for the system HO lies on the orbit represented by $L_1+L_2$. To see this, construct the operator $M_3$, from (2.14a), in the Heisenberg picture. This can be done by letting $t\to t_0$ and replacing $i\partial_t$ by the Hamiltonian.[25] For the systems HO, HOUDF, and RO,

$$M_3\to(p^2+\omega^2x^2)/2\omega\ .$$

TABLE I. Potentials and auxiliary time-dependent functions for model Hamiltonians.

| System | $V(x,t)$ | $\chi_1(t)$ | $\chi_2(t)$ | $\xi$ | $\varphi_3$ |
|---|---|---|---|---|---|
| HOUDF | $\frac{\omega^2x^2}{2}+g_1(t)x$ | $\frac{1}{\sqrt{\omega}}\cos[\omega(t-t_0)]$ | $\frac{1}{\sqrt{\omega}}\sin[\omega(t-t_0)]$ | $\frac{1}{\sqrt{2\omega}}e^{i\omega(t-t_0)}$ | $\frac{1}{\omega}$ |
| HO | $\frac{\omega^2x^2}{2}$ | $\frac{1}{\sqrt{\omega}}\cos[\omega(t-t_0)]$ | $\frac{1}{\sqrt{\omega}}\sin[\omega(t-t_0)]$ | $\frac{1}{\sqrt{2\omega}}e^{i\omega(t-t_0)}$ | $\frac{1}{\omega}$ |
| FP | 0 | 1 | $t-t_0$ | $\frac{1}{\sqrt{2}}[1+i(t-t_0)]$ | $1+(t-t_0)^2$ |
| LP | $\kappa x$ | 1 | $t-t_0$ | $\frac{1}{\sqrt{2}}[1+i(t-t_0)]$ | $1+(t-t_0)^2$ |
| RO | $-\frac{\omega^2x^2}{2}$ | $\frac{1}{\sqrt{\omega}}\cosh[\omega(t-t_0)]$ | $\frac{1}{\sqrt{\omega}}\sinh[\omega(t-t_0)]$ | $\frac{1}{\sqrt{2\omega}}\{\cosh[\omega(t-t_0)]+i\sinh[\omega(t-t_0)]\}$ | $\frac{1}{\omega}\cosh[2\omega(t-t_0)]$ |

For the systems FP and LP we get

$$M \to (p^2+x^2)/2 \; .$$

Note that for the system HO, $M_3=(i/\omega)\partial_t$ is the energy operator, i.e., in the Heisenberg picture $M_3$ is proportional to the Hamiltonian. With these remarks as background we can proceed to the analysis of coherent states in Sec. III.

## III. COHERENT STATES

We shall divide our discussion into three parts. Section III A will deal with the calculation of coherent states by the DOCS and AOCS methods. In Sec. III B, we shall calculate the time evolution of MUCS for the systems FP, LP, and RO. The computation of expectation values for $x$ and $p$ and the uncertainty product will be made in Sec. III for the number operator representation.

### A. DOCS and AOCS

According to Perelomov's[4] (or Gilmore's[7]) definition of coherent states, we require a Lie group $G$, which in our case is the group obtained from the exponentiation of the Lie algebra $\{M_3, J_-, J_+, I\}$. We need an irreducible representation of $G$, namely $\uparrow -\frac{1}{2},1$, and a fixed vector in the representation space; we choose the extremal weight function $h_0(x,t)$. The stationary group of $h_0(x,t)$ will consist of elements of the subgroup $\exp(i\sigma M_3+i\tau I)$:

$$\exp(i\sigma M_3+i\tau I)h_0=e^{i(\sigma/2+\tau)}h_0$$

by (2.19). The displacement operator is the coset representative

$$D(\alpha)=\exp(\alpha J_+ - \bar{\alpha} J_-) \tag{3.1}$$

which is a unitary operator and $\alpha=\mu+i\nu$ is a complex parameter.

The DOCS $f_\alpha(x,t)$ is defined to be

$$f_\alpha(x,t)=D(\alpha)h_0(x,t) \; . \tag{3.2}$$

By standard arguments,[11] it can be shown that $f_\alpha(x,t)$ is also an eigenfunction of the lowering operator $J_-$, with eigenvalues $\alpha$, i.e.,

$$J_- f_\alpha(x,t)=\alpha f_\alpha(x,t) \; . \tag{3.3}$$

This is the definition of the AOCS; the equivalence of the DOCS and AOCS definitions is peculiar to the structure of the Lie algebra $\{M_3, J_-, J_+, I\}$, and does not hold for all Lie groups.

We shall now construct a space-time realization of the DOCS, $f_\alpha(x,t)$. To begin, the Baker-Campbell-Hausdorff relation allows[18]

$$\begin{aligned} f_\alpha(x,t) &= \exp(-\tfrac{1}{2}\,|\,\alpha\,|^2)\exp(\alpha J_+)\exp(\bar{\alpha} J_-)h_0(x,t) \\ &= \exp(-\tfrac{1}{2}\,|\,\alpha\,|^2)\exp(\alpha J_+)h_0(x,t) \end{aligned} \tag{3.4}$$

since $J_- h_0=0$. From (2.20) we have an expression for $h_0(x,t)$:

$$\begin{aligned} h_0(x,t) &= (\pi\varphi_3)^{-1/4} \\ &\times \exp\left[ i \left[ \frac{x^2\dot{\varphi}_3}{4\varphi_3} + \frac{x\mathscr{A}_3}{\varphi_3} - \Lambda - \frac{\Phi}{2} \right] \right] \\ &\times \exp\left[ -\frac{1}{2} \left[ \frac{x}{\varphi_3^{1/2}} - \mathscr{B} \right]^2 \right] , \end{aligned} \tag{3.5}$$

where $\varphi_3$, $\mathscr{B}$, $\Lambda$, $\Phi$, and $\mathscr{A}_3$ are given by (2.11), (2.21), and (2.12). Now the action of the group element $\exp(\alpha J_+)$ on $h_0(x,t)$ [multiplier representation, Eq. (2.1)] can be determined[26] from the form of $J_+$ [see (2.14b)]. We have (see the Appendix)

$$f_\alpha(x,t)=\exp\left[ -\frac{|\,\alpha\,|^2}{2} - \frac{\alpha^2\bar{\xi}}{2\xi} + \frac{x\alpha}{\xi} - \frac{2\alpha\bar{\xi}\mathscr{B}}{\varphi_3^{1/2}} \right] h_0(x,t) \; , \tag{3.6}$$

where the properties of the Wronskian (2.10) have been used. That $f_\alpha(x,t)$ also satisfies (3.3) and the Schrödinger equation (2.3) can be seen by substitution. Since $D(\alpha)$ is unitary, we get

$$\begin{aligned} \langle \alpha \,|\, \alpha \rangle &= \int_{-\infty}^{+\infty} \bar{f}_\alpha(x,t) f_\alpha(x,t)\,dx \\ &= \int_{-\infty}^{+\infty} \bar{h}_0(x,t) D^\dagger(\alpha) D(\alpha) h_0(x,t)\,dx \\ &= \int_{-\infty}^{+\infty} \bar{h}_0(x,t) h_0(x,t)\,dx = 1 \; , \end{aligned} \tag{3.7}$$

and $f_\alpha(x,t)$ is normalized.

The AOCS approach requires that we solve the eigenvalue problem (3.3). In this case, (3.3) reduces to a first-order partial differential equation for $f_\alpha$,

$$\xi f_{\alpha,x} + (i\mathscr{C} - i\dot{\xi}x - \alpha) f_\alpha = 0 \; . \tag{3.8}$$

This equation can be solved, in part, by the method of characteristics,[27] where the characteristic equations are

$$\frac{dt}{0} = \frac{dx}{\xi} = -\frac{df}{(i\mathscr{C} - i\dot{\xi}x - \alpha) f_\alpha} \; .$$

Integrating we obtain

$$f_\alpha(x,t)=g(t)\exp\left[ \frac{i\dot{\xi}}{2\xi}x^2 - \frac{i\mathscr{C}x}{\xi} + \frac{\alpha x}{\xi} \right] , \tag{3.9}$$

where $g(t)$ is an arbitrary function of $t$. To fix $g(t)$, we demand that $f_\alpha$ satisfy the Schrödinger equation (2.3). This places the constraint

$$2i\dot{g} = \left[ -\frac{1}{2}\frac{\dot{\xi}}{\xi} - \frac{i}{2\xi^2}(\alpha - i\mathscr{C})^2 \right] g$$

on $g$. Integrating this equation and substituting the result into (3.9) yields the AOCS

$$\begin{aligned} f_\alpha(x,t) = N\xi^{-1/2}\exp\Bigg[ &\frac{i\dot{\xi}}{2\xi}x^2 - \frac{i\mathscr{C}x}{\xi} + \frac{\alpha x}{\xi} \\ &- \frac{i}{2}\int_{t_0}^{t} \frac{(\alpha - i\mathscr{C})^2}{\xi^2}\,ds \Bigg] , \end{aligned} \tag{3.10}$$

where $N$ is the normalization constant. That this expression for $f_\alpha$ is equivalent to the DOCS $f_\alpha$, up to normalization, can be seen from exploiting the properties of the time-dependent functions $\xi$, $\mathscr{C}$, $\mathscr{A}_3$, $\Lambda$, $\mathscr{B}$, and $\Phi$ (see Appendix).

Here we have two nice methods suitable for obtaining wave functions for coherent states. Their value lies in their generalization to other groups and to differential equations of higher dimensions.

Let us establish some properties of the states $f_\alpha(x,t)$. From (3.4), we can expand $\exp(\alpha J_+)$ and from (2.19) we get

$$f_\alpha(x,t)=\exp\left[-\frac{|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\frac{(\alpha J_+)^n}{n!}h_0(x,t)$$
$$=\exp\left[-\frac{|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}h_n(x,t)\ , \qquad (3.11)$$

which corresponds to the usual expansion of the oscillator coherent state[2,11] in the basis for the number operator representation. Naturally, (3.11) and (3.6) provide us with an identity

$$\sum_{n=0}^{\infty}\frac{\alpha^n e^{-in\Phi}}{2^{n/2}n!}H_n\left[\frac{x}{\varphi_3^{1/2}}-\mathscr{B}\right]=\exp\left[-\frac{\alpha^2\bar{\xi}}{2\xi}+\frac{x\alpha}{\xi}-\frac{2\alpha\bar{\xi}\mathscr{B}}{\varphi_3^{1/2}}\right]$$

which generalizes the standard identity for Hermite polynomials.[22] From (3.7), the DOCS are normalized. However, they are not orthogonal since

$$\int_{-\infty}^{+\infty}\bar{f}_\alpha(x,t)f_\beta(x,t)dx=\int_{-\infty}^{+\infty}h_0(x,t)D^\dagger(\alpha)D(\beta)h_0(x,t)dx$$
$$=e^{\bar{\alpha}\beta-\alpha\bar{\beta}}\int_{-\infty}^{+\infty}\bar{h}_0(x,t)D(\beta-\alpha)h_0(x,t)dx=e^{-(|\alpha|^2+|\beta|^2-2\bar{\alpha}\beta)/2}$$

which arises because of the overcompleteness of the DOCS.[29]

Let us work in the bra-ket notation and set

$$|\alpha\rangle=D(\alpha)|0\rangle\ .$$

By (2.18), the DOCS $|\alpha\rangle$ satisfies the Schrödinger equation, for

$$C|\alpha\rangle=CD(\alpha)|0\rangle=D(\alpha)C|0\rangle=-\tfrac{1}{2}|\alpha\rangle$$
$$=-\tfrac{1}{2}(\varphi_3 Q+1)|\alpha\rangle \Longrightarrow Q|\alpha\rangle=0\ ,$$

since $\varphi_3\neq 0$ for any $t$.

We shall need a number of matrix elements which can be obtained from the property $J_+^\dagger=J_-$

$$\langle\alpha|J_+|\alpha\rangle=\bar{\alpha}\ . \qquad (3.12)$$

From the casimir operator (2.18).

$$\langle\alpha|M_3|\alpha\rangle=\langle\alpha|(J_+J_- -C)|\alpha\rangle=|\alpha|^2+\tfrac{1}{2}\ . \qquad (3.13)$$

Now we wish to evaluate some properties of the DOCS, (3.6); first, in general, and then for the special systems HO, HOUDF, FP, LP, and RO. Specifically, under what conditions will the DOCS give rise to a wave packet which preserves its shape in time, which remains localized in position and momentum about their classical values and which follows the classical equations of motion and has the classical, phase-space trajectory. To discover this, we must evaluate expectation values for $x(t)$, $x^2(t)$, $p(t)$, and $p^2(t)$ as well as the variances $(\Delta x)^2$ and $(\Delta p)^2$. The operators $x$, $p$, $x^2$, and $p^2$ can be expressed in terms of the ladder operators $J_-$ and $J_+$ as follows:

$$x=\xi J_+ +\bar{\xi}J_- +i(\xi\overline{\mathscr{C}}-\bar{\xi}\mathscr{C})\ , \qquad (3.14)$$
$$x^2=\xi^2J_+^2+\bar{\xi}^2J_- -(\xi\overline{\mathscr{C}}-\bar{\xi}\mathscr{C})^2+2\xi\bar{\xi}J_+J_- +\xi\bar{\xi}$$
$$+2i\xi(\xi\overline{\mathscr{C}}-\bar{\xi}\mathscr{C})J_+ +2i\bar{\xi}(\xi\overline{\mathscr{C}}-\bar{\xi}\mathscr{C})J_-\ , \qquad (3.15)$$

$$p=\dot{\xi}J_+ +\dot{\bar{\xi}}J_- +i(\dot{\xi}\mathscr{C}-\dot{\bar{\xi}}\mathscr{C})\ , \qquad (3.16)$$
$$p^2=\dot{\xi}^2J_+^2+\dot{\bar{\xi}}_2J_2-(\dot{\xi}\overline{\mathscr{C}}-\dot{\bar{\xi}}\mathscr{C})^2+2\dot{\bar{\xi}}\xi J_+J_- +\dot{\xi}\dot{\bar{\xi}}$$
$$+2i\dot{\xi}(\dot{\xi}\overline{\mathscr{C}}-\dot{\bar{\xi}}\mathscr{C})J_+ +2i\dot{\bar{\xi}}(\dot{\bar{\xi}}\overline{\mathscr{C}}-\dot{\bar{\xi}}\mathscr{C})J_-\ . \qquad (3.17)$$

From the properties of the coherent states computed above we have

$$\langle x(t)\rangle=\langle\alpha|x|\alpha\rangle=\bar{\alpha}\xi+\alpha\bar{\xi}+i(\xi\overline{\mathscr{C}}-\bar{\xi}\mathscr{C})$$
$$=\chi_2(\sqrt{2}\mu+\mathscr{C}_2)+\chi_2(\sqrt{2}\nu-\mathscr{C}_1)\ , \qquad (3.18)$$

where

$$\mathscr{C}_\sigma(t)=\int_{t_0}^{t}g_1(s)\chi_\sigma(s)ds,\quad \sigma=1,2 \qquad (3.19)$$
$$\langle x^2(t)\rangle=\langle\alpha|x^2|\alpha\rangle=\langle x(t)\rangle^2+\xi\bar{\xi}\ , \qquad (3.20)$$
$$\langle p(t)\rangle=\langle\alpha|p|\alpha\rangle=\bar{\alpha}\dot{\xi}+\alpha\dot{\bar{\xi}}+i(\dot{\xi}\overline{\mathscr{C}}-\dot{\bar{\xi}}\mathscr{C})$$
$$=\dot{\chi}_1(\sqrt{2}\mu+\mathscr{C}_2)+\dot{\chi}_2(\sqrt{2}\nu-\mathscr{C}_1)\ , \qquad (3.21)$$
$$\langle p^2(t)\rangle=\langle\alpha|p^2|\alpha\rangle=\langle p(t)\rangle^2+\dot{\xi}\dot{\bar{\xi}}\ . \qquad (3.22)$$

Therefore, from (3.20) and (3.22) we get

$$(\Delta x)^2=\langle x^2(t)\rangle-\langle x(t)\rangle^2=\xi\bar{\xi}=\tfrac{1}{2}\varphi_3\ , \qquad (3.23a)$$
$$(\Delta p)^2=\langle p^2(t)\rangle-\langle p(t)\rangle^2=\dot{\xi}\dot{\bar{\xi}}\ , \qquad (3.23b)$$

and the uncertainty product is

$$(\Delta x)^2(\Delta p)^2=\xi\bar{\xi}\dot{\xi}\dot{\bar{\xi}}=\tfrac{1}{4}(\chi_1^2+\chi_2^2)(\dot{\chi}_1^2+\dot{\chi}_2^2)\ . \qquad (3.24)$$

But the Wronskian $W(\chi_1,\chi_2)=\chi_1\dot{\chi}_2-\dot{\chi}_1\chi_2=1$. Expanding (3.24), squaring the Wronskian, and substituting into the expanding (3.24) we get

$$(\Delta x)^2(\Delta p)^2=\left[1+\left[\frac{d}{dt}(\chi_1^2+\chi_2^2)\right]^2\right]\geq\frac{1}{4}\ , \qquad (3.25)$$

and see that the uncertainty relation is satisfied and, in particular, if $\chi_1^2+\chi_2^2=\xi\bar{\xi}$ is a constant then the minimum uncertainty holds. What are the conditions for $\chi_1^2+\chi_2^2$ to remain constant in time? Since $\chi_1^2+\chi_2^2$ is a solution of (2.7), we get a necessary condition $\dot{g}_2=0$ or $g_2(t)=\text{const}$. If $g_2(t)$ is constant, it can be positive (HO), zero (FP), or negative (RO). In what follows, we show that the sufficient condition for the minimum uncertainty is that $g_2(t)$ be a positive constant. Surprisingly, the uncertainty relation is independent of $g_1(t)$. Therefore, only for the systems HO and HOUDF will the minimum uncertainty condition be realized for all time, i.e., the coherent states are nondispersive.

Expression (3.25) is analogous to that obtained by Hartley and Ray[12] for the potential $V(x,t)=\omega^2(t)x^2$ using the method of Lewis-Riesenfeld invariants. They concluded, as we have, that the wave packet is dispersive in time, but they did not address, other than for the system HO, the nature of the coefficient $\omega(t)$ and the conditions for coherence. Nor was the impact of a uniform driving force on coherence examined.[12,14]

We can readily establish that the classical equations (Newton's equations) of motion are obtained. From (3.18)

$$\frac{d\langle x(t)\rangle}{dt}=\dot{\chi}_1(\sqrt{2}\mu+\mathscr{C}_2)+\dot{\chi}_1(\sqrt{2}\nu-\mathscr{C}_1)+\chi_1\dot{\mathscr{C}}_2-\nu_2\dot{\mathscr{C}}_1$$
$$=\langle p(t)\rangle$$

since $\chi_1\dot{\mathscr{C}}_2-\chi_2\dot{\mathscr{C}}_1=0$ by Eq. (3.19). The force is

$$\begin{aligned}\frac{d\langle p(t)\rangle}{dt}&=\ddot{\chi}_1(\sqrt{2}\mu+\mathscr{C}_2)+\ddot{\chi}_2(\sqrt{2}\nu-\mathscr{C}_1)+\dot{\chi}_1\dot{\mathscr{C}}_2-\dot{\chi}_2\dot{\mathscr{C}}_1\\&=-2g_2(t)[\chi_1(\sqrt{2}\mu+\mathscr{C}_2)+\chi_2(\sqrt{2}\nu-\mathscr{C}_1)]-(\chi_1\dot{\chi}_2-\dot{\chi}_1\chi_2)g_1(t)\\&=-2g_2(t)\langle x(t)\rangle-g_1(t)=-\left\langle\frac{\partial V}{\partial x}\right\rangle,\end{aligned}$$

since by (2.6), $\ddot{\chi}_\sigma=-2g_2(t)\chi_\sigma$ for $\sigma=1,2$ and the property (2.8) of the Wronskian, $W(\chi^1,\chi^2)$.

To obtain the sufficient condition for minimum uncertainty we require the detailed solutions for three specific cases: the systems HO, FP, and RO. In addition, we treat two other examples: the systems HOUDF and LP. The first three have $g_1(t)=0$ and the latter two have $g_1(t)\neq 0$. We refer to Table I for appropriate values of the time-dependent functions $\chi_\sigma(t)$, $\xi(t)$, and $\varphi_3(t)$. We treat HO and FP in detail and sketch the derivations for the remaining examples. The results are summarized in Table II.

*HO.* For the harmonic oscillator the coherent-state wave function has the form

$$\begin{aligned}f_\alpha(x,t)=&(\pi/\omega)^{-1/4}\\&\times\exp(-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}\alpha^2e^{-2i\omega(t-t_0)}\\&+\sqrt{2}\omega x\alpha e^{-i\omega(t-t_0)}-\tfrac{1}{2}\omega x^2)\ .\end{aligned}\qquad(3.26)$$

This has a form similar to that given by Nieto and Simmons[2] and Santhanam[8] if $(\Delta x)^2=1/2\omega$ and $(\Delta p)^2=\omega/2$ is substituted.

The minimum uncertainty product is satisfied:

$$(\Delta x)^2(\Delta p)^2=\tfrac{1}{4}\ .\qquad(3.27)$$

The average position, from (3.18), is

$$\langle x(t)\rangle=\left[\frac{2}{\omega}\right]^{1/2}\{\mu\cos[\omega(t-t_0)]+\nu\sin[\omega(t-t_0)]\}\qquad(3.28)$$

and the momentum

$$\langle p(t)\rangle=\sqrt{2}\omega\{-\mu\sin[\omega(t-t_0)]+\nu\cos[\omega(t-t_0)]\}\ .\qquad(3.29)$$

At $t=t_0$, the initial values are obtained

$$\langle x(t_0)\rangle=x_0=\left[\frac{2}{\omega}\right]^{1/2}\mu\Longrightarrow\mu=x_0\frac{\omega}{2}=\frac{1}{2}\frac{x_0}{\Delta x}\qquad(3.30)$$

and

$$\langle p(t_0)\rangle=p_0=\sqrt{2}\omega\nu\Longrightarrow\nu=\frac{p_0}{\sqrt{2}\omega}=\frac{1}{2}\frac{p_0}{\Delta p}\ .\qquad(3.31)$$

If we set $\tan\sigma=\mu/\nu$, then

$$\langle x(t)\rangle=\left[\frac{2}{\omega}|\alpha|^2\right]^{1/2}\sin(\omega t+\sigma)\ ,\qquad(3.32)$$

where

$$|\alpha|^2=\mu^2+\nu^2=\frac{1}{\omega}\left[\frac{p_0^2}{2}+\frac{\omega^2x_0^2}{2}\right]\qquad(3.33\text{a})$$

$$=\langle\alpha|M_3|\alpha\rangle-\langle 0|\mu_3|0\rangle\ .\qquad(3.33\text{b})$$

The analogous expression for the momentum, from Eq. (3.21), is

$$\langle p(t)\rangle=(2\omega|\alpha|^2)^{1/2}\cos(\omega t+\varphi)\ .\qquad(3.34)$$

Eliminating the time dependence from (3.34) and (3.32) we have

$$\langle p(t)\rangle^2+\omega^2\langle x(t)\rangle^2=2\omega|\alpha|^2=p_0^2+\omega^2x_0^2\ ,\qquad(3.35)$$

a constant. Equation (3.35) is the equation for an ellipse in phase space and the trajectory is closed. This is the classical trajectory. The HO satisfies the definition of coherent states in all its aspects.[2]

*HOUDF.* This case resembles in many respects the previous one. The variances or uncertainties $(\Delta x)^2$ and $(\Delta p)^2$ (Table II) are independent of time and the minimum un-

TABLE II. DOCS predicted values for $\langle x(t)\rangle$, $\langle p(t)\rangle$, $(\Delta x)^2$, $(\Delta p)^2$, and uncertainty products.

| System | $\langle x(t)\rangle$, $\langle p(t)\rangle$ | $(\Delta x)^2$, $(\Delta p)^2$ | $(\Delta x)^2(\Delta p)^2$ | Phase-space trajectory |
|---|---|---|---|---|
| HO | $\langle x(t)\rangle=\left[\frac{2\vert\alpha\vert^2}{\omega}\right]^{1/2}\sin[\omega(t-t_0)+\sigma]$ [a] <br> $\langle p(t)\rangle=(2\omega\vert\alpha\vert^2)\cos[\omega(t-t_0)+\sigma]$ | $(\Delta x)^2=\frac{1}{2\omega}$ <br> $(\Delta p)^2=\frac{\omega}{2}$ | $\frac{1}{4}$ | Ellipse |
| HOUDF | $\langle x(t)\rangle=A(t)\sin[\omega(t-t_0)+\sigma(t)]$ [b] <br> $\langle p(t)\rangle=\omega A(t)\cos[\omega(t-t_0)+\sigma(t)]$ | $(\Delta\omega)^2=\frac{1}{2\omega}$ <br> $(\Delta p)^2=\frac{\omega}{2}$ | $\frac{1}{4}$ | Depends on $g_1(t)$ |
| FP | $\langle x(t)\rangle=x_0+p_0(t-t_0)$ <br> $\langle p(t)\rangle=p_0$ | $(\Delta x)^2=\frac{1}{2}[1+(t-t_0)^2]$ <br> $(\Delta p)^2=\frac{1}{2}$ | $\frac{1}{4}[1+(t-t_0)^2]$ | Straight line |
| LP | $\langle x(t)\rangle=x_0+p_0(t-t_0)-\frac{\kappa(t-t_0)^2}{2}$ <br> $\langle p(t)\rangle=p_0-\kappa(t-t_0)$ | $(\Delta x)^2=[1+(t-t_0)^2]$ <br> $(\Delta p)^2=\frac{1}{2}$ | $\frac{1}{4}[1+(t-t_0)^2]$ | Parabola |
| RO | $\langle x(t)\rangle=\left[\frac{2}{\omega}\mathrm{Re}(\alpha^2)\right]^{1/2}\cosh[\omega(t-t_0)+\sigma]$ [c] <br> $\langle p(t)\rangle=[2\omega\,\mathrm{Re}(\alpha^2)]^{1/2}\sinh[\omega(t-t_0)+\sigma]$ | $(\Delta x)^2=\frac{1}{2\omega}\cosh(2\omega t)$ <br> $(\Delta p)^2=\frac{\omega}{2}\cosh(2\omega t)$ | $\frac{1}{4}\cosh^2(2\omega t)$ | Hyperbola |

[a] The phase $\sigma$ is given by $\tan\sigma=\mu/\nu$.
[b] The amplitude $A(t)$ is given by (3.39); the phase $\sigma(t)$ by $\tan\sigma(t)=(\sqrt{2}\mu+\mathscr{C}_2)/(\sqrt{2}\nu-\mathscr{C}_1)$.
[c] The phase $\sigma$ is given by $\tanh\sigma=\nu/\mu$.

certainty condition holds. From (3.18) and (3.21) we get

$$\langle x(t)\rangle = A(t)\sin[\omega(t-t_0)+\sigma(t)] \tag{3.36}$$

and

$$\langle p(t)\rangle = \omega A(t)\cos[\omega(t-t_0)+\sigma(t)]\ , \tag{3.37}$$

where

$$\tan\sigma(t) = (\sqrt{2}\mu+\mathscr{C}_2)/(\sqrt{2}\nu-\mathscr{C}_1) \tag{3.38}$$

and

$$A^2(t) = \frac{1}{\omega}[(\sqrt{2}\mu+\mathscr{C}_2)^2+(\sqrt{2}\nu-\mathscr{C}_1)^2]\ . \tag{3.39}$$

The phase $\sigma(t)$ and the amplitude $A(t)$ are clearly time dependent. The trajectory

$$\langle p(t)\rangle^2+\omega^2\langle x(t)\rangle^2 = A^2(t)$$

will depend upon the nature of $g_1(t)$. The trajectory will be closed if $g_1(t)$ is periodic.

*FP.* The wave function for the coherent state can be represented by

$$f_\alpha(x,t) = \{\pi[1+(t-t_0)^2]\}^{-1/4}\exp\left[-\frac{|\alpha|^2}{2}-\frac{\alpha^2[1-i(t-t_0)]}{[1+i(t-t_0)]}+\frac{\sqrt{2}x\alpha}{[1+i(t-t_0)]}-\frac{x^2}{2[1+(t-t_0)^2]}\right]\exp\left[i\frac{x^2(t-t_0)}{2[1+(t-t_0)^2]}-\frac{i\tan^{-1}(t-t_0)}{2}\right]\ . \tag{3.40}$$

The variances from (3.23) are shown in Table II with the uncertainty product. It is clear that initially the state is coherent, satisfying the minimum uncertainty condition, but with the progression of time the coherent state dissipates (Fig. 1).

The $\langle x(t)\rangle$ and $\langle p(t)\rangle$ given by (3.18) and (3.21) have the form

$$\langle x(t)\rangle = x_0+p_0(t-t_0)\ ,$$

$$\langle p(t)\rangle = p_0\ ,$$

where $x_0=\sqrt{2}\mu$ and $p_0=\sqrt{2}\nu$; the equations are for a particle moving in uniform rectilinear motion with constant momentum $p_0$.

The requirements of the coherent state are not met in full; we have the classical equations of motion, the classical trajectories, but the dispersion is time dependent. The same observations hold for the next two examples: the systems LP and RO.

*LP and RO.* The calculations for these cases are the same as above and a summary of results can be found in Table II. The dispersions are time dependent as are the uncertainty products. The initial state is coherent in the full sense of the term but dissipates in time.

The trajectory predicted for the system LP is parabolic, which is the classical one. For the RO the trajectory is hyperbolic as can be seen by eliminating the time dependence from $\langle x(t)\rangle$ and $\langle p(t)\rangle$.

This brings us to the evolution of the MUCS, the details of which are provided next. To anticipate, we shall find similar predictions for the systems FP, LP, and RO from those calculations.

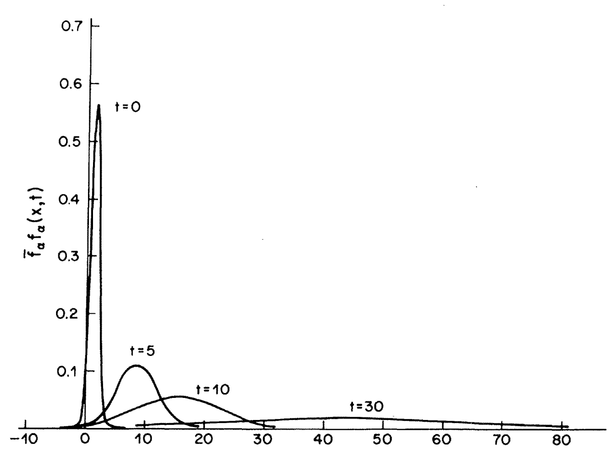


FIG. 1. A plot of the DOCS density as a function of $x$ and $t$.

### B. MUCS

For the algebra $\{x,p,I\}$ with commutation relations

$$[x,\ p] = iI\ , \tag{3.41}$$

we can compute the minimum uncertainty states according to the equation[2]

$$\left[\frac{x}{(\Delta x)_0}+\frac{ip}{(\Delta p)_0}\right]\psi_{CS} = \left[\frac{x_0}{(\Delta x)_0}+\frac{ip_0}{(\Delta p)_0}\right]\psi_{CS}\ , \tag{3.42}$$

where $p=-i\partial_x$ and the subscript 0 refers to the time $t_0$. The coherent-state function calculated from Eq. (3.42) has the form[2]

$$\psi_{CS} = [2\pi(\Delta x)_0^2]^{-1/4}\exp\left[-\left[\frac{x-x_0}{2(\Delta x)_0}\right]^2+ip_0x\right]\ . \tag{3.43}$$

For any time-independent Hamiltonian (for example, the systems FP, LP, HO, and RO), we can obtain the dynamics from the wave function

$$\Psi_{CS}(x,t) = e^{-iH(t-t_0)}\psi_{CS}(x)\ , \tag{3.44}$$

where $H$ is the appropriate Hamiltonian. Let $O$ be some operator. Then its time dependence may be expressed

$$\langle O(t)\rangle = \langle\Psi_{CS}|O|\Psi_{CS}\rangle = \langle\psi_{CS}|O(t)|\psi_{CS}\rangle$$

where[13]

$$O(t)=e^{iH(t-t_0)}Oe^{-iH(t-t_0)}$$
$$=O+i(t-t_0)[H,\,O]$$
$$+\frac{[i(t-t_0)]^2}{2!}[H,\,[H,\,O]]\;\cdots\;.$$

Nieto and Simmons[2] have performed these calculations for the systems HO and we have done them for the systems FP, LP, and RO when $O$ is $x$, $p$, $x^2$, and $p^2$. The uncertainty products were then calculated. The results are displayed in Table III.

In each case the phase-space trajectories are identical to those obtained by the displacement operator method. Except for the system HO which is truly coherent, the MUCS are "dissipative," the amplitude of the coherent state spreads in time. We can see by comparing Tables II and III that the DOCS and MUCS predict the correct classical equations of motion in each case.

It is instructive to contrast the dynamics of the DOCS and MUCS to those predicted by the number operator representation of Sec. II. This we do next in Sec. III C.

### C. Number operator representation

Details of the calculations for $\langle x\rangle$ and $\langle p\rangle$ may be found in Ref. 15. We have

$$\langle x(t)\rangle=i(\xi\overline{\mathscr{C}}-\overline{\xi}\mathscr{C})\,, \tag{3.45}$$

$$\langle p(t)\rangle=i(\dot{\xi}\overline{\mathscr{C}}-\dot{\overline{\xi}}\mathscr{C})\,, \tag{3.46}$$

and the force is given by

$$\frac{d\langle p\rangle}{dt}=2ig_2(\overline{\xi}\mathscr{C}-\xi\overline{\mathscr{C}})-g_1=-\left\langle\frac{\partial V}{\partial x}\right\rangle$$
$$=2g_2\langle x(t)\rangle-g_1 \tag{3.47}$$

which is Newton's equation of motion. The uncertainty relation is

$$(\Delta x)^2(\Delta p)^2=(2n+1)^2\xi\overline{\xi}\dot{\xi}\dot{\overline{\xi}}\geq\tfrac{1}{4}\,, \tag{3.48}$$

where

$$(\Delta x)^2=(2n+1)\xi\overline{\xi},\;\;(\Delta p)^2=(2n+1)\dot{\xi}\dot{\overline{\xi}}\,. \tag{3.49}$$

The correct classical trajectories cannot be calculated from (3.45) and (3.46). For example, for the systems HO, FP, and RO, $n=0$ and so

$$\langle x(t)\rangle=0,\;\;\langle p(t)\rangle=0\,.$$

In these cases, the trivial solution of Newton's equation of motion is obtained. Even for the HOUDF the classical trajectory is not obtained. The minimum uncertainty is obtained only for the state $n=0$ and only for the harmonic oscillator.

## IV. CONCLUSION

We require of a particle in a coherent state that it, first of all, follow the classical equations of motion and the classical phase-space trajectory. In addition, we expect that the wave packet describing the particle would be nondispersing in both position and momentum about the classical values as well as the minimum uncertainty product. If the first requirement is met but the wave packet, initially coherent, disperses with time, we call it a dissipative coherent state. It is dissipative in the sense that the probability of locating the particle in a small volume at a point in space decreases with time but the total probability over all space at any time remains unity.

We have clearly shown for a Schrödinger equation with potential $V(x,t)=g_2(t)x^2+g_1(t)x$ that $\Delta x$, $\Delta p$, and the uncertainty relations depend only on the nature of the function $g_2(t)$ and not on $g_1(t)$. Furthermore, the necessary and sufficient condition for $\Delta x$, $\Delta p$, and the minimum uncertainty product to be independent of time is that $g_2(t)=\text{const}>0$, regardless of $g_1(t)$. Thus for the systems HO and HOUDF we get a nondispersive coherent state; for the everywhere nonconfining potentials (FP, LP, and RO), we have dissipative coherent states.

Also, it is apparent that even dissipative coherent states, such as those described above, are quantum analogs of their classical counterparts. Therefore, this property is not tied to the minimum uncertainty condition. Thus, in each of our examples, dispersive or nondispersive, the classical picture is reproduced by the DOCS or MUCS method. This conclusion is supported by the work of Hartley and Ray,[12] and by Roy and Singh.[14]

In each case, in our analysis, the constant of motion in which the number operator representation is diagonal has, in the Heisenberg picture, the form $p^2+cx^2$ where $c=\omega^2$ for the systems HO, HOUDF, and RO or $c=1$ for the systems FP and LP. For the system HO, $p^2+\omega^2x^2$ corresponds to the Hamiltonian whose spectrum is discrete with equispaced eigenvalues. For the other systems, $p^2+cx^2$ does not correspond to their respective Hamiltonians which have continuous spectra.

In summary, we have constructed coherent states for a system subjected to a potential of the form $V(x,t)=g_2(t)x^2+g_1(t)x$ by the displacement operator method. This technique provides a natural mathematical framework because the number operators and the ladder operators which define them are symmetries of the time-dependent Schrödinger equation (2.3). Indeed, they are constants of motion. This displacement operator formalism is readily extended to other groups and to differential equations in higher spatial dimensions.

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## APPENDIX

To obtain the DOCS wave function (3.6) we must compute the action of the group element $\exp(\alpha J_+)$ on the ground-state function (3.5). By using Eq. (2.1) we have

$$\exp(\alpha J_+)h_0(x_0,t_0)=\nu(x_0,t_0;\alpha)h_0(\vec{\mathrm{f}}(x_0,t_0;\alpha))\,, \tag{A1}$$

where

$$J_+=-\overline{\xi}\partial_x+ix\dot{\overline{\xi}}-i\overline{\mathscr{C}} \tag{A2}$$

TABLE III. MUCS predictions for $\langle x(t)\rangle$, $\langle p(t)\rangle$, $(\Delta x)^2$, $(\Delta p)^2$, and uncertainty products.

| System | $\langle x(t)\rangle$, $\langle p(t)\rangle$ | $(\Delta x)^2$, $(\Delta p)^2$ | $(\Delta x)^2(\Delta p)^2$ | Phase-space trajectory |
|---|---|---|---|---|
| HO | $\langle x(t)\rangle = x_0\cos[\omega(t-t_0)] + \frac{p_0}{\omega}\sin[\omega(t-t_0)]$<br>$\langle p(t)\rangle = p_0\cos[\omega(t-t_0)] - x_0\omega\sin[\omega(t-t_0)]$ | $(\Delta x)^2 = \frac{\sin^2[\omega t - t_0]}{4\omega^2(\Delta x)^2} + (\Delta x)^2\cos^2[\omega(t-t_0)]$<br>$(\Delta p)^2 = \omega^2(\Delta x)^2\sin^2[\omega(t-t_0)] + \frac{\cos^2[\omega(t-t_0)]}{4(\Delta x)^2}$ | $\frac{1}{4}$ | Ellipse |
| FP | $\langle x(t)\rangle = x_0 + p_0(t-t_0)$<br>$\langle x(t)\rangle = p_0$ | $(\Delta x)^2 = (\Delta x)^2 + \frac{(t-t_0)^2}{4(\Delta x)^2}$<br>$(\Delta p)^2 = \frac{1}{4(\Delta x)^2}$ | $\frac{1}{4}\left[1 + \frac{(t-t_0)^2}{4(\Delta x)^4}\right]$ | Straight line |
| LP | $\langle x(t)\rangle = x_0 + p_0(t-t_0) - \frac{\kappa(t-t_0)^2}{2}$<br>$\langle p(t)\rangle = p_0 - \kappa(t-t_0)$ | $(\Delta x)^2 = (\Delta x)^2 - \frac{(t-t_0)^2}{4(\Delta x)^2}$<br>$(\Delta p)^2 = \frac{1}{4(\Delta x)^2}$ | $\frac{1}{4}\left[1 + \frac{(t-t_0)^2}{4(\Delta x)^4}\right]$ | Parabola |
| RO | $\langle x(t)\rangle = x_0\cosh[\omega(t-t_0)] + \frac{p_0}{\omega}\sinh[\omega(t-t_0)]$<br>$\langle p(t)\rangle = p_0\cosh[\omega(t-t_0)] + \omega x_0\sinh[\omega(t-t_0)]$ | $(\Delta x)^2 = (\Delta x)^2\cosh^2[\omega(t-t_0)] + \frac{\sinh^2[\omega(t-t_0)]}{4\omega^2(\Delta x)^2}$<br>$(\Delta p)^2 = \frac{\cosh^2[\omega(t-t_0)]}{4(\Delta x)^2} - \omega^2(\Delta x)^2\sinh^2[\omega(t-t_0)]$ | $\frac{1}{4}\cosh^4[\omega(t-t_0)] + \frac{1}{4}\sinh^4[\omega(t-t_0)] + \sinh^2[\omega(t-t_0)] \times\cosh^2[\omega(t-t_0)] \times[\omega^2(\Delta x)^4 + \frac{1}{16\omega^2}(\Delta x)^4]$ | Hyperbola |

and $x_0, t_0$ are some "initial values." We shall drop the subscript 0 later. The group action can be calculated by integrating the equations[18]

$$\frac{dx}{d\alpha} = -\bar{\xi}\ , \tag{A3a}$$

$$\frac{dt}{d\alpha} = 0\ , \tag{A3b}$$

$$\frac{d\ln\nu}{d\alpha} = ix\dot{\bar{\xi}} - i\bar{\mathscr{C}}\ . \tag{A3c}$$

First we integrate (A3b) to get $t = t_0$. Then, Eq. (A3a) has the solution

$$x = x_0 - \bar{\xi}\alpha = f_1(x_0, t_0, \alpha)\ . \tag{A4}$$

Substituting $t = t_0$ and (A4) into (A3c) we have

$$\frac{d\ln\nu}{d\alpha} = i(x_0 - \bar{\xi}\alpha)\dot{\bar{\xi}} - i\bar{\mathscr{C}}\ .$$

Hence

$$\ln\nu = i(x_0\alpha - \tfrac{1}{2}\bar{\xi}\alpha^2)\dot{\bar{\xi}} - i\bar{\mathscr{C}}\alpha\ . \tag{A5}$$

Going back to (A1)

$$\exp(\alpha J_+)h_0(x_0, t_0) = \exp\{i[(x_0\alpha - \tfrac{1}{2}\bar{\xi}\alpha^2)\dot{\bar{\xi}} - \mathscr{C}\alpha]\}h_0(x_0 - \bar{\xi}\alpha, t_0)\ . \tag{A6}$$

Dropping the subscripts on the variables $x$ and $t$ and making use of (3.4) and (3.5) we obtain the DOCS, (3.6).

To be able to identify the DOCS, Eq. (3.6), and the AOCS, (3.10), we need some relationships between the auxiliary time-dependent functions $\xi$, $\varphi_3$, $\mathscr{C}$, $\mathscr{B}$, $\Lambda$, and $\Phi$. In full form, the DOCS, (3.6), is

$$\begin{aligned} f_\alpha(x,t) &= (\pi\varphi_3)^{-1/4}\exp\left[-\frac{|\alpha|^2}{2} - \frac{\alpha^2\bar{\xi}}{2\xi} + \frac{x\alpha}{\xi} - \frac{2\alpha\bar{\xi}\mathscr{B}}{\varphi_3^{1/2}}\right]\exp\left[i\left(\frac{x^2\dot{\varphi}_3}{4\varphi_3} + \frac{x\mathscr{A}_3}{\varphi_3} - \Lambda - \tfrac{1}{2}\Phi\right)\right]\exp\left[-\left(\frac{x}{\varphi_3^{1/2}} - \mathscr{B}\right)^2\Big/2\right] \\ &= (\pi\varphi_3)^{-1/4}\exp\left[-\tfrac{1}{2}|\alpha|^2 - \frac{\alpha^2\bar{\xi}}{2\xi} + \frac{x\alpha}{\xi} - \frac{2\alpha\bar{\xi}\mathscr{B}}{\varphi_3^{1/2}}\right] \\ &\quad\times\exp\left[\frac{x^2}{2}\left(\frac{i\dot{\varphi}_3}{2\varphi_3} - \frac{1}{\varphi_3}\right) + x\left(\frac{i\mathscr{A}_3}{\varphi_3} + \frac{\mathscr{B}}{\varphi_3^{1/2}}\right) + (\tfrac{1}{2}\mathscr{B}^2 - i\Lambda) - \frac{i}{2}\Phi\right]\ . \end{aligned} \tag{A7}$$

We now develop the needed identities.

(i) First,

$$\frac{i\dot{\varphi}_3}{2\varphi_3} - \frac{1}{\varphi_3} = \frac{1}{\varphi_3}[i(\xi\dot{\bar{\xi}} + \dot{\xi}\bar{\xi}) - i(\xi\dot{\bar{\xi}} - \dot{\xi}\bar{\xi})] = \frac{i\dot{\xi}}{\xi}\ , \tag{A8}$$

where we have used the Wronskian (2.10).

(ii) From (2.12) and (2.21),

$$\frac{i\mathscr{A}_3}{\varphi_3} + \frac{\mathscr{B}}{\varphi_3^{1/2}} = \frac{-i(\xi\bar{\mathscr{C}} + \bar{\xi}\mathscr{C})}{\varphi_3} + \frac{i}{\varphi_3}(\xi\bar{\mathscr{C}} - \bar{\xi}\mathscr{C}) = \frac{-i\mathscr{C}}{\xi}\ . \tag{A9}$$

(iii) $\Lambda$ is defined in (2.21) which when expanded and using (2.12) we get

$$\Lambda = \int_{t_0}^{t}\frac{(\xi^2\bar{\mathscr{C}}^2 + \bar{\xi}^2\mathscr{C}^2)}{\varphi_3^2}ds$$

and so we have by integration by parts

$$-\tfrac{1}{2}\mathscr{B}^2 - i\Lambda = -\int_{t_0}^{t}ds\,\dot{\mathscr{B}}\mathscr{B} - i\int_{t_0}^{t}\frac{(\xi^2\bar{C}^2 + \bar{\xi}^2\mathscr{C}^2)}{\varphi_3^2}ds\ .$$

Substituting (2.21) and the definition of $\varphi_3$ we have

$$\tfrac{1}{2}\mathscr{B}^2 - i\Lambda = \frac{-i}{2}\int_{t_0}^{t}ds\frac{\mathscr{C}^2}{\xi^2}\ . \tag{A10}$$

(iv) Next,

$$\exp\left[-\frac{i}{2}\Phi\right] = \left[\frac{\bar{\xi}}{\xi}\right]^{1/2}\ , \tag{A11}$$

by (2.21). Substituting (A8)—(A11) into (A7) we obtain

$$\begin{aligned} f_\alpha(x,t) &= \frac{1}{(2\pi\xi\bar{\xi})^{1/4}}\left[\frac{\bar{\xi}}{\xi}\right]^{1/4} \\ &\quad\times\exp\left[-\frac{|\alpha|^2}{2} - \frac{\alpha^2\bar{\xi}}{2\xi} + \frac{x\alpha}{\xi} - \frac{2\alpha\bar{\xi}\mathscr{B}}{\varphi_3^{1/2}}\right] \\ &\quad\times\exp\left[\frac{ix^2\dot{\xi}}{2\xi} - \frac{ix\mathscr{C}}{\xi} - \frac{i}{2}\int_{t_0}^{t}ds\frac{\mathscr{C}^2}{\xi^2}\right]\ . \end{aligned} \tag{A12}$$

Other identities we require are the following.

(v) From (2.10),

$$\int_{t_0}^{t}\frac{ds}{\xi^2} = i\int_{t_0}^{t}\frac{ds(\xi\dot{\bar{\xi}} - \dot{\xi}\bar{\xi})}{\xi^2} = \frac{i\bar{\xi}}{\xi}\ . \tag{A13}$$

(vi) Integrating by parts, and using (2.10),

$$\frac{2\bar{\xi}\mathscr{B}}{\varphi_3^{1/2}}=2\int_{t_0}^{t}\left[\left[\frac{\varphi_3\dot{\bar{\xi}}-\frac{1}{2}\dot{\varphi}_3\bar{\xi}}{\varphi_3^{3/2}}\right]+\frac{\xi\mathscr{A}_3}{\varphi_3^2}\right]ds$$

$$=2\int_{t_0}^{t}\left[\frac{-i\bar{\xi}\mathscr{B}}{\varphi_3^{3/2}}+\frac{\xi\mathscr{A}_3}{\varphi_3^2}\right]ds$$

$$=-\int_{t_0}^{t}\frac{\mathscr{C}}{\xi^2}ds\ . \tag{A14}$$

Putting (A13) and (A14) back into (A12) and rearranging we have

$$f_\alpha(x,t)=(2\pi)^{-1/4}\xi^{-1/2}\exp(-\tfrac{1}{2}\,|\alpha\,|^2)\times\exp\left[\frac{ix^2\dot{\xi}}{2\xi}-\frac{ix\mathscr{C}}{\xi}+\frac{x\alpha}{\xi}-\frac{i}{2}\int_{t_0}^{t}ds\frac{(\alpha-i\mathscr{C})^2}{\xi^2}\right] \tag{A15}$$

which is equivalent to the AOCS (3.10) up to normalization.

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